Regularity properties and tree-forcings

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Brief Introduction

Over the years, several notions of regularity have been studied in set theory. The most popular ones are certainly the Baire property and the Lebesgue measurability.

Definition

A set of reals X is *Lebesgue measurable* iff there exists a Borel set B such that $X \triangle B$ is null. Analogously one can define the Baire property by replacing "null" with "meager".

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Another important notion of regularity comes from Ramsey theory.

Definition

$X \subseteq [\omega]^\omega$ is *completely Ramsey* iff for every $s \in [\omega]^{<\omega}$ and $H \in [\omega]^\omega$, $H \supset s$, there exists $H' \subseteq H$ such that either $[s, H']^\omega \subseteq X$ or $[s, H']^\omega \cap X = \emptyset$.

Definition

$T \subseteq \omega^{<\omega}$ is called *perfect tree* iff it is closed under initial segments and for every $s \in T$ there exist $t \supseteq s$ in T and $n_0, n_1 \in \omega$ such that both $t^\frown n_0$ and $t^\frown n_1$ are in T.
A poset $\mathbb{P}$ is called *tree-forcing* iff every $T \in \mathbb{P}$ is a perfect tree and for all $t \in T$ one has $T_t := \{s \in T : s \subseteq t \vee t \subseteq s\} \in \mathbb{P}$.
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Some examples:

- Cohen forcing $\mathbb{C} := \{s \in 2^{<\omega}\}$
- random forcing $\mathbb{B} := \{T : T \text{ perfect tree} \wedge \mu([T]) > 0\}$
- Mathias forcing

$$\mathbb{MA} := \{T \subseteq 2^{<\omega} : \forall s \supseteq \mathrm{STEM}(T)(s^\frown 1 \in T \Rightarrow s^\frown 0 \in T)\}.$$

$\mathbb{P}$-measurability

Definition

A set of reals X is called $\mathbb{P}$-*null* iff for every $T \in \mathbb{P}$ there exists $T' \in \mathbb{P}$ such that $T' \subseteq T$ and $X \cap [T'] = \emptyset$. Furthermore, we define $I_{\mathbb{P}}$ to be the σ-ideal σ-generated by the $\mathbb{P}$-null sets.
A set of reals X is said to be $\mathbb{P}$-measurable iff
$\forall T \in \mathbb{P} \exists T' \in \mathbb{P}, T' \subseteq T (X \cap [T'] \in I_{\mathbb{P}} \vee [T'] \setminus X \in I_{\mathbb{P}})$.

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The following are well-known:

- X has the Baire property iff X is $\mathbb{C}$-measurable;
- X is Lebesgue measurable iff X is $\mathbb{B}$-measurable;
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We use the following notation

$$\Gamma(\mathbb{P}) :\equiv \text{all sets of reals are } \mathbb{P}\text{-measurable}.$$

Silver and Miller

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- A perfet tree $T \subseteq 2^{<\omega}$ is *Silver* iff for every $s, t \in T$, with $|s| = |t|$, one has

$$s^\frown 0 \in T \Leftrightarrow t^\frown 0 \in T \wedge s^\frown 1 \in T \Leftrightarrow t^\frown 1 \in T.$$

- A perfet tree $T \subseteq \omega^{<\omega}$ is *Miller* iff every splitting node has infinitely many immediate successors.

Historical background

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- (Shelah, 1984-1985) $\mathrm{Con}(ZF + DC + \Gamma(\mathbb{B})) \to \mathrm{Con}(ZFC + \exists\kappa \text{ inaccessible})$, while $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZF + DC + \Gamma(\mathbb{C}))$.

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More recently, the study of regularity properties has been continued by other set theorists: Brendle, Löwe, Spinas, Schrittesser, Friedman, Ikegami and Khomskii.

Regularity properties diagram

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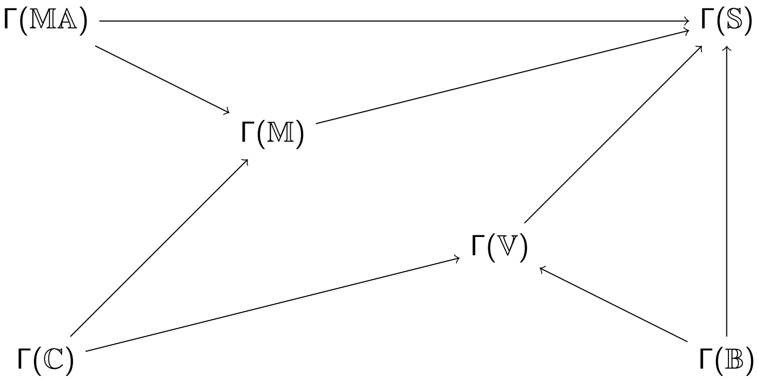

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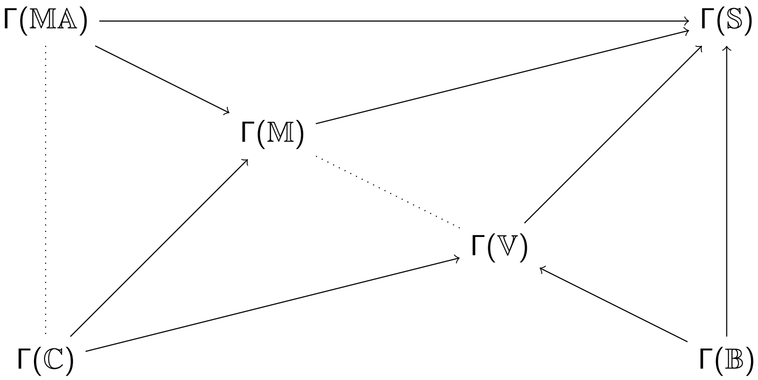

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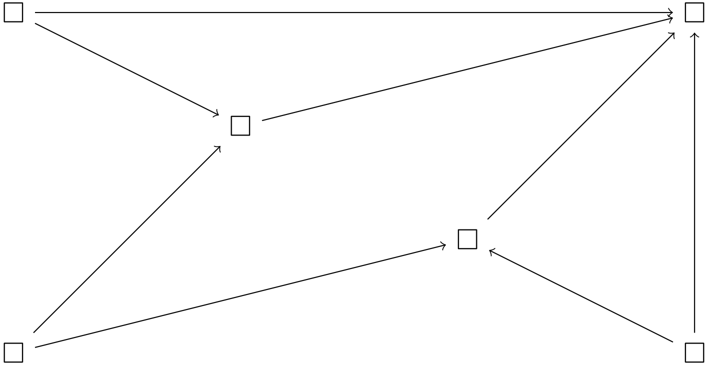

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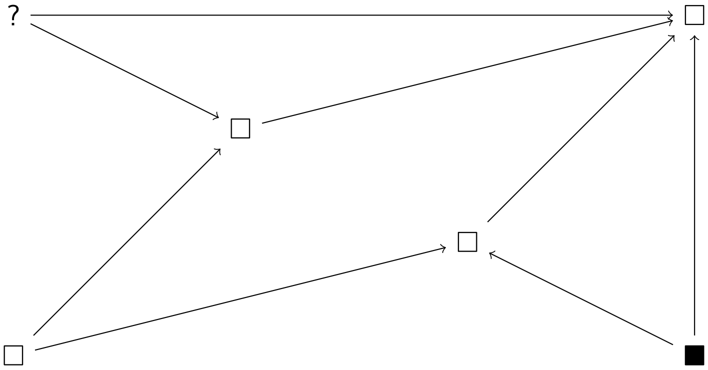

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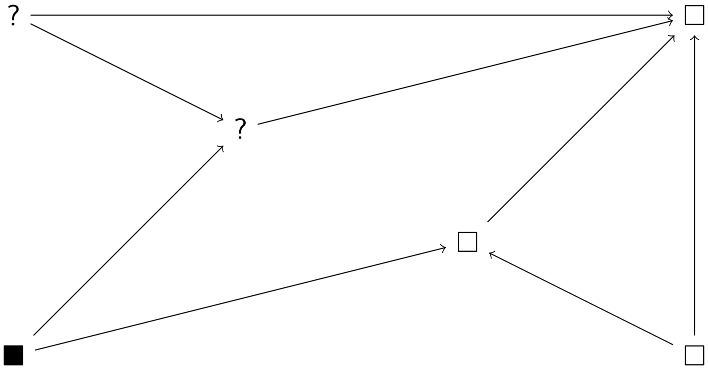

$\mathbb{P}$-homogeneity

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To force all sets to be $\mathbb{P}$-measurable, Solovay's proof needs a complete boolean algebra satisfying the following key property.

Definition

A complete boolean algebra B is $\mathbb{P}$-homogeneous iff for every formula $\phi(x)$ with parameters in the ground model, and B-name τ for a $\mathbb{P}$-generic real, one has $||\phi(\tau)||_B \in B_\tau$, where B_τ is the complete subalgebra generated by τ.

In particular, if B satisfies the following:

for every $B_0, B_1 \lessdot B$ such that $B_0 \cong B_1 \cong \mathbb{P}$ and $f : B_0 \to B_1$ there exists $f^* \supseteq f$ such that f^* is an automorphism of B,

then B is $\mathbb{P}$-homogeneous.

$(\mathbb{P}, Y)$-homogeneity

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Definition

Let B be a complete boolean algebra and Y a B-name for a set of reals. We say that B is $(\mathbb{P}, Y)$-homogeneous iff for every formula $\phi(Y, x)$ and B-name τ for a $\mathbb{P}$-generic real, one has $||\phi(Y, \tau)||_B \in B_\tau$, where B_τ is the complete subalgebra generated by τ.

To obtain this property, the B-name Y needs to be a fixed point of the automorphisms f^*, i.e., $\Vdash_B f^*(Y) = Y$.

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Question. Why $(\mathbb{P}, Y)$-homogeneity?

Solovay's proof: $\mathbb{P}$-homogeneity gives

1. $V[G] \models$ all On^ω-definable sets are $\mathbb{P}$-measurable.
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Analogously, $(\mathbb{P}, Y)$-homogeneity gives

1. $V[G] \models$ all (On^ω, Y)-definable sets are $\mathbb{P}$-measurable.
2. Moreover, Y can be constructed in order to get $V[G] \models Y$ is not $\mathbb{Q}$-measurable.
3. Hence, $L(\mathbb{R}, \{Y\})^{V[G]} \models \Gamma(\mathbb{P}) \wedge \neg\Gamma(\mathbb{Q})$.

Shelah's amalgamation

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Given a Boolean algebra A, $A^0, A^1 \lessdot A$ and $f : A^0 \to A^1$ an isomorphism, the amalgamation provides us with a machinery to build a complete Boolean algebra $A^* \supseteq A$ and an automorphism $f^* : A^* \to A^*$ such that $f^* \supseteq f$.

Then, we can iterate this process and use a book-keeping argument in order to obtain a complete Boolean algebra $B \supseteq A$ such that *for each isomorphic pair* $A^0, A^1 \lessdot B$ and $f : A^0 \to A^1$ there exists $f^* \supseteq f$, $f^* : B \to B$ automorphism.

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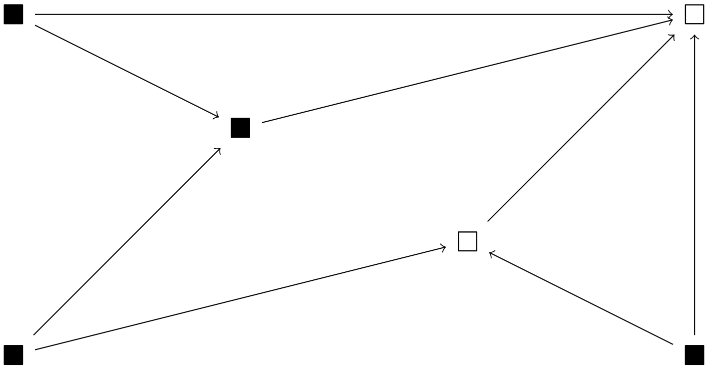
key point. We want to define Y is order to obtain:

- $f^*(Y) = Y$, for every automorphism generated by the amalgamation, and
- Y is not $\mathbb{Q}$-measurable.

$\Gamma(\mathbb{V}) \wedge \neg\Gamma(\mathbb{M}) \wedge \neg\Gamma(\mathbb{B})$

Let us focus on the following diagram:
(L., 2012)



In this particular situation we need to build two different sets of B-names Y and Z:

- Y will be non-Miller measurable;
- Z will be non-Lebesgue measurable.

We want to recursively construct $\langle B_\alpha : \alpha < \kappa \rangle$, $\langle Y_\alpha : \alpha < \kappa \rangle$ and $\langle Z_\alpha : \alpha < \kappa \rangle$ and put

- $B := \lim_{\alpha<\kappa} B_\alpha$
- $Y := \bigcup_{\alpha<\kappa} Y_\alpha$
- $Z := \bigcup_{\alpha<\kappa} Z_\alpha$.

Let us see a sketch of the construction.

- If $f : A_0 \to A_1$ is an isomorphism, $A_0 \cong A_1 \cong \mathbb{V}$, $\langle B_{\alpha_\eta} : \eta < \kappa \rangle$ is an *increasing cofinal* sequence of complete Boolean algebras and $\langle f_\eta : \eta < \kappa \rangle$ is a sequence of isomorphisms generated by the amalgamation, with $\mathrm{dom}(f_\eta) = B_{\alpha_\eta}$ and $f_\eta \supseteq f$, then we put

$$\dot{Y}_{\alpha_\eta+1} := \dot{Y}_{\alpha_\eta} \cup \{f_\eta^j(\dot{y}), f_\eta^{-j}(\dot{y}) : \dot{y} \in \dot{Y}_{\alpha_\eta}, j \in \omega\},$$
$$\dot{Z}_{\alpha_\eta+1} := \dot{Z}_{\alpha_\eta} \cup \{f_\eta^j(\dot{z}), f_\eta^{-j}(\dot{z}) : \dot{z} \in \dot{Z}_{\alpha_\eta}, j \in \omega\};$$

- for cofinally many α's,

$$B_{\alpha+1} = B_\alpha * \dot{\mathbb{AV}}.$$

In this case, put $\dot{Y}_{\alpha+1} = \dot{Y}_\alpha$ and $\dot{Z}_{\alpha+1} = \dot{Z}_\alpha$.

- for cofinally many α's, $B_{\alpha+1} = B_\alpha * \dot{\mathbb{M}}$ and

$$\dot{Y}_{\alpha+1} = \dot{Y}_\alpha \cup \{\dot{y}_T : T \in \mathbb{M}\},$$

where $\dot{y}_T$ is a name for a Miller real over N^{B_α} through $T \in \mathrm{N}^{B_\alpha}$,

- for cofinally many α's, $B_{\alpha+1} = B_\alpha * \dot{\mathbb{B}}$ and

$$\dot{Z}_{\alpha+1} = \dot{Z}_\alpha \cup \{\dot{z}_T : T \in \mathbb{B}\},$$

and z_T is a name for a random real through the positive measure tree $T \in \mathrm{N}^{B_\alpha}$.

By using $(\mathbb{V}, Y, Z)$-homogeneity, together with the amoeba Silver $\mathbb{AV}$, a pretty standard argument gives

$$\mathrm{N}[G] \models \text{all } (\mathrm{On}^\omega, Y, Z)\text{-definable sets are } \mathbb{V}\text{-measurable.}$$

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What is more complicate is to prove that Y and Z are not *regular*.
We need to find two combinatorial properties for the names in Z and Y, respectively, which are:

- preserved by amalgamation;
- preserved by Silver extension;
- satisfied by random reals and Miller reals, respectively.

Unboundedness

For Y, the suitable property is:

$$\dot{x} \text{ is unbounded over the ground model } \mathrm{N},$$

i.e., $\forall y \in \omega^\omega \cap \mathrm{N} \exists^\infty n(y(n) < \dot{x}(n))$. Note that Miller reals are unbounded over the ground model. Such a property was also used by Shelah to get $\Gamma(\mathbb{B}) \wedge \neg\Gamma(\mathbb{C})$.

Unreachability

For Z, we need to introduce a different property: the *unreachability*. **A real x is unreachable iff it is not captured by any ground model slalom**.

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- $\Gamma_k = \{\sigma \in \mathbf{HF}^\omega : \forall n \in (|\sigma(n)| \leq 2^{kn})\}\}$ and $\Gamma = \bigcup_{k\in\omega} \Gamma_k$, where **HF** denotes the hereditary finite sets;
- let $g(n) = 2^n$ and $\{I_n : n \in \omega\}$ be the partition of ω such that $I_0 = \{0\}$ and $I_{n+1} = \left[\sum_{j\leq n} g(j), \sum_{j\leq n+1} g(j)\right)$, for every $n \in \omega$;
- given $x \in 2^\omega$, define $h_x(n) = x{\restriction}I_n$.

Definition

One says that $z \in 2^\omega$ is *unreachable over* N iff

$$\forall \sigma \in \Gamma \cap \mathrm{N} \exists n \in \omega (h_z(n) \notin \sigma(n)).$$

The following hold:

- If x is random over N, then x is unreachable over N.
- If x is unreachable over N and r is $\mathbb{V}$-generic over N, then x is unreachable over $\mathrm{N}[r]$.
- The property "x is unreachable over the ground model" is preserved by amalgamation.

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Corollary

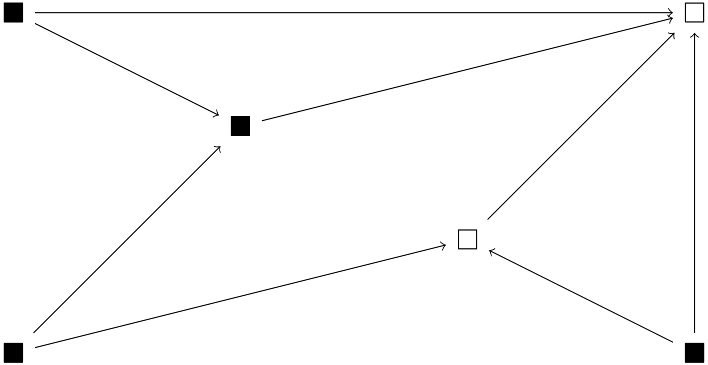
Z is not Lebesgue measurable.

$\Gamma(\mathbb{V}) \wedge \neg\Gamma(\mathbb{M}) \wedge \neg\Gamma(\mathbb{B})$

Hence, we obtain

$$L(\mathbb{R}, \{Y\}, \{Z\})^{\mathrm{N}[G]} \models \Gamma(\mathbb{V}) \wedge \neg\Gamma(\mathbb{B}) \wedge \neg\Gamma(\mathbb{M}),$$

which gives us the desired diagram



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  (I conjecture that one can construct a model for $\Gamma(\mathbb{MA}) \wedge \neg\Gamma(\mathbb{C})$.)
- **Main open problem**: can one build a model for $\Gamma(\mathbb{MA})$ without using inaccessible cardinals?

Grazie per la vostra attenzione!

Thanks for your attention!