Seminar: Special Holonomy

Sebastian Goette
Anda Degeratu

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Abstract

The focus of this seminar are Riemannian manifolds with special holonomy, and, in particular, Joyce’s construction of compact $G_2$-manifolds.

Introduction

Our main goal is to understand Joyce’s construction of the first compact manifold with $G_2$-holonomy.

We start by introducing the concept of holonomy associated to a connection, then talk about the holonomy of a Riemannian manifold as the holonomy of the Levi-Civita connection. For a generic Riemannian metric on an $n$-dimensional manifold, the holonomy is the full group $SO(n)$. A smaller holonomy group corresponds to extra-symmetries of the Riemannian metric. For example, $U(n)$-holonomy corresponds to a metric which is Kähler, $SU(n)$-holonomy to Ricci-flat Kähler metrics (also called Calabi-Yau metrics), $Sp(n)$-holonomy to hyperkähler metrics.

We then review a few “classical” things about $SU(n)$-holonomy: Yau’s proof of the Calabi conjecture, and examples of manifolds with $SU(2)$-holonomy. Then we delve into the definition of the group $G_2$, the properties of manifolds with $G_2$-holonomy, and Joyce’s construction.

1 Overview

Date: 20.10.2014 Speaker: Anda Degeratu

2 The concept of holonomy I

Date: 27.10.2014
1. Principal bundles and vector bundles (very briefly)
2. The construction of the tangent bundle from the frame bundle
3. connection on a vector bundle, connection on a principal bundle; the case of the tangent bundle and the frame bundle.
4. connection on a manifold (i.e. on its tangent bundle), the torsion and the holonomy of a connection
5. $G$-structure on a manifold and its torsion

Bibliography: [Joy00, Chapter 2], [KN63, II.1-4]

3 The concept of holonomy II

Date: 3.11.2014

1. Riemannian manifold, the Levi-Civita connection, curvature and Ricci curvature
2. Riemannian holonomy; state Berger’s Theorem (Theorem 3.4.1 in [Joy00])
3. Riemannian symmetric spaces and their holonomy; the example of Grassmannian
4. The meaning of holonomy:
   • $U(n)$ Kähler metrics
   • $SU(n)$ Ricci-flat Kähler metrics (Calabi-Yau metrics)
   • $Sp(n)$ hyperkähler metrics

Bibliography: [Joy00, Chapter 3]

4 SU($n$)-Holonomy and the Calabi conjecture

Date: 10.11.2014

1. Theorem: Let $(M, g, \omega, J)$ be a closed connected Kähler manifold. If $c_1(M) = 0$, then $M$ has a unique Ricci-flat Kähler metric $g'$ with Kähler class $\omega'$ satisfying $[\omega'] = [\omega]$.

2. Sketch the proof following [Bal06, Chapter 8].

Additional bibliography: [Joy00, Chapter 5]
5 SU(2)-Holonomy I: the Eguchi-Hanson space

Date: 17.11.2014

1. SU(2) = Sp(1); A metric has holonomy SU(2) if and only if it is hyperkähler. All hyperkähler metrics are Ricci-flat.
2. $\mathbb{R}^4$, $T^4$ with the flat metric
3. the Eguchi-Hanson space: give the metric, describe the topology, show that it is a complete hyperkähler metric.
4. generalize to higher dimension $T^*\mathbb{C}P^n$.

Bibliography: [Joy00, Sections 7.1 and 7.2], [Cal79]

6 SU(2)-Holonomy II

Date: 24.11.2014

1. Define a $K3$ surface; Example 1: the Fermat quartic
2. $K3$ surfaces are hyperkähler 4-manifold.
3. the Kummer construction

Bibliography: [Joy00, Sections 7.3.1 and 7.3.3] and [Joy96, I, Section 1.3]

7 G$_2$-Holonomy

Date: 1.12.2014

1. Define the group G$_2$.
2. G$_2$-structures and G$_2$-manifolds

Bibliography: [Joy00, Section 10.1] and [Joy96, I, Section 1.1]

7b G$_2$-Holonomy II

Date: 8.12.2014

1. G$_2$-holonomy and G$_2$-metric
2. the torus $T^7$ with the flat G$_2$-structure
8 Joyce’s construction I

Date: 12.01.2015

1. the action of $\Gamma = \mathbb{Z}_2^3$ on $T^7$ preserving the flat $G_2$-structure
2. description of the singularities of $T^7/\Gamma$.
3. desingularization of $T^7/\Gamma$ to a simply-connected manifold $X$
4. a family $\phi_t$ of $G_2$-structures on $X$

Bibliography: [Joy96, I]

9 Joyce’s construction II

Date: 15.12.2014

Theorem A: a $G_2$-structure with small torsion can be deformed to a torsion-free $G_2$-structure

Bibliography: [Joy96, I]

10 Joyce’s construction III

Date: 19.01.2015

Theorem B: Theorem A applies to the family $\phi_t$
Corollary: $X$ has a $G_2$-metric

Bibliography: [Joy96, I]

11 Joyce’s construction IV

Date: 26.01.2015

Theorem C: Deformation of $G_2$-metrics

Theorem: $X$ admits a 43-dimensional family of metrics with $G_2$-holonomy.

Bibliography: [Joy96, I]

12 The topology of $G_2$-manifolds

Date: 2.02.2015
The topology of compact $G_2$-Manifolds

Bibliography: [Joy96, II, Section 1.1] [Joy00, Section 10.2]

13 The moduli space of $G_2$-manifolds

Date: 9.02.2015

References


