A null ideal for inaccessibles (?)

Giorgio Laguzzi (joint work with Sy Friedman)

Universität Freiburg

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Generalized Cantor space

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- **Question.** What about the *generalized* Lebesgue measure?

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Definition

Let $\mathbb{P}$ be a tree-like forcing. A set $X \subseteq 2^\kappa$ is said to be $\mathbb{P}$-*null* iff

$$\forall T \in \mathbb{P} \exists T' \in \mathbb{P}(T' \leq T \wedge [T'] \cap X = \emptyset).$$

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Note that $\mathbb{C}$-null sets correspond to κ-nowhere dense sets.
When dealing with the ω-case, **random null sets correspond to measure zero sets**.

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Shelah himself gives an answer to such a question, but assuming that κ be **weakly compact**.
Our method is different and provides us with an answer for κ inaccessible (weak compactness is not needed).

The main construction

We recursively define, for $\lambda < \kappa^+$, an increasing sequence of families of trees $\{\mathbb{F}_\lambda : \lambda < \kappa^+\}$ satisfying the following properties:

(P1) $\mathbb{F}_\lambda \subseteq \mathbb{S}^{\text{club}}$ and $|\mathbb{F}_\lambda| \leq \kappa$;

(P2) $\forall T \in \mathbb{F}_{<\lambda} \forall \gamma < \kappa \exists T' \leq_\gamma T \forall T'' \leq T' (T' \in \mathbb{F}_\lambda \wedge T'' \notin \mathbb{F}_{<\lambda})$;

(P3) $\forall T \in \mathbb{F}_\lambda \forall t \in T (T_t \in \mathbb{F}_\lambda)$;

(P4) $\mathbb{F}_\lambda$ is closed under descending $< \kappa$-sequences;

(P5) $\forall \alpha < \lambda \forall T \in \mathbb{F}_\lambda \setminus \mathbb{F}_\alpha \exists \bar{\gamma} < \kappa \forall \gamma \geq \bar{\gamma} \forall t \in \text{Split}_\gamma(T) \exists S \in \mathbb{F}_\alpha \setminus \mathbb{F}_{<\alpha} (T_t \subseteq S)$.

(Remind that $T \in \mathbb{S}^{\text{club}}$ iff T is **Sacks** and for all $x \in [T]$ one has $\{\alpha < \kappa : x{\restriction}\alpha \text{ splits }\}$ is **closed unbounded**.)

Finally, we define our forcing as follows:

$$\mathbb{F} := \bigcup_{\lambda < \kappa^+} \mathbb{F}_\lambda.$$

In our construction we assume $\diamondsuit_{\kappa^+}(S^{\kappa}_{\kappa^+})$, where $S^{\kappa}_{\kappa^+} := \{\lambda < \kappa^+ : \mathrm{cf}(\lambda) = \kappa\}$.

1. $\mathbb{F}_0 := \{(2^{<\kappa})_t : t \in 2^{<\kappa}\}$.
2. Case $\lambda + 1$: For every $T \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, pick $T' \in \mathbb{S}^{\text{club}}$ such that $T' \leq_\gamma T$ and T' does not contain subtrees in $\mathbb{F}_\lambda$. Then for all $t \in T'$ we add T'_t to $\mathbb{F}_{\lambda+1}$. We then close $\mathbb{F}_{\lambda+1}$ under descending $< \kappa$-sequences, i.e., for every descending $\{T^i : i < \delta\}$, with $\delta < \kappa$, we put $T^* := \bigcap_{i<\delta} T^i$ into $\mathbb{F}_{\lambda+1}$.
3. Case $\mathrm{cf}(\lambda) < \kappa$: let $\{T^i : i < \mathrm{cf}(\lambda)\} \subseteq \mathbb{F}_{<\lambda}$ be descending with $\{\mathrm{Rank}(T^i) : i < \mathrm{cf}(\lambda)\}$ cofinal in λ.
Then put $T^* := \bigcap_{i<\mathrm{cf}(\lambda)} T^i$ into $\mathbb{F}_\lambda$. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

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   - 4.a Suppose $D_\lambda \subseteq \lambda$ codes a maximal antichain A_λ in $\mathbb{F}_{<\lambda}$. For every $T \in \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, construct a "κ-fusion" sequence $\{T^i : i < \kappa\}$ of trees in $\mathbb{S}^{\text{club}}$ such that
     1. $T =: T^0 \geq_\gamma T^1 \geq_{\gamma+1} T^2 \geq_{\gamma+2} \cdots \geq_{\gamma+i} T^{i+1} \geq_{\gamma+i+1} \ldots$
     2. T^i_t belongs to $\mathbb{F}_{<\lambda}$ with $\mathrm{Rank}(T^i_t)$ at least λ_i for each t in $\mathrm{Split}_{\gamma+i}(T)$.
     3. $T^1 := \bigcup\{S_t : t \in \mathrm{Split}_\gamma(T)\}$, where each $S_t \leq T_t$ and S_t hits A_λ, i.e., there exists $S^* \in A_\lambda$ such that $S_t \leq S^*$.

     Then add $T^* := \bigcap_{i<\kappa} T^i$ to $\mathbb{F}_\lambda$. Moreover, for every $t \in T^*$, add T^*_t to $\mathbb{F}_\lambda$ too. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

4.b Suppose that $D_\lambda \subseteq \lambda$ codes $\{A_{i,j} : i < \kappa, j < \kappa\}$, where for each $i < \kappa$, $\bigcup_{j<\kappa} A_{i,j}$ is a maximal antichain in $\mathbb{F}_{<\lambda}$ and $j_0 \neq j_1 \Rightarrow A_{i,j_0} \cap A_{i,j_1} = \emptyset$. For every $T \in \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, build a κ-fusion sequence $\{T^i : i < \kappa\}$ of trees in $\mathbb{S}^{\text{club}}$ such that

1. $T =: T^0 \geq_\gamma T^1 \geq_{\gamma+1} T^2 \geq_{\gamma+2} \cdots \geq_{\gamma+i} T^{i+1} \geq_{\gamma+i+1} \cdots$
2. T^i_t belongs to $\mathbb{F}_{<\lambda}$ with $\text{Rank}(T^i_t)$ at least λ_i for t in $\text{Split}_{\gamma+i}(T^i)$.
3. for every $i < \kappa$, $T^{i+1} := \bigcup\{S^{i+1}_t : t \in \text{Split}_{\gamma+i}(T^i)\}$, where each $S^{i+1}_t \leq T^i_t$ and S^{i+1}_t hits $\bigcup_{j<\kappa} A_{i,j}$.

Then add $T^* := \bigcap_{i<\kappa} T^i$ to $\mathbb{F}_\lambda$. Moreover, for every $t \in T^*$, add T^*_t to $\mathbb{F}_\lambda$ too. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

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Then add $T^* := \bigcap_{i<\kappa} T^i$ to $\mathbb{F}_\lambda$. Moreover, for every $t \in T^*$, add T^*_t to $\mathbb{F}_\lambda$ too. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

4.c If D_λ neither codes a maximal antichain (case (a)) nor an instance of κ^κ-bounding (case (b)), then proceed as in case (a) without its item iii.

Proposition

$\mathbb{F}$ is $< \kappa$-closed, κ^+-cc and κ^κ-bounding.

Proof.

The $< \kappa$-closure follows from point 3 of the construction.
To prove κ^+-cc we argue as follows. Let $A \subseteq \mathbb{F}$ be a maximal antichain and pick λ such that $\mathrm{cf}(\lambda) = \kappa$ and $A \cap \mathbb{F}_{<\lambda}$ is coded by D_λ, using $\diamondsuit_{\kappa^+}(S^\kappa_{\kappa^+})$. By 4.(a) of the construction, for every $T \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$, there is γ' such that for every $\gamma \geq \gamma'$ for every $t \in \mathrm{Split}_\gamma(T)$, T_t is a subtree of some element of $A \cap \mathbb{F}_{<\lambda}$. By P5, if $T \in \mathbb{F} \setminus \mathbb{F}_\lambda$, there is $\gamma'' \geq \gamma'$ such that for every $\gamma \geq \gamma''$ for every $t \in \mathrm{Split}_\gamma(T)$, T_t is a subtree of some element of $\mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$. It follows that for any $T \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$ there is $t \in T$ such that T_t is a subtree of some element of $A \cap \mathbb{F}_{<\lambda}$, and therefore $A \cap \mathbb{F}_{<\lambda}$ is a maximal antichain in $\mathbb{F}$. So $A \cap \mathbb{F}_{<\lambda} = A$, which finishes the proof as $|\mathbb{F}_{<\lambda}| = \kappa$.

. . . .

For κ^κ-bounding we argue as follows. Let $\dot{x}$ be an $\mathbb{F}$-name for an element of κ^κ and $T \in \mathbb{F}$. Choose $\{A_{ij} : i < \kappa, j < \kappa\}$ so that for each $i < \kappa$, $\bigcup_{j<\kappa} A_{ij}$ is a maximal antichain and elements of A_{ij} force $\dot{x}(i) = j$. Pick $\lambda < \kappa$ such that T belongs to $\mathbb{F}_{<\lambda}$, $\mathrm{cf}(\lambda) = \kappa$ and D_λ codes such a sequence of antichains. By 4.(b) of the construction, we can then build a κ-fusion sequence in order to get $T' \leq T$ such that for each $i < \kappa$, T' forces the generic to hit $\bigcup_{j \in J_i} A_{ij}$, where each $J_i \subseteq \kappa$ has size $\leq 2^i$. Define $z \in \kappa^\kappa \cap V$ by $z(i) = \sup J_i$; then $T' \Vdash \forall i < \kappa, \dot{x}(i) \leq z(i)$. $\square$

Some results

$\mathbb{F}$-null VS meager

Proposition

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Proof.

Let $A := \{A_i : i < \kappa\}$ be a maximal antichain in $\mathbb{F}$. Clearly, $X := \bigcup_{i<\kappa}[A_i]$ is $\mathbb{F}$-conull, since for every $T \in \mathbb{F}$, there is $i < \kappa$ such that $A_i \parallel T$, and so there is $T' \leq A_i$ such that $T' \leq T$. It is then sufficient to show that we can find such an antichain A with the further property that any $[A_i]$ is nowhere dense. But note that by property P2, any $T \in \mathbb{F}$ can be extended to contain no subtree of the form $(2^{<\kappa})_s$ for $s \in 2^{<\kappa}$ and $[T]$ is nowhere dense for such a tree T. Now let $\mathbb{F}^* \subseteq \mathbb{F}$ be the dense set of such trees, and pick A a maximal antichain in $\mathbb{F}^*$. Then A remains a maximal antichain in $\mathbb{F}$ as well, and it is then enough for our purpose. $\square$

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There are essentially two possible notions of regularity related to $\mathbb{F}$.

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Definition

A set $X \subseteq 2^\kappa$ is said to be:

1. $\mathbb{F}$-*measurable* iff for every $T \in \mathbb{F}$ there exists $T' \in \mathbb{F}$, $T' \leq T$ such that $[T'] \setminus X \in \mathcal{I}_{\mathbb{F}}$ or $X \cap [T'] \in \mathcal{I}_{\mathbb{F}}$.
2. $\mathbb{F}$-*regular* iff there exists $B \in$ Bor such that $X \triangle B \in \mathcal{I}_{\mathbb{F}}$.

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Let $X \subseteq 2^\kappa$. X is $\mathbb{F}$-measurable iff X is $\mathbb{F}$-regular.

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Proposition

Let $X \subseteq 2^\kappa$. X is $\mathbb{F}$-measurable iff X is $\mathbb{F}$-regular.

Proposition (Friedman - L. / Friedman - Khomskii - Kulikov)

The club filter Cub is not $\mathbb{F}$-measurable. So, $\mathbf{\Sigma}^1_1(\mathbb{F})$ fails in ZFC.

Shelah's forcing $\mathbb{Q}$ VS $\mathbb{F}$

Theorem (Friedman- L.)

Let $V = L$ and $\mathbb{F}_{\kappa^+}$ be a κ^+-iteration with $\leq$ k-size support. Then

$$\mathbb{F}_{\kappa^+} \Vdash \mathbf{\Delta}^1_1(\mathbb{F}) \wedge \neg\mathbf{\Delta}^1_1(\mathbb{Q}).$$

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Proof.

The proof that $\mathbf{\Delta}^1_1(\mathbb{F})$ holds is rather standard. To prove $\neg \mathbf{\Delta}^1_1(\mathbb{Q})$ the key point is to check that $\mathbb{Q}$ does not satisfy the generalized Sacks property. To this aim, we prove that the $\mathbb{Q}$-generic is not captured by any ground model $\bar{\lambda}$-slalom $S = \{a_i : i < \kappa\}$, for a fixed $\bar{\lambda} = \{\lambda_i : i < \kappa\}$.

. . . .

Let $\langle \kappa_i : i < \kappa \rangle$ list all inaccessibles below κ (remind κ is weakly compact here). Let $\bar{\lambda} := \langle \lambda_i : i < \kappa \rangle$ and κ_{α_i} be the least inaccessible $> \lambda_i$. Given $x \in 2^\kappa$, we define $h_x \in \kappa^\kappa$ so that $h_x(i) = c(x \restriction I_i)$ has size $\leq \lambda_i$, where $c : 2^{<\kappa} \to \kappa$ is some coding map, $I_0 := [0, \kappa_{\alpha_1})$ and for all $i < \kappa$, $I_i := [\kappa_{\alpha_i}, \kappa_{\alpha_{i+1}})$. Let $S \in ([\kappa]^{<\kappa})^\kappa$ be a $\bar{\lambda}$-slalom. A rather technical proof shows that

$$A_S := \{x \in 2^\kappa : \forall i < \kappa (h_x(i) \in a_i)\}$$

is $\mathbb{Q}$-null, which concludes the proof. □

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What is not so good: $\mathbb{F}$ seems not to behave like random is some cases: for instance it satisfies the generalized Sacks property.

Thank you for listening!