κ-trees and Cohen κ-sequences

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Tree-forcing

In *set theory of the reals* some rather important objects are the so-called *tree-forcings*. This kind of forcings plays a relevant role in many applications regarding cardinal characteristics and regularity properties.

Definition

- $T \subseteq 2^{<\kappa}$ is called *perfect tree* iff T is closed under initial segments, T is closed under increasing $< \kappa$-sequences of nodes and $\forall s \in T \exists t \in T$ such that $s \subseteq t$ and t is *splitting*.
- $\mathbb{P}$ is called *tree-forcing* iff every $p \in \mathbb{P}$ is a perfect tree and for every $t \in \mathbb{P}$, $T_t \in \mathbb{P}$ too, and we define $q \leq p \Leftrightarrow q \subseteq p$.

Tree-ideals and tree-measurability

Definition

- X is $\mathbb{P}$-*open dense* iff $\forall T \in \mathbb{P} \exists T' \subseteq T (T' \in \mathbb{P} \wedge [T'] \subseteq X)$. The complement of a $\mathbb{P}$-open dense set is called $\mathbb{P}$-*nowhere dense*. X is $\mathbb{P}$-*meager* iff it can be covered by a $\leq \kappa$-size union of $\mathbb{P}$-nowhere dense sets. The ideals of $\mathbb{P}$-nowhere dense sets and $\mathbb{P}$-meager sets are respectively denoted by $\mathcal{N}_{\mathbb{P}}$ and $\mathcal{I}_{\mathbb{P}}$.
- X is $\mathbb{P}$-measurable iff for every $T \in \mathbb{P}$ there is $T' \subseteq T$, $T' \in \mathbb{P}$ such that $[T'] \cap X \in \mathcal{I}_{\mathbb{P}}$ or $[T'] \setminus X \in \mathcal{I}_{\mathbb{P}}$.

Some results about regularity properties at κ

- (Schlicht) The Levy collapse of an inaccessible to κ^+ gives a model where all projective sets have the perfect set property.
- (Lücke, Motto Ros, Schlicht) The Levy collapse of an inaccessible to κ^+ gives a model where all $\mathbf{\Sigma}^1_1$ sets have the Hurewicz dichotomy.
- (Friedman, Khomskii, Kulikov) Let $\mathbb{P}$ be a tree forcing which is either κ^+-cc or satisfies κ-axiom A. Then a κ^+-iteration of $\mathbb{P}$ with support of size κ yields $\mathbf{\Delta}^1_1(\mathbb{P})$.
- (L.) A κ^+ iteration with support of size $< \kappa$ of κ-Cohen forcing gives a model where all projective sets are Silver measurable.

κ-Mathias forcing and Cohen κ-sequences

What we know from the standard ω-case:

- $\mathbb{MA}$ has pure decision, satifies the Laver property (and so does not add Cohen reals)
- $\mathbb{MA}_{\omega_2} \Vdash \mathfrak{b} > \text{cov}(\mathcal{M})$
- $\mathbb{MA}_{\omega_1} \Vdash \mathbf{\Sigma}^1_2(\mathbb{MA}) \wedge \neg \mathbf{\Delta}^1_2(\mathbb{C})$.

κ-Mathias forcing $\mathbb{MA}$ for κ uncountable is defined as the poset of pairs (s, A), where $s \subseteq \kappa$ of size $< \kappa$ and $A \subseteq \kappa$ of size κ such that $\sup(s) < \min(A)$,with $(s, A) \geq (t, B) \Leftrightarrow t \supseteq s \wedge A \subseteq B \wedge t \setminus s \subseteq A$.

Remark

Note that for $\mathbb{MA}^{Club}$ we have the following two straightforward facts:

1. *$\mathbb{MA}^{Club}$ adds Cohen κ-reals. Let z be the canonical $\mathbb{MA}^{Club}$-generic set and then define $c \in 2^\kappa$ by: $c(\alpha) = 0$ iff the $\alpha + 1$-st element of z is in S (where $S \subseteq \kappa$ is stationary and co-stationary). One can easily check that c is κ-Cohen.*
2. *$\mathbb{MA}^{Club}$ does not have pure decision. In fact, let $T \in \mathbb{MA}^{Club}$ and $\alpha \in \kappa$ such that T does not decide the α-th element of z. Consider the formula $\varphi =$ "the α-th element of z is in S"; then φ cannot be purely decided by T.*

We now want to show that we can build a somehow more general κ-Cohen sequence which occurs even in cases when the set of splitting levels of a κ-Mathias condition is not a club. Let $\{\gamma_i : i < \kappa\}$ enumerate the limit ordinals $< \kappa$.

Fix a stationary and costationary set $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$. Given $x \in [\kappa]^\kappa$, $x := \{\alpha_\gamma : \gamma < \kappa\}$, we define, for $i < \kappa$:

$$C_x(i) := \begin{cases} 0 & \text{iff } \sup\{\alpha_{\gamma_i+n} : n \in \omega\} \in S. \\ 1 & \text{else.} \end{cases}$$

Claim

If x is $\mathbb{MA}$-generic, then C_x is κ-Cohen.

Sketch of the proof.

Let $(s, A) \in \mathbb{MA}$, $\bar{c}$ the part of C_x decided by (s, A) and $\sigma \in 2^{<\kappa}$ arbitrarily fixed. It is enough to show that there exists $(t, B) \leq (s, A)$ such that $(t, B) \Vdash \bar{c}^\frown\sigma \subseteq C_x$.
Let $A := \{\alpha^A_\gamma : \gamma < \kappa\}$. Define $\beta^A_j := \sup\{\alpha^A_{\gamma_j+n} : n \in \omega\}$.
Then we can freely remove elements from A in order to find $B \subseteq A$ such that

$$\beta^B_j \in S \Leftrightarrow \sigma(j) = 0.$$

Hence by fixing $t \subseteq B$ sufficiently long, we get (t, B) as desired. □

Two important topological differences from the ω-case

Proposition (Proposition 1)

$\mathcal{N}_{\mathbb{MA}} \neq \mathcal{I}_{\mathbb{MA}}$.

Proposition (Proposition 2)

Let Γ *be a topologically reasonable family of subsets of* κ*-reals. Then* $\Gamma(\mathbb{MA}) \Rightarrow \Gamma(\mathbb{C})$.

(Remind that $\Gamma(\mathbb{P}) :=$ every set in Γ is $\mathbb{P}$-measurable.)

Proof of Proposition 1.

Define

$$X_i := \{x \in [\kappa]^\kappa : \forall j > i, C_x(j) = 0\}.$$

It is clear that for each $i < \kappa$, $X_i \in \mathcal{N}_{\mathbb{MA}}$; indeed, let $T \in \mathbb{MA}$ and pick $j > i$ so large that $\gamma_j > ot(\{\alpha : \textsc{Stem}(T)(\alpha) = 1\})$. Then, one can easily shrink T in order to get $[T'] \cap X_i = \emptyset$, in a similar way as we argued before to prove that C_x was κ-Cohen.
But $X := \bigcup_{i<\kappa} X_i \notin \mathcal{N}_{\mathbb{MA}}$; indeed, for every $T \in \mathbb{MA}$, we can always find $z \in [T]$ and $i < \kappa$ such that for all $j > i$, $C_z(j) = 0$, which proves $[T] \cap X \neq \emptyset$ (e.g., it is enough to pick $\gamma_i > ot(\{\alpha : \textsc{Stem}(T)(\alpha) = 1\})$.) □

Proof of Proposition 2.

W.l.o.g. assume every $T_{(s,A)} \in \mathbb{MA}$ is such that $ot(s)$ is a limit ordinal. Define a map $\varphi : 2^\kappa \to 2^\kappa$ as follows: for every $x \in 2^\kappa$,

$$\varphi(x)(j) := \begin{cases} 0 & \text{iff } \sup \beta_j^x \in S \\ 1 & \text{else,} \end{cases}$$

where remind $\beta_j^x := \sup\{\alpha^x_{\gamma_j+n} : n \in \omega\}$. Moreover let $\varphi^* : 2^{<\kappa} \to 2^{<\kappa}$ be its associated *approximating function*. Let $X \in \Gamma$ and $Y := \varphi^{-1}[X]$. By assumption Y is $\mathbb{MA}$-measurable. We aim to prove X has the Baire property.

. . . .

Sublemma: Let $T \in \mathbb{MA}$. If $Y \cap [T]$ is $\mathbb{MA}$-comeager in $[T]$, then $X \cap [\varphi^*(\textsc{Stem}(T))]$ is comeager in $[\varphi^*(\textsc{Stem}(T))]$.
We give a sketch of the proof of this Sublemma.
Pick $\{B_\alpha : \alpha < \kappa\}$ be a $\subseteq$-decresing sequence of $\mathbb{MA}$-open dense sets in $[T]$ such that $\bigcap_{\alpha<\kappa} B_\alpha \subseteq Y \cap [T]$.
$\textsc{Aim}$: Find $\{U_\alpha : \alpha < \kappa\}$ open dense sets in $[\varphi^*(\textsc{Stem}(T))]$ such that $\bigcap_{\alpha<\kappa} U_\alpha \subseteq X \cap [\varphi^*(\textsc{Stem}(T))]$.
The set U_α are obtained as
$U_\alpha := \bigcup\{[\varphi^*(\textsc{Stem}(T(t)))] : t \in \kappa^{<\alpha}\}$, where the $T(t)$'s are recursively built as follows.
Fix an enumeration $\{\sigma_j : j < \kappa\}$ of all $\sigma \in 2^{<\kappa}$. Given $t \in \kappa^\alpha$ we can find $S(t^\frown j) \leq T(t)$ so that
$\varphi^*(\textsc{Stem}(S(t^\frown j)) \supseteq \varphi^*(\textsc{Stem}(T(t))^\frown \sigma_j$.
Then we pick $T(t^\frown j) \leq S(t^\frown j)$ so that $T(t^\frown j) \in B_{|t|}$.
For $t \in \kappa^\alpha$ with α limit ordinal, simply put $T(t) := \bigcap_{\xi<\alpha} T(t{\restriction}\xi)$.

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The construction then satisfies the following points:

- for every $t \in \kappa^{\alpha+1}$, we have $[T(t)] \subseteq B_\alpha$;
- $U_{\alpha+1} := \bigcup_{t \in \kappa^{\alpha+1}} [\varphi^*(\textsc{Stem}(T)(t))]$ is dense in U_α.

Note also that we can refine the choice of the $T(t^\frown j)$ in order to get for every $i \neq j$, $[\textsc{Stem}(T(t^\frown i))] \cap [\textsc{Stem}(T(t^\frown j))] = \emptyset$.
Hence $\bigcap_{\alpha<\kappa} U_\alpha$ is dense in $[\varphi^*(\textsc{Stem}(T))]$.
Finally, to show $\bigcap_{\alpha<\kappa} U_\alpha \subseteq X \cap [\varphi^*(\textsc{Stem}(T))]$ we argue as follows: pick $x \in \bigcap_\alpha U_\alpha$, $\eta \in \kappa^\kappa$ (unique) so that $x \in [\varphi^*(\textsc{Stem}(T_{\eta \restriction \alpha}))]$, for every $\alpha < \kappa$. Then pick $y \in \bigcap_{\alpha<\kappa} [T_{\eta\restriction\alpha}]$ so that $\varphi(y) = x$. By construction $y \in \bigcap_{\alpha<\kappa} B_\alpha \subseteq Y \cap [T]$, and so $\varphi(y) := x \in \varphi[Y] := X$. q.e.d (Sublemma)

. . . .

Now for every $t \in 2^{<\kappa}$, pick $T \in \mathbb{MA}$ so that $\varphi[[T]] = [t]$ (i.e., $\varphi^*(\textsc{Stem}(T)) = t$). Since we are assuming Y is $\mathbb{MA}$-measurable, it follows:

- there is $T' \leq T$ such that $Y \cap [T']$ is $\mathbb{MA}$-comeager, and so $\varphi^*(\textsc{Stem}(T')) := t' \supseteq t$ is such that $X \cap [t']$ is comeager in $[t']$ by the Sublemma applied to Y, or
- there is $T' \leq T$ such that $Y \cap [T']$ is $\mathbb{MA}$-meager, and so $\varphi^*(\textsc{Stem}(T')) := t' \supseteq t$ is such that $X \cap [t']$ is meager in $[t']$ by the Sublemma applied to $\kappa^\kappa \setminus Y$.

Hence, we get

$$\forall t \in 2^{<\kappa} \exists t' \supseteq t([t'] \cap X \text{ is meager} \vee [t'] \cap X \text{ is comeager}),$$

which means X has the Baire property. □

add($\mathcal{I}_{\mathbb{S}}$) vs cov($\mathcal{M}$)

Amoeba for Sacks

We start with an example for Sacks forcing in the ω-case.

Definition

We define $\mathbb{AS}$ the poset consisting of pairs (n, T), with T perfect tree and $n \in \omega$. The ordering is given by:

$$(n', T') \leq (n, T) \Leftrightarrow n' \geq n \wedge T' \subseteq T \wedge T' \restriction n = T \restriction n.$$

Given a generic filter G for $\mathbb{AS}$, one may easily check that $T_G := \bigcap \{T : (n, T) \in G\}$ is a perfect tree such that each branch is Sacks generic. From now on we refer to T_G as a **generic tree**.

Remark

Let T_G be the generic tree and $\{t_n : n \in \omega\}$ be the sequence of its leftmost splitting nodes. Define $z \in 2^\omega$ so that $z(n) = 0$ iff $|t_{n+1}| \leq \min\{|t| : t \in \textsc{Split}(T_G) \wedge t \supsetneq t_n{}^\frown 0\}$.

It easy to check that z is a Cohen real.
However, there is way to define finer versions of Sacks-amoeba in order to *kill* this kind (and all other kinds) of Cohen reals. This in particolar implies that one can force

$$\text{add}(\mathcal{I}_{\mathbb{S}}) > \text{cov}(\mathcal{M}).$$

(Similar situations occur for Miller and Laver forcing (Spinas, 1995).)

But what about the generalized context?

Amoeba for Sacks in 2^κ

Club Sacks forcing

There are several ways to generalize Sacks forcing. For instance one can consider the following.

Definition

Let $T \subseteq 2^{<\kappa}$ be a perfect tree. We say T is club Sacks iff for each $x \in [T]$ one has

$$\{\alpha < \kappa : x{\restriction}\alpha \text{ is splitting}\} \text{ is club.}$$

The forcing consisting of this kind of trees is called **club Sacks forcing** and denoted by $\mathbb{S}^{\text{Club}}$.

Amoeba for club Sacks

Definition

Define $\mathbb{AS}^{\text{Club}}$ as the poset consisting of pairs (p, T), with T club Sacks tree in $2^{<\kappa}$ and $p \subset T$ club subtree with size $< \kappa$. The order is:

$$(p', T') \leq (p, T) \Leftrightarrow p' \supseteq^{\text{end}} p \wedge T' \subseteq T.$$

As in the ω-case, given a generic filter G for $\mathbb{AS}^{\text{Club}}$, one can check that $T_G := \bigcap\{T : (\alpha, T) \in G\}$ is a club Sacks tree of generic branches.

Some important properties:

- $\mathbb{AS}^{\text{Club}}$ satisfies κ-axiom A, for κ inaccessible.
- $\mathbb{AS}^{\text{Club}}$ satisfies **quasi pure decision**: for every $D \subseteq \mathbb{AS}$ dense, $(p, T) \in \mathbb{AS}$, there is T' such that for every $(q, S) \leq (p, T')$,

$$(q, S) \in D \Rightarrow (q, T' \restriction q) \in D.$$

As in the ω-case, $\mathbb{AS}^{\text{Club}}$ adds κ-Cohen reals.
Here we are going to prove a much stronger result, showing that when you have a Sacks tree (not necessarily with club splitting) one can always code a sort of κ-Cohen sequence *inside* the tree.
This will then yield to the proof that add($\mathcal{I}_{\mathbb{S}}$) $\leq$ cov($\mathcal{M}$).

Coding by stationary sets

Let κ be inaccessible. Fix $\{S_\tau : \tau \in 2^{<\kappa}\}$ family of pairwise disjoint stationary subsets of $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$.

Lemma (Pruning Lemma)

Let $\{D_\alpha : \alpha < \kappa\}$ be a $\subseteq$-decreasing family of open subsets of 2^κ, $T \in \mathbb{S}$. There exists $T^* \leq_\alpha T$ such that for all $\alpha \in \lim(\kappa)$ there is $\tau_\alpha \in 2^{<\kappa}$ such that:

- $\forall \sigma \in 2^{\leq\alpha}$, $[\sigma^\frown\tau_\alpha] \subseteq D_\alpha$, *and*
- $\sup\{|r_n^\alpha| : n \in \omega\} \in S_{\tau_\alpha}$, *where for every $n \in \omega$, r_n^α is the leftmost splitnode in* $\mathrm{SPLIT}_{\alpha+n}(T^*)$.

The key step to prove the Pruning Lemma is the following result.

Lemma (Coding Lemma)

Let $\alpha \in \lim(\kappa)$, $T \in \mathbb{S}$ and $\tau \in 2^{<\kappa}$. There exists $T' \leq_\alpha T$ such that $\sup\{|r_n^\alpha| : n \in \omega\} \in S_\tau$ (where the r_n^α's are the leftmost splitnodes in $\textsc{Split}_{\alpha+n}(T')$).

Sketch of the proof.

Consider the stationary subset S_τ. Let s be the leftmost in $\textsc{Split}_{\alpha+1}(T)$. Then one can find $r_n^\alpha \supset s$, $r_n^\alpha \in \textsc{Split}(T)$ in such a way that $\sup\{|r_n^\alpha| : n \in \omega\} \in S_\tau$, and then let T' be the subtree obtained by setting r_n^α as the "new" leftmost nodes in $\textsc{Split}_{\alpha+n}(T')$, by removing the exceeding splitnodes. □

Proof of Pruning Lemma.

We build a κ-fusion sequence $\{T_\alpha : \alpha < \kappa\}$ as follows. We start from $T_0 = T$; for $\alpha < \kappa$ we recursively construct:

- $\alpha \notin \lim(\kappa)$: $T_{\alpha+1} = T_\alpha$.
- $\alpha \in \lim(\kappa)$: First put $S_\alpha = \bigcap_{\beta<\alpha} T_\beta$. Pick $\tau_\alpha \in 2^{<\kappa}$ such that:

$$\forall \sigma \in H_\alpha, \sigma^\frown \tau_\alpha \in \bigcap_{\beta<\alpha} D_\beta.$$

  Note this can be done as $2^{\leq\alpha}$ has size $< \kappa$ and each D_β is open dense. Then apply the sublemma with $\tau = \tau_\alpha$ and $T = S_\alpha$, and set $T_\alpha = T'$.

Finally put $T^* = \bigcap_{\alpha<\kappa} T_\alpha$. By construction T^* clearly satisfies the required properties. □

Definition

Given $T \in \mathbb{S}$ we define the *coding sequence associated with* T $\{\tau_\alpha : \alpha \in \lim(\kappa)\}$ in such a way that for every $\alpha \in \lim(\kappa)$, τ_α is chosen so that $\sup\{|t_n^\alpha| : n \in \omega\} \in S_{\tau_\alpha}$, where $t_n^\alpha \in \textsc{Split}_{\alpha+n}(T)$.

Theorem

add$(\mathcal{I}_{\mathbb{S}}) \leq$ *cov*$(\mathcal{M}_\kappa)$.

Sketch of proof.

Let $\lambda < \text{add}(\mathcal{I}_{\mathbb{S}})$ and $\{D_i : i < \lambda\}$ family of open dense subsets of 2^κ. We aim at finding $x \in 2^\kappa$ such that $x \in \bigcap_{i<\lambda} D_i$. Let $\{S_\tau : \tau \in 2^{<\kappa}\}$ be a family of pairwise disjoint stationary subsets of $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ as above. First of all recursively construct a family of maximal antichains $\{A_i : i < \lambda\}$ such that for every $i < \lambda$, every $T \in A_i$ satisfies:

$$\forall \beta \in \lim(\kappa) \forall w \in H_\beta, [w^\frown \tau_\beta] \subseteq D_i, \tag{1}$$

where $\sup\{|t_n^\beta| : n \in \omega\} \in S_{\tau_\beta}$ and t_n^β is the leftmost node in $\text{SPLIT}_{\beta+n}(T)$. Put $X_i := 2^\kappa \setminus \bigcup\{[T] : T \in A_i\}$. Since $\mathcal{N}_{\mathbb{S}} = \mathcal{I}_{\mathbb{S}}$, it follows there is $T^* \in \mathbb{S}$ such that $[T^*] \cap X_i = \emptyset$ for all $i < \lambda$.

. . . .

Now let $f : 2^{<\kappa} \to \textsc{Split}(T^*)$ the canonical isomorphism and $F : 2^\kappa \to [T^*]$ its induced one. Let c be a Cohen κ-real over the ground model and put $x = F(c)$. Note that since $F(c) \in [T^*]^{V[c]}$, T^* is coded in V, each A_i is a maximal antichain, it follows that for every $i < \lambda$ there exists $T^i \in A_i$ such that $V[c] \models x \in [T^i]$, and so there exists $\sigma \in 2^{<\kappa}$ such that $\sigma \Vdash x \in [T^i]$.
For $\sigma \in 2^{<\kappa}$, put $B(\sigma) := \bigcap\{T^i \in A_i : i < \lambda \wedge \sigma \Vdash x \in [T^i]\}$. A density argument shows that there exists $T(\sigma) \in \mathbb{S}$ such that $T(\sigma) \subseteq B(\sigma)$.

. . . .

Let $\{T^\eta : \eta < \kappa\}$ enumerate all such $T(\sigma)$'s and let $\{\tau^\eta_\xi : \xi \in \lim(\kappa)\}$ be the coding sequence associated with T^η. Define recursively $\{\rho_j : j < \kappa\}$ as follows:

- $\rho_0 := \emptyset$
- $\rho_{j+1} := \rho_j^\frown \tau^{\rho}_{\xi_{j+1}}$, where ξ_{j+1} is chosen in such a way that $2^{\leq \xi_{j+1}} \ni \rho_j$
- $\rho_j := \bigcup_{j'<j} \rho_{j'}$, for j limit ordinal.

and then put $x := \bigcup_{j<\kappa} \rho_j$. By construction $x \in \bigcap_{i<\lambda} D_i$ as desired. □

A couple of open questions

Some final questions

Question

Let $\mathcal{M}_\kappa$ be the ideal of κ-meager sets, $\mathcal{I}_{\mathbb{S}}$ is the ideal of $\mathbb{S}$-meager sets, and $\leq_T$ denotes Tukey embedding. Is $\mathcal{M}_\kappa \leq_T \mathcal{I}_{\mathbb{S}}$?

Question

Can one prove κ-axiom A for amoebas and tree-forcings when κ is successor?

THANK YOU FOR YOUR ATTENTION!