SEMINAR K-THEORY AND INDEX THEORY

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We want to understand the Atiyah-Singer Index Theorem from a topological point of view following [AS], [LM], and [Sh]. We will need some background in topological K-theory, and in the analysis of pseudodifferential operators. We leave out cohomological formulas, except where explicitly stated.

1. Vector Bundles and K-Theory

1.1. **Vector Bundles — 28. 4.** Vector bundles, sections, Whitney sum and product [Ha, Section 1.1]. Differential operators on sections, principal symbol $\sigma(D)$ [LM, III 1.2–1.4, 1.7].

1.2. **Dirac Bundles and Dirac Operators — 5. 5.** Dirac bundles and operators. Examples: Euler operator, signature operator, Dolbeault operator, spin structures and the spin Dirac operator, and their analytic indices.

1.3. **K-Theory — 12. 5.** Topological K-theory for compact spaces [LM, I 9.1–9.13]. For locally compact spaces: description by complexes of vector bundles, example: symbol of an elliptic differential operator [Sh, pp. 61–67], see also [Se, pp. 148–151].

Classifying Spaces, Characteristic Classes and the Splitting Principle — entfällt. Classification of vector bundles [Ha, Section 1.2], Cohomology of Grassmannians, Splitting principle.

1.4. **Thom isomorphism and Bott periodicity — 19. 5.** Bott periodicity, just sketch the proof [LM, 9.14–9.20]. Thom isomorphism theorem for complex vector bundles [Sh, pp. 67–69] with proof [LM, App. C].

Transversality and Embeddings — entfällt. Explain Sard's Lemma and transversality. Prove the Whitney Embedding Theorem. Explain the "stable normal bundle" of a smooth manifold.

1.5. **The Topological Index — 26. 5.** Define the topological index and prove the axiomatic characterisation [Sh, Section 9].

1.6. **The Cohomological Index — 2. 6.** Multiplicative sequences and the Chern character. Compare Thom-Isomorphisms in cohomology and K-theory. Give the cohomological index in terms of the principal symbol, and discuss the special case of Dirac operators [LM, III Sections 11, 12 and 13.8–13.15].

2. Pseudodifferential Operators and the Index Theorem

2.1. **Sobolev Spaces — 16. 6.** Recall the basics of Fourier analysis, define Sobolev spaces of sections, and prove the Sobolev theorem and Rellich lemma [LM, Section 2].

2.2. **Pseudodifferential operators — 23. 6.** Motivate and define pseudodifferential operators (ΨDOs) [Sh, Section 10] [LM, III Section 3].

2.3. **Elliptic Operators — 30. 6.** Define elliptic ΨDOs, parametrices. Prove the Fredholm property and define the index for elliptic differential operators [LM, III 4.1–5.4].

2.4. **Spectrum and the Heat Equation — 7. 7.** Construct the spectrum of an elliptic differential operator, prove the Hodge theorem [LM, III 5.5–5.9]. Define the heat operator and sketch the heat kernel proof of the index theorem [LM, III Section 6].

2.5. **The Index of Elliptic Pseudodifferential Operators — 14. 7.** Define the index of an elliptic ΨDO and show that it depends only on the principal symbol [Sh, Section 11].

2.6. **The Atiyah-Singer Index Theorem — 21. 7.** Prove the Atiyah-Singer index theorem [LM, III Section 13] or [Sh, Section 12].

Generalisations of the Index Theorem — 28. 7. State the Atiyah-Singer index theorem for manifolds with a group action and the fixpoint formula [LM, III Sections 9 and 14] [Sh, Sections 13–15], or for families of manifolds [LM, III Sections 8 and 15].

Literatur

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