Rosłanowski and Spinas dichotomies

Giorgio Laguzzi
(joint with Yurii Khomskii)

Universität Hamburg

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$\mathbb{P}$-dichotomies

Let $\mathcal{I}$ be a σ-ideal over the reals, and let $\mathbb{P}$ be a forcing with tree conditions.

Definition

We say that a set of reals X satisfies the $(\mathcal{I}, \mathbb{P})$-dichotomy iff either $X \in \mathcal{I}$ or there exists $T \in \mathbb{P}$ such that $[T] \subseteq X$.

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Well-known examples:

- Perfect set property: $\mathcal{I}$ = ideal of countable sets / $\mathbb{P}$ = Sacks forcing
- K_σ-regularity: $\mathcal{I}$ = ideal of bounded sets / $\mathbb{P}$ = Miller forcing

Such dichotomies provide a dense embedding

$$\mathbb{P} \hookrightarrow \textsc{Borel} \setminus \mathcal{I}.$$

They are useful because we can use both the combinatorial properties of trees and the properties of the σ-ideal for studying the forcing notion associated. As an example, one can consider the following result of Zapletal.

Theorem (Zapletal)

If $\mathcal{I}$ is a σ-ideal on ω^ω σ-generated by closed sets then the forcing $\textsc{Borel} \setminus \mathcal{I}$ is proper and preserves Baire category (non-meager ground-model sets remain non-meager in the extension).

The perfect set property is related to Davis' game on the Cantor space 2^ω. The analogous game played on the Baire space ω^ω gives rise to the following dichotomy.

Definition

- Given $f : \omega^{<\omega} \to \omega$, let $D_f := \{x \in \omega^\omega : \forall^\infty n (f(x{\restriction}n) \neq x(n))\}$ and then $\mathfrak{D}_\omega := \{D_f : f : \omega^{<\omega} \to \omega\}$.
- We say that a tree $T \subseteq \omega^{<\omega}$ is *full-splitting* iff for every splitting node $t \in T$ for all $n \in \omega$, $t^\frown n \in T$.
- We say that a set $X \subseteq \omega^\omega$ satisfies the $(\mathfrak{D}_\omega, \mathbb{FM})$-dichotomy (or *Rosłanowski dichotomy*) iff either $X \in \mathfrak{D}_\omega$ or there exists $T \in \mathbb{FM}$ such that $[T] \subseteq X$.

Theorem (Rosłanowski)

Every $\mathbf{\Sigma}^1_1$ set satisfies the Rosłanowski dichotomy.

A slightly different σ-ideal has been studied by Spinas.

Definition

For every $x \in \omega^\omega$ let $K_x := \{y \in \omega^\omega \mid \forall^\infty n(x(n) \neq y(n))\}$, and let $\mathfrak{I}_{\mathrm{ioe}}$ be the σ-ideal generated by K_x, for $x \in \omega^\omega$.

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The two ideals are very similar, and in fact the following equalities hold.

Proposition

1. $cov(\mathfrak{I}_{\mathrm{ioe}}) = cov(\mathfrak{D}_\omega) = cov(\mathcal{M})$
2. $non(\mathfrak{I}_{\mathrm{ioe}}) = non(\mathfrak{D}_\omega) = non(\mathcal{M})$.
3. $add(\mathfrak{I}_{\mathrm{ioe}}) = add(\mathfrak{D}_\omega) = \omega_1$
4. $cof(\mathfrak{I}_{\mathrm{ioe}}) = cof(\mathfrak{D}_\omega) = \mathfrak{c}$.

The notion of a full-splitting Miller tree is not sufficient to get the right dichotomy for $\mathfrak{I}_{\mathrm{ioe}}$, as the following example shows.

Example

Let T be the tree on $\omega^{<\omega}$ defined as follows:

- If $|s|$ is even then $\textsc{Succ}_T(s) = \{0, 1\}$.
- If $|s|$ is odd then

$$\textsc{Succ}_T(s) = \begin{cases} 2\mathbb{N} & \text{if} \quad s(|s|-1) = 0 \\ 2\mathbb{N}+1 & \text{if} \quad s(|s|-1) = 1 \end{cases}$$

where $\textsc{Succ}_T(s) := \{n \mid s^\frown \langle n \rangle \in T\}$. Clearly T is $\mathfrak{I}_{\mathrm{ioe}}$-positive but cannot contain a full-splitting subtree.

The right dichotomy for $\mathfrak{I}_{\mathrm{ioe}}$ involves a subtle modification of the notion of a full-splitting Miller tree.

Definition

A tree $T \subseteq \omega^\omega$ is called an *infinitely often equal tree*, or simply *ioe-tree*, if for each $t \in T$ there exists $N > |t|$, such that for every $k \in \omega$ there exists $s \in T$ extending t such that $s(N) = k$. Let $\mathbb{IE}$ denote the partial order of ioe-trees ordered by inclusion.

Definition

We say that a set $X \subseteq \omega^\omega$ satisfies the $(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE})$-dichotomy (or *Spinas dichotomy*) iff either $X \in \mathfrak{I}_{\mathrm{ioe}}$ or there exists $T \in \mathbb{IE}$ such that $[T] \subseteq X$.

Theorem (Spinas)

Every $\mathbf{\Sigma}^1_1$ set satisfies the Spinas dichotomy.

Simple remarks about $\mathbb{FM}$ and $\mathbb{IE}$

- $\mathbb{FM}$ adds a Cohen real (let $\{s_n : n \in \omega\}$ be a fixed enumeration of $\omega^{<\omega}$ and consider the function φ defined by $\varphi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown s_{x(2)}{}^\frown \dots$).
- $\mathbb{IE} * \mathbb{IE}$ adds a Cohen real.
- $\mathbb{IE}$ below a certain condition is equivalent to $\mathbb{FM}$. Such a condition is constructed in the following way:
  1. If $s \neq t$ are splitting nodes of T^{GS} then $|s| \neq |t|$.
  2. If $t \in T^{\mathrm{GS}}$ is a non-splitting node of T then $t(|t|-1) = 0$.

Hence, $\mathbb{IE}$ also adds Cohen reals.

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We now want to investigate the behaviour of Rosłanowski and Spinas dichotomies for higher projective class, i.e., statements of the form $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$, $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$, etc. Note that such statements have a rather unpredictable behaviour:

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- in yet other cases, involving the Silver forcing, the related dichotomy for $\mathbf{\Sigma}^1_2$ sets is actually inconsistent.

Spinas and Rosłanowski dichotomies will fall into the second category.

A useful characterization is the Mansfield-Solovay style theorem for $\mathbb{FM}$ and $\mathbb{IE}$.

Proposition (Khomskii - L.)

1. *For any $\Sigma^1_2(r)$ set A, either there exists an $\mathbb{FM}$-tree $U \in L[r]$, such that $[U] \subseteq A$, or A can be covered by $\mathfrak{D}_\omega$-small Borel sets coded in $L[r]$.*
2. *For any $\Sigma^1_2(r)$ set A, either there exists an $\mathbb{IE}$-tree $U \in L[r]$, such that $[U] \subseteq A$, or A can be covered by $\mathfrak{I}_{\mathrm{ioe}}$-small Borel sets coded in $L[r]$.*

The proof uses a standard Cantor-Bendixson analysis.

We can then use such a characterization to prove the following result.

Theorem (Khomskii - L.)

1. $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$
2. $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$
3. $\forall r \in \omega^\omega \ \{x \mid x \textit{ is not iof over } L[r]\} \in \mathfrak{D}_\omega$
4. $\forall r \in \omega^\omega \ \{x \mid x \textit{ is not ioe over } L[r]\} \in \mathfrak{I}_{\mathrm{ioe}}$
5. $\forall r \in \omega^\omega \ (\omega_1^{L[r]} < \omega_1)$

Proof.

- (1) $\Rightarrow$ (3). Fix an arbitrary r and let $X := \{x \mid x \text{ is not iof over } L[r]\}$. It is not hard to see that X is a $\Sigma^1_2(r)$ set, so by assumption either $X \in \mathfrak{D}_\omega$ or there is some $T \in \mathbb{FM}$ such that $[T] \subseteq X$. We will show that the second option is impossible. From $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ we have $\mathbf{\Sigma}^1_2(\mathbb{FM})$, which can be proven to be equivalent to $\mathbf{\Sigma}^1_2(\text{Baire})$. In particular, there is a Cohen real c, which is an iof real, over $L[r]$. Let $T \in \mathbb{FM}$ and recall that there is a homeomorphism $\psi : \omega^\omega \cong [T]$ such that ψ-preimages of $\mathfrak{D}_\omega$-small sets are $\mathfrak{D}_\omega$-small. Since being an iof real is the same as being $\mathfrak{D}_\omega$-quasigeneric, it easily follows that $\psi(c)$ is an iof real in $[T]$. This contradicts $[T] \subseteq X$.

. . . .

- (3) $\Rightarrow$ (1). By the previous proposition, we know that every $\mathbf{\Sigma}^1_2$ set A either contains $[T]$ for $T \in \mathbb{FM}$ or $A \subseteq \{x \mid x$ is not $\mathfrak{D}_\omega$-quasigeneric over $L[r]\} = \{x \mid x$ is not iof over $L[r]\}$, from which the result follows.
- (5) $\Rightarrow$ (3). If $\omega_1^{L[r]} < \omega_1$ then $\{x \mid x$ is not iof over $L[r]\} = \bigcup\{B \mid B$ is a Borel $\mathfrak{D}_\omega$-small set coded in $L[r]\}$ is a countable union of $\mathfrak{D}_\omega$-small sets.

. . . .

- (3) $\Rightarrow$ (5). A result of Newelski and Rosłanowski implies that for any family $F = \{x_\alpha \mid \alpha < \omega_1\}$ of reals satisfying $\forall \alpha \neq \beta \, \exists^\infty n \, (x_\alpha(n) \neq x_\beta(n))$, and letting $X_\alpha := \{x \mid \forall n \, (x(n) \neq x_\alpha(n))\}$, we have
  - $X_\alpha \in \mathfrak{I}_{\text{ioe}} \subseteq \mathfrak{D}_\omega$ for all $\alpha < \omega_1$, and
  - $\bigcup_{\alpha<\omega_1} X_\alpha \notin \mathfrak{D}_\omega$.

  If $\omega_1^{L[r]} = \omega_1$ for some r, then we have an F as above satisfying $F \subseteq \omega^\omega \cap L[r]$. But then $\{x \mid x$ is not iof over $L[r]\} = \bigcup\{B \mid B$ is a Borel $\mathfrak{D}_\omega$-small coded in $L[r]\} \supseteq \bigcup\{X_\alpha \mid \alpha < \omega_1\}$ cannot be $\mathfrak{D}_\omega$-small.

$\square$

Solovay's model and higher projective levels

Theorem (Khomskii - L.)

Let κ be inaccessible and let G be $Coll(\omega, < \kappa)$-generic over V. Then in $V[G]$ all sets definable from countable sequences of ordinals satisfy the $\mathbb{FM}$- and the $\mathbb{IE}$-dichotomy, and in $L(\mathbb{R})^{V[G]}$ all sets of reals satisfy the $\mathbb{FM}$- and $\mathbb{IE}$-dichotomy.

A game for $\mathbb{IE}$-dichotomy

Definition

Let $G^{\mathbb{IE}}(A)$ be the game in which players I and II play as follows:

I:	N_0		(s_0, N_1)		(s_1, N_2)		$\dots$	
II:		k_0		k_1		k_2		$\dots$

where $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$, $N_i \geq 1$, $k_i \in \omega$, and the following rules must be obeyed for all i:

- $|s_i| = N_i$,
- $s_i(N_i - 1) = k_i$.

Then player I wins iff $z := s_0{}^\frown s_1 \frown s_2{}^\frown \cdots \in A$.

Theorem (Khomskii - L.)

1. *Player I has a winning strategy in* $G^{\mathbb{IE}}(A)$ *iff there is an* $\mathbb{IE}$*-tree* T *such that* $[T] \subseteq A$.
2. *Player II has a winning strategy in* $G^{\mathbb{IE}}(A)$ *iff* $A \in \mathfrak{I}_{\mathrm{ioe}}$.

Open questions

A couple of open questions

1. Is there a $T \in \mathbb{IE}$ forcing that "there are no Cohen reals"?
2. Are $\mathbf{\Pi}^1_1(\mathbb{FM}\text{-dich})$ and $\mathbf{\Pi}^1_1(\mathbb{IE}\text{-dich})$ equivalent to $\forall r \in \omega^\omega\ (\omega_1^{L[r]} < \omega_1)$?
3. Investigate the ideal σ-generated by the sets X satisfying $\forall T \in \mathbb{P} \exists T' \in \mathbb{P}(T' \leq T \wedge [T'] \cap X = \emptyset)$, where $\mathbb{P} \in \{\mathbb{FM}, \mathbb{IE}\}$.

Thanks for your attention!