WEDNESDAY SEMINAR SUMMER 2017
"INDEX THEORY"

The Atiyah-Singer index theorem is certainly one of the most influential collection of results from the last century. The theorem itself relates an analytic invariant, the Fredholm index of an elliptic differential operator, to a topological invariant, the topological index of its symbol. Applications of this and related theorem range from the computation of virtual dimensions of moduli spaces in gauge theory over obstructions against positive curvature metrics to integrality results for certain characteristic numbers. A close cousin is the theorem of (Hirzebruch/Grothendieck-) Riemann-Roch in algebraic geometry.

Early precursors of the Atiyah-Singer index theorem are the Riemann-Roch theorem (Roch, 1865) and the "global" Gauß-Bonnet theorem (v. Dyck, 1888), both in (real) dimension 2. Higher dimensional versions appeared much later. In fact, the most important special cases all predate the Atiyah-Singer index theorem, namely the Gauß-Bonnet-Chern theorem (Allendoerfer-Weil, 1943; Chern, 1944), the Hirzebruch signature theorem (Hirzebruch, 1953), and the Hirzebruch-Riemann-Roch theorem (Hirzebruch, 1954). Gel'fand realised in 1960 that all these theorems express Fredholm indices of elliptic operators in terms of topological data. He asked for a general topological index formula, which was found by Atiyah and Singer. Their original proof (announced 1963, published by Palais and Cartan-Schwartz both in 1965) relied on cobordism theory and the Hirzebruch-Riemann-Roch theorem, the published version from 1968 uses topological K-theory instead. Atiyah and Singer noticed that spin geometry plays an important role in the background, and they also introduced the Dirac operator in mathematics. Atiyah and Segal in 1968 provided an equivariant generalisation, extending the Lefschetz fixpoint theorem (Lefschetz, 1926). Atiyah and Singer also proved a families' version in 1971, similar in spirit to the Grothendieck-Riemann-Roch theorem (Grothendieck, 1957), and a version for real K-theory.

There are many ways to formulate and prove the Atiyah-Singer index theorem, and each of them has its own merits and limitations. In this seminar, we will concentrate on the special case of Dirac operators. Though "geometric" Dirac operators are rather special differential operators, they are rich enough to generate all elliptic differential operators in a K-theoretic sense on one hand, and have a nice description in terms of elementary Riemannian geometry on the other. The following list of results can all be expressed in terms of geometric Dirac operators.

If d^* denote the formal adjoint of the exterior differential d on the smooth de Rham complex on an n-dimensional smooth manifold M, then $D = d + d^*$ is the *Hodge-Dirac* operator. Because D exchanges parity of the degree of forms, we can define the index of $D\colon \Omega^{\mathrm{ev}} \to \Omega^{\mathrm{odd}}$ as

$$\begin{aligned}\operatorname{ind}\bigl(D\colon \Omega^{\mathrm{ev}} \to \Omega^{\mathrm{odd}}\bigr) &= \dim\ker(D) - \dim\operatorname{coker}(D)\\ &= \dim\ker\bigl(D\colon \Omega^{\mathrm{ev}} \to \Omega^{\mathrm{odd}}\bigr) - \dim\ker\bigl(D\colon \Omega^{\mathrm{odd}} \to \Omega^{\mathrm{ev}}\bigr)\ . \qquad (1)\end{aligned}$$

By the Hodge theorem, the kernel of the full operator D represents cohomology with real coefficients. We get the *Gauß-Bonnet-Chern theorem* for the Euler characteristic

$$\begin{aligned}\chi(M) = \sum_{k=0}^{n}(-1)^k \dim H^k(M;\mathbb{R}) &= \operatorname{ind}\bigl(D\colon \Omega^{\mathrm{ev}} \to \Omega^{\mathrm{odd}}\bigr)\\ &= e(TM)[M] = \int_M e\bigl(\nabla^{TM}\bigr)\ . \qquad (2)\end{aligned}$$

If M is oriented, Poicaré duality gives a bilinear form $H^k(M;\mathbb{R}) \times H^{n-k}(M;\mathbb{R}) \to \mathbb{R}$ by $(a,b) \mapsto (a \smile b)[M]$. If $n = 4\ell$, this form is symmetric and nondegerate on the middle cohomology group $H^{2\ell}(M;\mathbb{R})$; its signature is called the *signature* of M. On the level of forms, we have the bilinear form $(\alpha,\beta) \mapsto \int_M \alpha\wedge\beta = \langle\alpha, *\beta\rangle_{L^2}$, where "$*$" is the Hodge star operator. The ±1-eigenspaces of "$*$" are denote by $\Omega^{\pm}(M)$. We have the *Hirzebruch signature theorem*

$$\operatorname{sign}(M) = \operatorname{ind}\bigl(D\colon \Omega^{+}(M) \to \Omega^{-}(M)\bigr) = L(TM)[M] = \int_M L\bigl(\nabla^{TM}\bigr)\ . \qquad (3)$$

If M is a Kähler manifold with holomorphic tangent bundle $T'M$ (for example, a smooth projective variety over $\mathbb{C}$ with its algebraic tangent bundle), and $W \to M$ is a Hermitian holomorphic vector bundle, then we similarly get the Dolbeault operator $D = \bar\partial + \bar\partial^*$ acting on antiholomorphic differential forms, which is a Dirac operator, too. By the Hodge theorem, its kernel represents the cohomology of the sheaf of holomorphic sections of W. We have the *Hirzebruch-Riemann-Roch theorem* for the holomorphic Euler characteristic

$$\begin{aligned}\chi(M;W) &= \operatorname{ind}\bigl(D\colon \Omega^{0,\mathrm{ev}}(M;W) \to \Omega^{0,\mathrm{odd}}(M;W)\bigr)\\ &= (\operatorname{Td}(T'M)\operatorname{ch}(W))[M] = \int_M \operatorname{Td}\bigl(\nabla^{T'M}\bigr)\ \operatorname{ch}\bigl(\nabla^{W}\bigr)\ . \qquad (4)\end{aligned}$$

This Dolbeault operator also appears in an analytic proof of the Kodaira embedding theorem.

Finally, if M is oriented and spin (with a fixed spin structure), there is the Atiyah-Singer (or *untwisted*) Dirac operator D on the spinor bundle $S \to M$. If $n = \dim M$ is even, this bundle splits as $S = S^+ \oplus S^-$. We may twist it with any vector bundle $W \to M$ with connection if we like (or with the trivial vector bundle $\underline{\mathbb{C}}$ to keep it untwisted). This way we can recover the Hodge-Dirac operator and the Dolbeault operator (if M is also Kähler) among others. By the *Atiyah-Singer index theorem for Dirac operators,*

$$\begin{aligned}&\operatorname{ind}\bigl(D\colon \Gamma(S^+\otimes W) \to \Gamma(S^-\otimes W)\bigr)\\ &\qquad= \bigl(\hat A(TM)\ \operatorname{ch}(W)\bigr)[M] = \int_M \hat A\bigl(\nabla^{TM}\bigr)\ \operatorname{ch}\bigl(\nabla^{W}\bigr)\ . \qquad (5)\end{aligned}$$

This is our primary goal, the other theorems above can be deduced from it. The untwisted Dirac operator also appears in the Lichnerowicz theorem on metrics of positive scalar curvature.

For the proof, we note that the index equals the supertrace of the heat operator (McKean-Singer, 1967), which can be expressed in terms of local geometric data (Gilkey, 1974; Atiyah-Bott-Patodi, 1975). While the heat kernel itself becomes singular in the small time limit, by some "miraculous cancellations", its supertrace converges. For geometric Dirac operators, these miraculous cancellations have been explained by Getzler in 1983 using ideas by Witten and Alvarez-Gaumé. This is the method of proof we will use. The heat kernel proof bypasses K-theory and directly produces a formula in de Rham-cohomology. This is well for the index theorem itself, but not quite enough for some of its extensions, for example to families. On the other hand, it gives rise to refinements in other directions, for example for manifolds with boundary. And Bismut and others have used the heat kernel method to prove cohomological versions also of the equivarant and family generalisations of the index theorem.

The first part of the programme introduces Dirac operators and their properties (talks 1–4) and proves the index theorem (talks 5–8). Some of the talks can nicely be split among two speakers, just ask. The last part of the seminar (5 talks) is devoted to applications, reformulations and extensions of the index theorem. Because there are many interesting topics, some of these talks are optional at the moment. The participants are invited to establish a nice mixture of different topics by choosing from the suggested list. Some talks can be combined into one overview talk if needed (e.g. 9 and 10 or 12–**??**).

There is no ideal book for the whole seminar. Roe's book [Roe] is short and elementary, but uses ad hoc methods every now and then. The book [BGV] by Berline, Getzler and Vergne is rather comprehensive and often too general for our purposes, but is *the* book if one wants to work on local index theory. The books [BB] by Bleecker and Booss-Bavnbek, [LM] by Lawson and Michelsohn and [Sh] favour the K-theoretic embedding proof (see optional talks 9 and 10), but can still be helpful.

1. Dirac Operators, 26.4., Lye, Vetere

Dirac operators are a very special kind of differential operators. This talk should recall some basic concepts from Riemannian geometry and then introduce *geometric* Dirac operators, which will be needed particularly in talks 7, 8, and give many nice examples.

Recall Riemannian manifolds, introduce the **Levi-Civita connection** [BGV, (1.18)] (we don't need Riemannian curvature yet). Recall differential forms, the exterior derivative d, and Stokes' theorem. Introduce the Hodge star operator and the formal adjoint d^* of d [Roe, Def 1.21–Prop 1.23], the **Clifford algebra**, (graded) **Clifford bundles** and **Dirac operators** [Roe, Defs 3.1–3.5], [BGV, Def 3.1–Prop3.5]. Note that Roe's definition gives exactly the "geometric" Dirac operators that we will need later, whereas [BGV, Def 3.36, Prop 3.38] is more general. As a motivation, Roe shows that the square of the Dirac operator on flat $\mathbb{R}^n$ is the Laplacian. If you like, relate this to

Dirac's original definition on Minkowski space-time, but don't compute squares of more general Dirac operators yet. Show that Dirac operators are symmetric [Roe, Prop 3.11].

The second half introduces some of the "classical" Dirac operators. We will focus on their geometric construction. To interpret their indices, please assume the Hodge theorem both in the real and in the complex setting; it will be proved later. Also take $\operatorname{ind}(D) = \dim\ker(D^+) - \dim\ker(D^-)$ as in (1) as a preliminary definition of the index. Cover as many of the following examples as you like: The index of the de Rham operator is the Euler characteristic, computed by the Euler class (needed for **Gauß-Bonnet-Chern** (2)) [Roe, Ex 3.19–(3.23)],[BGV, Prop 3.53, Cor 3.55]. A different grading gives the signature, computed by the L-genus (figuring in the **Hirzebruch signature theorem** (3)) [BGV, Def 3.57–first half of Prop 3.61]. Note that [BGV] and [Roe] follow different conventions for the Hodge star operator. The index of the Dolbeault operator on a Kähler manifold is the holomorphic Euler characteristic (as in **Hirzebruch-Riemann-Roch** (4)) [Roe, Ex 3.25–Prop 3.27]. If there is time, follow [BGV, Def 3.63–Cor 3.69] instead, where cohomology of holomorphic vector bundles is considered.

2. Spin geometry, 3.5., Bergner, Eberhardt

On so-called spin manifolds, there is a fundamental Dirac operator acting on spinors. It can be used for example to exclude the existence of positive scalar curvature metrics on certain spin manifolds. All other Dirac operators can be interpreted as "twisted" versions of the spin Dirac operator.

Recall principal bundles and induced vector bundles, describe connections [Roe, Ex 2.1–Prop 2.7], [BGV, Def 1.1–Prop 1.4, Prop 1.7–Def 1.10, Defs 1.12, 1.14, Prop 1.16]. Introduce the groups (Pin(n) and) **Spin**(n) [BGV, Def 2.3, Thm 2.9; Prop 3.7–Prop 3.10], [Roe, Def 4.5, Prop 4.7]. Discuss **representations of the Clifford algebra**: [Roe, Prop 4.9] uses representation theory of finite groups, [BGV, Lemma 3.17–Def 3.20] just gives an explicit description.

Introduce **spin structures,** mention the obstruction $w_2(TM)$ [BGV, Def 3.33, Prop 3.34]. If M has a spin structure, then there is a basic Clifford bundle, the **spinor bundle** S, and each Clifford bundle is of the form $S \otimes W$ [BGV, Prop 3.35]. If M is not spin, this construction still works locally. If M is even-dimensional, the spinor bundle **splits** as $S = S^+ \oplus S^-$, and the bundle of **exterior forms** is isomorphic to $S \otimes S$, in other words, the spinor bundle is the square root of the bundle of forms. Exhibit the **two gradings** on $\Lambda^\bullet T^*M$ leading to the Euler operator and the signature operator. On a **Kähler manifold** with spin structure, show that $\Lambda^{0,\bullet}M \cong S \otimes L$, where $L \otimes L = K$ is the canonical bundle. If there is time, also mention Spin^c-structures [Roe, Def 4.26–4.28], these may be helpful in talks 9 and 10.

3. The square of a Dirac operator, 10.5., Zaccanelli

The square of a Dirac operator D is the sum of a connection Laplacian and a curvature term. Depending on the context, this formula is known as

Lichnerowicz, Bochner, Schr"odinger or Weitzenböck formula. This formula is not only a starting point for the heat equation proof. If the curvature term is positive, then $\ker D = 0$. Applications include obstructions against positive scalar curvature and the Kodaira embedding theorem.

Introduce the **curvature** of a connection, discuss the **Riemannian curvature tensor** on TM with its symmetries [BGV, Prop 1.26] and introduce Ricci and scalar curvature. To give some feeling for these notions, it might be nice to quote comparison theorems like Alexandrov-Toponogov and Bishop-Gromov without proof [Go2, Sätze 2.20, 2.42]. Show that the curvature of a Clifford connection splits into the **spin curvature** and a **twist curvature** [Roe, Lem 3.13–Prop 3.16]. Introduce the **connection Laplacian** and show that it is non-negative [Roe, Lem 3.9], [LM, Prop II.8.1]. Derive the **Schrödinger-Bochner-Lichnerowicz-Weitzenböck formula** for the square of the Dirac operator. [Roe, Prop 3.18].

Depending on time, discuss as many of the following applications as possible (or as you like). The index of the spin Dirac operator is an obstruction against **positive scalar curvature** [LM, Thm II.8.8, Cor II.8.9], [Roe, Thm 13.1]. The **Kodaira vanishing** and **embedding theorems** rely on a similar argument [GH]. The **Bochner trick** shows that $H^1(M;\mathbb{R})$ is represented by parallel forms if the Ricci curvature is nonnegative [LM, Thms II.8.4, II.8.5], [Roe, Thm 6.9].

4. Spectral Theory, 17.5., Fornasin

Dirac operators on Riemannian manifolds are elliptic. This has important consequences for their analytical behaviour. In particular, they have compact resolvent and therefore discrete spectrum on compact manifolds, and one may talk about their Fredholm indices.

Explain the **symbol** of a differential operator D and define **elliptic** [LM, Sect. III.1]. Explain **Sobolev spaces** and the theorems of **Sobolev** and **Rellich** [Roe, Def 5.1–5.12]. Prove the **Gårding inequality** and the **elliptic estimate**, and use them to prove that the Dirac operator has discrete spectrum [Roe, 5.14–Thm 5.27] (this is also done in [LM, Sect. III.2–5], but in greater generality). Define the **Fredholm index** and prove equation (1) above [LM, III.§5–Cor 5.3], mention stability [LM, Thm III.7.10]. If there is time, introduce the heat operator following [Roe, Prop 5.29–Rem 5.32]. As an application of this theory, we now prove the **Hodge theorem** [LM, Cor 5.6] [Roe, Def 6.1–Cor 6.3], which was already used in talk 1.

5. Chern-Weil theory, 24.5., Lye, Schmidtke

The analytic proof of the index theorem will spit out a differential form constructed from a certain polynomial in the curvatures of the bundles involved. The purpose of this talk is to explain Chern-Weil theory, which interprets such expressions as characteristic classes in de Rham cohomology.

Recall vector bundles with structure group G and G-connections. From G-invariant polynomials on $\mathfrak{g}$, construct closed differential forms, prove independence of the G-connection [MS, App C, Fund. Lemma, Cor.], [Zh, Chap 1], [Go2, Abs 4.3], [Roe, Def 2.17–Def 2.21],. Construct **Chern classes** in $H^\bullet(-;\mathbb{R})$ that satisfy the usual axioms [MS, App C, Lem 6 and Thm on p 306, Cor 1], and **Pontryagin classes** [MS, Cor 1 on p 308], [Roe, (2.26)]. Note that these classes are typically defined in integral cohomology, but for evaluating against a fundamental class, $\mathbb{R}$-coefficients suffice. If you like, mention cobordism invariance of Pontryagin numbers and describe the rational oriented bordism ring [MS, Thm 18.8–Cor 18.10] using Pontryagin numbers (without proof). These facts were used in the original proof of the Atiyah-Singer theorem.

Introduce the **Â-class** [Roe, Ex 2.28] and the **Chern character** [Roe, (2.25)]. Also explain the notion of the **twist** or **relative Chern character** [BGV, p. 146], [Roe, (4.25)]. Then explain as many of the following relations as time permits (or as you like). By computing the Chern character of the (graded or ungraded) spinor bundle, explain the **L-class** and the **Euler class** as products of the $\hat{A}$-class and the corresponding twist Chern character for $\Lambda^\bullet T^*M$ [BGV, Lem 4.4, Prop 4.5], [Roe, Prop 13.6], [LM, Prop III.11.24]. Similarly, exhibit the **Todd-class** as $\hat{A}(\nabla^{TM})\,\mathrm{ch}(\nabla^{K^{1/2}})$ [BGV, p. 152], see also [LM, III.(12.10, 11)]. All these classes are needed for the classical index theorems (2)–(5) in talk 8.

6. Heat operators, 31.5., Kertels

To prove the index theorem, we start with the McKean-Singer trick. Thus we rewrite the Fredholm index of a Dirac-type operator as the supertrace of an associated heat operator e^{-tD^2}, independent of the time t. The index itself naturally corresponds to the large time limit. In the small time limit, we get an asymptotic expansion of the heat operator that is locally computable from the Riemannian metric and the coefficients of D.

Introduce heat operators and **heat kernels** as fundamental solutions of the heat equation [Roe, Def 7.1–Rem 7.7], [BGV, Def 2.15–Prop 2.17], treat the heat kernel on flat $\mathbb{R}^n$ explicitly [BGV, Sect 2.2]. If you like, give a little physical motivation. Define trace class operators and show that the **heat supertrace** can be computed as an integral [Roe, Prop 8.1–Thm 8.12, Prop 11.2], [BGV, (2.9)]. Explain the **McKean-Singer trick** $\mathrm{ind}(D) = \mathrm{str}(e^{-tD^2})$ [Roe, Prop 11.9–(11.11)], [LM, III.§6].

Describe **formal solutions** of the heat equation by an ODE, show that the leading order term is parallel translation [Roe, Lem 7.12–Prop 7.19], [BGV, Sect 2.5]. Explain that for the McKean-Singer trick, the supertrace of a formal solution **suffices,** and note that $\mathrm{str}(e^{-tD^2})$ depends on a **higher order term.** Then sketch how to obtain an actual heat kernel [BGV, Sect 2.4]. Maybe consider the flat torus T^n as an example to see that the formal solution can differ from the actual one.

7. Getzler rescaling, 14.6., Müller

We have seen that the heat operator e^{-tD^2} has an asymptotic expansion for small times t with leading term of order $t^{-\frac{\dim M}{2}}$. Nevertheless by McKean-Singer, its supertrace should be constant. Getzler rescaling is a trick to prove that even locally, the heat supertrace for *geometric* Dirac operators has a converging integrand as $t \to 0$. Getzler rescaling affects not only the *space* and *time* directions of the heat operator, but involves also an *internal rescaling* of the Clifford algebra.

Before we start the actual rescaling, we have to represent the operator D^2 in a suitable **trivialisation** of the bundle E over a **normal coordinate chart** of M and estimate its coefficients [BGV, Lem 4.13, 4.14]. Then explain **Getzler rescaling** [BGV, p. 161], some motivation is given after [BGV, Prop 4.16]. Note that the part of Getzler rescaling on the Clifford algebra does not come from a rescaling of the Dirac bundle.

Then show that under Getzler rescaling, the operator D^2 converges to a **model operator** on T_pM [BGV, Prop 4.19]. The formal solution for the heat kernel converges to a (at this point still unknown) **model solution** of the model operator's heat equation [BGV, Lem 4.18]. Alternatively, follow [Roe, Chap 12–Prop 12.24].

8. Mehler's formula, 21.6., Hein, Peternell

The heat kernel of the model operator from the last talk can be explicitly computed. The resulting supertrace can be interpreted as an integral of a Chern-Weil-theoretic characteristic class. This finishes the proof of the Atiyah-Singer index theorem.

If you like, start by motivating the harmonic oscillator from quantum mechanics (see [Roe, Chap 9] for a source). Interpret the model operator as a harmonic oscillator on T_pM. The heat operator of a harmonic oscillator is given by **Mehler's formula** [BGV, Sect 4.2], [Roe, (9.11)–Rem 9.13]. Conclude that Mehler's formula describes the formal solution of the model operator [BGV, Thms 4.12, 4.20], [Roe, Prop 12.25]. Interpret the supertrace as a Chern-Weil theoretic expression to prove the **Atiyah-Singer index theorem** (5) [BGV, Thms 4.1–4.3], [Roe, end of chap 12].

Derive from this the **Gauß-Bonnet-Chern theorem** (2), the **Hirzebruch signature theorem** (3), and the **Hirzebruch-Riemann-Roch theorem** (4) [BGV, Thm 4.6–4.9], [Roe, Thms 13.7, 13.13]. Some preliminary work on the Euler classe, L-class and Todd class should have been done in talk 5.

9. Topological K-Theory, 28.6., McDonnell

This talk should be given and prepared in conjuction with talk 10. Probably, we have to squeeze both into one talk. For a topological space X, the K-ring $K^0(X)$ is the group completion of the monoid of finite rank complex vector bundles on X under direct sum and tensor product. It behaves like a cohomology theory, and Bott periodicity is an important feature.

Define (complex) **topological K-theory** $K^0(X)$ for a compact Hausdorff space X as the Grothendieck group of the monoid of vector bundles on X, and define the ring structure [Ha, Sect 2.1], [LM, I.§9–Prop 9.4, Cor 9.9], don't introduce classifying spaces. Then explain **Bott periodicity** (without proof) [At, Thm 2.2.1], [Ha, Thm 2.11], [LM, Thm I.9.20]. Explain **relative** K-theory or K-theory with **compact support** following [LM, Def I.9.23–I.9.26]. For a complex vector bundle $E \to X$, introduce the **Thom space** $T(E)$ [MS, §18] and state the **Thom isomorphism theorem** $\tilde{K}^0(X) \cong \tilde{K}^0(T(E))$ as a generalisation of Bott periodicity [At, Cor 2.7.12], [LM, Thm C.8]. If time allows, exhibit Chern-Weil characteristic classes as natural transformations from the K-theory of a smooth manifold to its de Rham cohomology.

10. The K-theoretic index theorem, 5.7., Recktenwald

The symbol of an elliptic differential operator on a manifold M defines a K^0-class on the Thom space $T(T^*M)$ of its cotangent bundle. The topological index is constructed as a pushforward. It is easy to generalise to families and group actions.

Explain the **almost complex structure** on the total space of TM, and on the total space of the normal bundle of $TM \hookrightarrow TN$ for any smooth embedding $M \hookrightarrow N$ [LM, p. 241]; this is the hidden reason why it makes sense to consider the index in the framework of complex K-theory on real manifolds. State the **Whitney embedding theorem** (without proof) and construct the **topological index** [LM, Def III.13.1]

$$\tilde{K}^0(T(T^*M)) \longrightarrow \tilde{K}^0(T(T^*\mathbb{R}^N)) \cong \tilde{K}^0(S^{2N}) \xleftarrow{\cong} \tilde{K}^0(S^0) \cong \mathbb{Z}\,.$$

Explain the two axiomatic **properties** [LM, p. 247].

Only if talks 9 and 10 are given separately, explain how to deduce a **cohomological index theorem** for general elliptic operators, and (5) for Dirac operators [LM, Thm III.13.8–10]. If there is even more time, describe the topological and the analytical family index for proper submersions $p\colon E \to B$.

11. Index theory and modular forms, 12.7., Wendland

Witten composed new genera by taking formal sums over a countable family of Atiyah-Singer-type index formulas. If the underlying manifold admits particular structures (string structures in the real case, Calabi-Yau metrics in the complex case), one obtains modular forms. This talk should explain this result and present some applications. One may also touch upon the highly speculative Stolz programme on understanding manifolds of positive Ricci curvature, or on the interpretation of the Witten genus in terms of vertex operator algebras.

12. Integrality and applications, 19.7., Wang

The index is always an integer, but its cohomological representation is a priori only rational. These two facts can be combined to give interesting applications and construct subtle secondary invariants in differential topology. The Atiyah-Patodi-Singer theorem sheds more light on these secondary invariants.

Even-dimensional complex projective spaces have a non-integral $\hat{A}$-genus, hence cannot be spin. **Rokhlin's theorem** says that the signature of a 4-dimensional spin manifold is divisible by 16 (example: $K3$-surfaces) [Roe, Thm 13.9] [LM, Cor. IV.1.2].

The **Eells-Kuiper invariant** uses integrality of the $\hat{A}$-genus to detect all 28 smooth structures on the topological S^7. It would be nice to mention Donnelly's intrinsic description of the Eells-Kuiper invariant using η-invariants and its extension to a $\mathbb{Z}$-valued invariant for manifolds with positive scalar curvature by Kreck-Stolz [Go1, sect 4.c].

13. The Grothendieck-Riemann-Roch theorem, 26.7., Hörmann

The Grothendieck-Riemann-Roch theorem relates the pushforward functors in (algebraic) K-theory and cohomology for proper maps of algebraic varieties. The pushforward in K-theory is similar to the "topological index" used in the K-theoretic embedding proof of the index theorem (talk 10), indeed, this proof was probably inspired by Grothendieck-Riemann-Roch. It would be nice if the commutative diagram underlying the theorem could be explained and related to a similar diagram relating topological K-theory and cohomology.

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