ANALYTIC TORSION FOR FAMILIES

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These are notes for the series of lectures on "analytic torsion for families" I gave at the workshop "smooth fibre bundles and higher torsion" in Göttingen. These notes are basically a summary of the original article [BL] by Bismut and Lott, together with some introductory and motivational remarks. These notes are neither original nor complete — they should be read together with [BL]. Also, a copy of [BGV] will be helpful for standard constructions with Dirac operators and heat kernels.

1. Franz-Reidemeister torsion and Ray-Singer Torsion

We recall two possible definitions of "torsion" on single manifolds.

1.a. Franz-Reidemeister torsion. We recall the definition of Franz-Reidemeister torsion ([R], [F]). A compact manifold M, a flat bundle $F \to M$ with parallel metric, and a cell structure S on M give a finite complex $(C^\bullet(S;F), \partial)$ with metric g^C. Define

$$\begin{aligned}\tau(M;F) &= -\frac{1}{2} \sum_{i=0}^{\dim M} (-1)^i\, i\, \log\det\big((\partial+\partial^*)^2|_{\ker(\partial+\partial^*)^\perp \cap C^i(S;F)}\big) \\ &= -\frac{1}{2}\, \mathrm{str}_{\ker(\partial+\partial^*)^\perp}\Big(N^C\, \log\big((\partial+\partial^*)^2\big)\Big)\,. \end{aligned} \tag{1.1}$$

In the second line, N^C acts by multiplication with i on $C^i(S;F)$, and the supertrace "str" is an abbreviation for the alternating sum of traces.

Let $H^\bullet(M;F) \cong \ker(\partial+\partial^*)$ denote the cohomology of $C^\bullet(S;F)$, then $H^\bullet(M;F)$ is called the (cellular) cohomology of M with local coefficients in F. The cohomology is independent of S. If $H^\bullet(M;F)=0$, we call F *acyclic.*

1.2. Theorem (Reidemeister, Franz). *If the bundle $F \to M$ is acyclic, then $\tau(M;F)$ is independent of the metric on F and of the cellular structure S on M.*

1.3. Remark. (Reidemeister, Franz). Let $L_{q,p_1,\dots,p_k} = S^{2k-1}/\mathbb{Z}_q$ denote the quotient of $S^{2k-1} \subset \mathbb{C}^k$ by a cyclic group of order q generated by the matrix

$$\begin{pmatrix} e^{\frac{2\pi i\, p_1}{q}} & & \\ & \ddots & \\ & & e^{\frac{2\pi i\, p_k}{q}} \end{pmatrix},$$

where $p_1, \dots, p_k$ are relative prime to q. Then two such lens spaces $L_{q,p_1,\dots,p_k}$ and $L_{q,p'_1,\dots,p'_k}$ are homeomorphic iff they are isometric.

To prove this statement, consider the $q-1$ pairwise non-isomorphic flat acyclic one-dimensional complex bundles over $L_{q,p_1,\dots,p_k}$. These bundles give rise to a set of $q-1$ numbers. For non-isometric lens spaces, one can show that these sets of numbers are different.

Let us note that nevertheless, some $L_{q,p_1,\dots,p_k}$ and $L_{q,p'_1,\dots,p'_k}$ are homotopy equivalent even though they are not isometric and thus also not homeomorphic. This shows that Franz-Reidemeister torsion detects a difference between the category of topological spaces and the category of topological spaces up to homotopy equivalence. In fact, torsion is even a simple-homotopy invariant, not only a homeomorphism invariant.

The complex $(C^\bullet(S;F),\partial)$ computes the cohomology of M with local coefficients in the bundle F. Since many different cellular structures on M give many different complexes, but all these complexes give rise to the same value of τ, Ray and Singer tried to use a different complex to compute the torsion of a manifold with a flat bundle.

1.b. Vector bundles, connections and the de Rham complex. We recall some definitions from differential geometry/topology.

Let M be a differentiable manifold, and let TM and T^*M denote its tangent and cotangent bundle. The bundle $\Lambda^\bullet T^*M$ has fibres

$$\Lambda^\bullet T_p^*M = \bigoplus_{i=0}^{\dim M} \Lambda^i T_p^*M \;.$$

For the spaces of smooth sections, we write

$$\Omega^i(M) = \Gamma(\Lambda^i T^*M) \qquad \text{and} \qquad \Omega^\bullet(M) = \bigoplus_{i=0}^{\dim M} \Omega^i(M) \;.$$

1.4. Remark. Let $\mathfrak{X}(M) = \Gamma(TM)$ denote the space of smooth vector fields on M and let $C^\infty(M)$ denote the space of smooth functions on M, then

$$\Omega^i(M) = \Big\{\, \alpha\colon \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{i \text{ factors}} \longrightarrow C^\infty(M) \;\Big|\; \alpha \text{ is alternating and } C^\infty(M)\text{-multilinear} \,\Big\} \;.$$

There is a wedge product $\wedge\colon \Omega^i(M)\times\Omega^j(M)\to\Omega^{i+j}(M)$, and $(\Omega^\bullet(M),\wedge)$ is called the exterior algebra. If $F\colon M\to N$ is smooth, then there is a natural pullback $F^*\colon\Omega^i(N)\to\Omega^i(M)$ given for $\alpha\in\Omega^k(N)$ and $X_1, \dots, X_k\in\mathfrak{X}(M)$ by

$$(F^*\alpha)(X_1,\dots,X_k) = (\alpha\circ F)\big(dF(X_1),\dots,dF(X_k)\big) \;.$$

We recall that the exterior differential $d\colon\Omega^i(M)\to\Omega^{i+1}(M)$ is given by

$$(d\alpha)(X_0,\dots,X_i) = \sum_{j=0}^{i}(-1)^j X_j\Big(\alpha\big(X_0,\dots,\widehat{X_j},\dots,X_i\big)\Big) + \sum_{0\le j<k\le i}(-1)^{j+k}\alpha\Big([X_j,X_k],X_0,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_i\Big) \;, \tag{1.5}$$

where $X_0, \dots, X_i\in\mathfrak{X}(M)$ are vector fields on M.

1.6. Proposition.

(1) *for a function $f \in C^\infty(M) = \Omega^0(M)$, df is the total differential of f;*
(2) $d \circ d = 0$;
(3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$;
(4) *Let $F\colon M \to N$ be smooth, then $F^* \circ d = d \circ F^*$.*

The complex $(\Omega^\bullet(M), d)$ is called *de Rham complex,* it is a functor from the category of differentiable manifolds to the category of chain complexes over $\mathbb{R}$.

We also want to consider the de Rham complex twisted by flat vector bundles, in analogy with (1.1). Suppose that $V \to M$ is a vector bundle, then define

$$\Omega^i(M;V) = \Gamma\big(\Lambda^i T^*M \otimes V\big) \qquad \text{and} \qquad \Omega^\bullet(M;V) = \bigoplus_{i=0}^{\dim M} \Omega^i(M;V)\ .$$

Again,

$$\Omega^i(M;V) = \Big\{\, \alpha\colon \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{i \text{ factors}} \longrightarrow \Gamma(V) \;\Big|\; \alpha \text{ is alternating and } C^\infty(M)\text{-multilinear} \,\Big\}\ .$$

A *connection* on V is an operator

$$\nabla^V\colon\ \Gamma(V) = \Omega^0(M;V) \ \longrightarrow\ \Gamma(T^*M \otimes V) = \Omega^1(M;V)$$

satisfying the Leibniz rule

$$\nabla^V(fv) = df \otimes v + f \cdot \nabla^V v$$

for all $f \in C^\infty(M)$ and all $v \in \Gamma(V)$. Write

$$\nabla^V_X v = \big(\nabla^V v\big)(X) \in \Gamma(V)$$

for $X \in \mathfrak{X}(M)$ and $v \in \Gamma(V)$. We extend ∇^V to an operator $\Omega^i(M;V) \to \Omega^{i+1}(M;V)$ by

$$\nabla^V(\alpha \otimes v) = d\alpha \otimes v + (-1)^i \alpha \wedge \nabla^V v$$

for all $\alpha \in \Omega^i(M)$ and $v \in \Gamma(V)$. For $\eta \in \Omega^i(M;V)$,

$$\begin{aligned}(1.7)\qquad \big(\nabla^V \eta\big)(X_0, \dots, X_i) &= \sum_{j=0}^{i} (-1)^j \nabla^V_{X_j}\Big(\eta\big(X_0, \dots, \widehat{X_j}, \dots, X_i\big)\Big) \\ &\quad + \sum_{0 \le j < k \le i} (-1)^{j+k} \eta\Big(\big[X_j, X_k\big], X_0, \dots, \widehat{X_j}, \dots, \widehat{X_k}, \dots, X_i\Big)\ .\end{aligned}$$

The square $(\nabla^V)^2$ is $C^\infty(M)$-linear and can be regarded as an element of $\Omega^2(M; \operatorname{End} V)$. By (1.7),

$$(1.8)\qquad \big(\nabla^V\big)^2(v)(X,Y) = \nabla^V_X \nabla^V_Y v - \nabla^V_Y \nabla^V_X v - \nabla^V_{[X,Y]} v\ ,$$

and this is called the *curvature* of ∇^V.

1.9. Definition. A connection ∇^V on V is called *flat* if $(\nabla^V)^2 = 0$. In this case, (V, ∇^V) is called a *flat vector bundle.*

This is equivalent to any of the topological definitions by calling a local section $v \in \Gamma(V|_U)$ *locally constant* iff $\nabla^V v = 0$, and by associating to (V, ∇^V) the representation $\rho\colon \pi_1(M) \to GL(n, \Bbbk)$ given by parallel translation with respect to ∇^V.

If $F\colon M \to N$ is a smooth map and $(V, \nabla^V) \to N$ is a flat vector bundle, then there is a natural pullback $(F^*V, F^*\nabla^V)$ which is again a flat vector bundle. With this definition, the analogue of Proposition 1.6 holds. The complex $(\Omega^\bullet(M; V), \nabla^V)$ is called the *de Rham complex with local coefficients in the flat vector bundle* (V, ∇^V). Later on, we will use the letter F instead of V, and we will sometimes write d or d_M instead of ∇^V.

1.c. Ray-Singer torsion. We introduce the Ray-Singer analytic torsion ([RS]). The value of τ in (1.1) is independent of the choice of cellular structure S chosen on M and the complex $C^\bullet(S; F)$ defined by it. This suggests to replace $C^\bullet(S; F)$ by a different complex which is natural (does not depend on extra choices)—the de Rham complex. Unfortunately, we still need to choose a Riemannian metric on M.

Let g^{TM} be a Riemannian metric on M, and let $d\operatorname{vol}_M \in \Omega^n(M; o(M))$ denote the induced Lebesgue measure on M. Here, $o(M) = \Lambda^{\max} T^*M \to M$ is the orientation line bundle, which is trivial if M is orientable. Let g^F be a metric on the flat vector bundle $(F, \nabla^F) \to M$ which is parallel, i.e.,

$$\nabla^F g^F = 0\,, \qquad \text{or equivalently} \qquad X\big(g^F(v, w)\big) = g^F\big(\nabla^F_X v, w\big) + g^F\big(v, \nabla^F_X w\big) \tag{1.10}$$

for all $X \in \mathfrak{X}(M)$ and v, $w \in \Gamma(F)$. The metrics g^{TM} on TM and g^F on F induce a metric $g^{T^*M,F}$ on $\Lambda^\bullet T^*M \otimes F$, and an L_2-metric $g^{T^*M,F}_{L_2}$ on $\Omega^\bullet(M; F)$ by

$$g^{T^*M,F}_{L_2}(\eta, \theta) = \int_M g^{T^*M,F}(\eta_p, \theta_p)\, d\operatorname{vol}_M(p)\ .$$

Vector fields $X \in \mathfrak{X}(M)$ act on $\Omega^\bullet(M; F)$ in several ways. Let us define the operators ε (*exterior multiplication*) and ι (*interior multiplication*) by

$$\varepsilon_X \eta = g^{TM}(X, \,\cdot\,) \wedge \eta \qquad \text{and} \qquad \iota_X \eta = \eta(X, \dots)\ .$$

Then ι is independent of g^{TM}. If $e_1, \dots, e_n$ is a local orthonormal frame on M and $1 \le j \le n$, we abbreviate

$$\varepsilon^j = \varepsilon_{e_j} \qquad \text{and} \qquad \iota_j = \iota_{e_j}\ .$$

Let ∇^{TM} denote the Levi-Civita connection on M with respect to g^{TM}, then ∇^{TM} and ∇^F induce a connection $\nabla^{TM,F}$ on $\Lambda^\bullet T^*M \otimes F$, and we may rewrite (1.5) and (1.7) locally as

$$d\eta = \sum_{j=1}^n \varepsilon^j \nabla^{TM,F}_{e_j} \eta$$

for a local orthonormal base $e_1, \dots, e_n$ of TM. The adjoint d^* of $d = \nabla^F$ with respect to $g^{T^*M,F}_{L_2}$ is then given by the formula

$$d^*\eta = -\sum_{j=1}^n \iota_j \nabla^{T^*M,F}_{e_j} \eta\ , \tag{1.11}$$

because ε and ι are adjoint to each other, and $\nabla^{TM,F}$ is skewsymmetric by our choice of g^F in (1.10).

To find an analytic analogue of (1.1), we observe that for $\lambda > 0$,

$$\frac{\partial}{\partial s}\Big|_{s=0} \lambda^{-s} = -\log \lambda \,.$$

If $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of a non-negative formally self-adjoint operator Δ with discrete spectrum, we introduce the ζ-function

$$\zeta_\Delta(s) = \sum_{\lambda_i > 0} \lambda_i^{-s} \,. \tag{1.12}$$

Then at least formally,

$$-\zeta'_\Delta(0) = \sum_{\lambda_i > 0} \log \lambda_i \,.$$

For the Hodge-Laplacian $\Delta^F = (d + d^*)^2$, by Weyl's assymptotic formula, the sum in (1.12) converges for $\mathrm{Re}(s) \geq \frac{n}{2}$. Moreover it is known that ζ_Δ admits a meromorphic continuation to the whole complex plane that has no pole at $s = 0$.

1.13. Definition. Let $(F, \nabla^F) \to M$ be a flat vector bundle with parallel metric g^F, let g^{TM} be a Riemannian metric on M, and let $\Delta^i = (d + d^*)^2$ be the induced Hodge-Laplacian on $\Omega^i(M; F)$. Then the *Ray-Singer analytic torsion* is defined as

$$\mathcal{T}(M; F) = -\frac{1}{2} \sum_{i=0}^{n} (-1)^i i \, \zeta'_{\Delta^i}(0) \,.$$

To compute the Ray-Singer analytic torsion, one may apply the Mellin transform. Let

$$\begin{aligned} b_i(M; F) &= \dim H^i(M; F) \,, \\ \chi(M; F) &= \sum_{i=0}^{n} (-1)^i \dim H^i(M; F) \\ \text{and} \qquad \chi'(M; F) &= \sum_{i=0}^{n} (-1)^i i \dim H^i(M; F) \,. \end{aligned}$$

and let N^M act on $\Omega^i(M; F)$ by multiplication with i. Once again, the supertrace "str" is an abbreviation for $\mathrm{tr}\big((-1)^{N^M} \cdot\big)$.

1.14. Proposition. *For* $\mathrm{Re}(s) \gg 0$,

$$\zeta_{\Delta^i}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \Big(\mathrm{tr}\big(e^{-t\Delta^i}\big) - b_i(M; F)\Big) \, dt \,. \tag{1}$$

There exist a_0, $a_{-1} \in \mathbb{R}$ *such that as* $t \to 0$,

$$\mathrm{str}\Big(N^M \, e^{-t(d+d^*)^2}\Big) = a_{-1} \, t^{-\frac{1}{2}} + a_0 + O\Big(t^{\frac{1}{2}}\Big) \,. \tag{2}$$

The analytic torsion is given by

$$\begin{aligned}(3)\quad \mathcal{T}(M;F) = &-\int_1^\infty \Big(\operatorname{str}\big(N^M\, e^{-t(d+d^*)^2}\big) - \chi'(M;F)\Big)\,\frac{dt}{2t}\\ &-\int_0^1 \Big(\operatorname{str}\big(N^M\, e^{-t(d+d^*)^2}\big) - a_{-1}\,t^{-\frac12} - a_0\Big)\,\frac{dt}{2t} + a_{-1} + \frac{a_0-\chi'(M;F)}{2}\,\Gamma'(1)\ .\end{aligned}$$

We will encounter a slightly more elegant description in connection with higher torsion forms.

Proof. Let $0 \le \lambda_1 \le \lambda_2 \le \dots$ denote the eigenvalues of Δ^i. For $\operatorname{Re}(s) > \frac{n}{2}$, we have

$$\begin{aligned}\zeta_{\Delta^i}(s) &= \frac{1}{\Gamma(s)} \sum_{\lambda_j>0} \int_0^\infty \left(\frac{t}{\lambda_j}\right)^{s-1} e^{-t}\,\frac{dt}{\lambda_j}\\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{\lambda_j>0} e^{-t\lambda_j}\,dt\ .\end{aligned}$$

This proves (1). The second claim follows from local calculations, which we omit here.

By (2), we have for $\operatorname{Re}(s) > \frac12$,

$$\sum_{i=0}^n (-1)^i i\,\zeta_{\Delta^i}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \Big(\operatorname{str}\big(N^M\, e^{-t(d+d^*)^2}\big) - \chi'(M;F)\Big)\,\frac{dt}{2t}\ .$$

The integral over $[1,\infty)$ converges for all s, and since $\Gamma(s)$ has a pole with residuum 1 at $s=0$, we find

$$\begin{aligned}-\frac{\partial}{\partial s}\Big|_{s=0}&\left(\frac{1}{\Gamma(s)}\int_1^\infty \Big(\operatorname{str}\big(N^M\, e^{-t(d+d^*)^2}\big) - \chi'(M;F)\Big)\,\frac{dt}{2t}\right)\\ &= \int_1^\infty \Big(\operatorname{str}\big(N^M\, e^{-t(d+d^*)^2}\big) - \chi'(M;F)\Big)\,\frac{dt}{2t} :\ .\end{aligned}$$

For a function $f(t) = a_{-1}\,t^{-\frac12} + a_0 + O\big(t^{\frac12}\big)$, we find for $\operatorname{Re}(s) > \frac12$ that

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\,f(t)\,\frac{dt}{2t} = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\Big(f(t) - a_{-1}\,t^{-\frac12} - a_0\Big)\,dt + \frac{a_{-1}}{\big(s-\frac12\big)\,\Gamma(s)} + \frac{a_0}{\Gamma(s+1)}\ ,$$

and the left hand side describes the meromorphic continuation to $\operatorname{Re}(s) > -\frac12$ with a pole at $s=\frac12$. Together with (1) and (2), we get (3). $\square$

Ray and Singer conjectured the following result, which was later proved independently by Cheeger and Müller, and which was extended to the case of arbitrary flat vector bundles (neither acyclic nor with a parallel metric) by Bismut and Zhang.

1.15. Theorem (Cheeger [C], Müller [M1], [M2], Bismut-Zhang [BZ]). *Let $(F,\nabla^F) \to M$ be an acyclic flat vector bundle with a parallel metric g^F on a compact Riemannian manifold (M,g^{TM}), then*

$$\tau(M;F) = \mathcal{T}(M;F)\ .$$

Bismut and Zhang not only generalised Theorem 1.15 to bundles that do not carry parallel metrics, they also proved a variation formula for $\mathcal{T}$. In fact, assume that the metrics g^F and g^{TM} are parametrized by some parameter space B, then we get a function

$$\mathcal{T}(M;F)\colon\ B \longrightarrow \mathbb{R}\ .$$

The tangent bundle TM becomes a bundle over $M\times B$, with a nontrivial connection ∇^{TM} if the metric g^{TM} varies. Let $e(TM,\nabla^{TM}) \in \Omega^n(M\times B)$ denotes its Euler form. Also, the L^2-metric $g^{\mathcal{H}}$ on the cohomology varies over B. We describe the variation of $\mathcal{T}(M;F)$ by giving its total derivative.

1.16. Theorem (Bismut-Zhang [BZ]). *Under the assumptions above,*

$$d\mathcal{T}(M;F) = \int_M e\big(TM, \nabla^{TM}\big)\ \mathrm{tr}_F\big((g^F)^{-1}\,[\nabla^F, g^F]\big) - \mathrm{tr}_{\mathcal{H}}\big((g^{\mathcal{H}})^{-1}\,[d_B, g^{\mathcal{H}}]\big)\ .$$

Bismut-Lott's main theorem on higher analytic torsion (Theorem 3.16) is an extension of the theorem above involving higher powers of the logarithmic derivative of the metric.

1.17. Exercise. Prove (1.8) using (1.7). Check that the second Bianchi identity is equivalent to $[\nabla^V, (\nabla^V)^2] = 0$ using the fact that the Levi-Civita connection ∇^{TM} is torsion free.

2. Torsion forms for complexes of vector bundles

This chapter introduces Chern-Weil theory for flat vector bundles. Moreover, we define a "higher torsion" for complexes of vector bundles. This higher torsion should be seen as a "toy model" for the higher analytic torsion of smooth fibre bundles.

2.a. Chern-Weil theory for flat bundles. We recall the Chern character form for vector bundles with connection and define Kamber-Tondeur classes for flat vector bundles, following [BL] and [BG].

Let $(V, \nabla^V) \to M$ be a vector bundle over $\mathbb{C}$ of rank r with an arbitrary connection. We regard the curvature of ∇^V as

$$\big(\nabla^V\big)^2 \in \Omega^2(M; \mathrm{End}\, V) = \Gamma\big(\Lambda^2 T^*M \otimes \mathrm{End}\, V\big)\ .$$

We extend the usual trace $\mathrm{tr}\colon \mathrm{End}\, V \to \mathbb{C}$ to a map

$$\mathrm{tr}_V = \mathrm{id}_{\Lambda^\bullet T^*M} \otimes \mathrm{tr}\colon\ \Omega^\bullet(M; \mathrm{End}\, V) \longrightarrow \Omega^\bullet(M;\mathbb{C})\ .$$

Then the Chern character form of (V, ∇^V) is defined as

$$\mathrm{ch}\big(V, \nabla^V\big) = \mathrm{tr}_V\Big(e^{-\frac{(\nabla^V)^2}{2\pi i}}\Big) \in \Omega^{\mathrm{even}}(M)\ .$$

The homogeneous components of $\mathrm{ch}(V, \nabla^V)$ are given by polynomials

$$\mathrm{ch}\big(V, \nabla^V\big)^{[2j]} = \frac{(-1)^j}{(2\pi i)^j\, j!}\ \mathrm{tr}_V\big((\nabla^V)^{2j}\big) \quad \in \quad \Omega^{2j}(M)\ .$$

2.1. Proposition. *The form* $\mathrm{ch}(V, \nabla^V)$ *is closed, i.e.,*

(1) $$d\,\mathrm{ch}\big(V, \nabla^V\big) = 0 \in \Omega^{\mathrm{odd}}(M, \mathbb{C})\ .$$

There exist natural classes $\widetilde{\mathrm{ch}}(V, \nabla^{V,0}, \nabla^{V,1}) \in \Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M)$ *depending on two connections* $\nabla^{V,0}$ *and* $\nabla^{V,1}$ *on* V*, such that*

(2) $$d\widetilde{\mathrm{ch}}\big(V, \nabla^{V,0}, \nabla^{V,1}\big) = \mathrm{ch}\big(V, \nabla^{V,1}\big) - \mathrm{ch}\big(V, \nabla^{V,0}\big)\ .$$

Proof. To prove (1), we fix a local trivialisation of V over a sufficiently small open subset $U \subset M$ and let ∇^0 denote the local flat connection that preserves this trivialisation. The exterior differential of the trace of a matrix is the trace of the componentwise exterior differential of the matrix. For a matrix valued function $A\colon U \to M_r(\mathbb{C})$ and any vector valued function $v\colon U \to \mathbb{C}^r$, we have

$$(dA)\cdot v + A\cdot(dv) = d(A\cdot v) = \bigl(\nabla^0 \circ A\bigr)(v) = \bigl[\nabla^0, A\bigr](v) + \bigl(A\circ\nabla^0\bigr)(v) \;,$$

so in the language of operators on V, we have to write

$$dA = \bigl[\nabla^0, A\bigr] \;.$$

Hence

$$d\,\mathrm{ch}\bigl(V,\nabla^V\bigr) = \mathrm{tr}_V\Bigl(\Bigl[\nabla^0, e^{-\frac{(\nabla^V)^2}{2\pi i}}\Bigr]\Bigr) \;. \tag{*}$$

Note that $\nabla^0 - \nabla^V$ contains no more derivatives, hence

$$\nabla^0 - \nabla^V \in \Omega^1(M;\operatorname{End}V) \;.$$

Because the coefficients Λ^1T^*M commute with $\Lambda^{\mathrm{even}}T^*M$, and because the trace of a commutator of matrices with commuting entries vanishes, we find by (*) that

$$\begin{aligned} d\,\mathrm{ch}\bigl(V,\nabla^V\bigr) &= \mathrm{tr}_V\Bigl(\Bigl[\nabla^V, e^{-\frac{(\nabla^V)^2}{2\pi i}}\Bigr]\Bigr) + \mathrm{tr}_V\Bigl(\Bigl[\nabla^0 - \nabla^V, e^{-\frac{(\nabla^V)^2}{2\pi i}}\Bigr]\Bigr) \\ &= \mathrm{tr}_V\Bigl(\Bigl[\nabla^V, e^{-\frac{(\nabla^V)^2}{2\pi i}}\Bigr]\Bigr) \;. \end{aligned}$$

The right hand side vanishes because ∇^V obviously commutes with $(\nabla^V)^2$ and its powers (cf. second Bianchi identity). This proves (1).

Next, we construct $\widetilde{\mathrm{ch}}$. The following construction is universal and will occur more often in the future. Replace $V \to M$ by its pullback

$$\overline{V} = V\times[0,1] \;\longrightarrow\; \overline{M} = M\times[0,1]$$

to $M\times[0,1]$. Then we can find a connection $\nabla^{\overline{V}}$ such that

$$\nabla^{\overline{V}}|_{M\times\{0\}} = \nabla^{V,0} \qquad\text{and}\qquad \nabla^{\overline{V}}|_{M\times\{1\}} = \nabla^{V,1} \;,$$

and this is in fact the decisive step in the proof.

Let s denote the extra coordinate on $[0,1]$. We put

$$\widetilde{\mathrm{ch}}\bigl(V,\nabla^{V,0},\nabla^{V,1}\bigr) = \int_{\overline{M}/M}\mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr) = \int_0^1\Bigl(\iota_{\frac{\partial}{\partial s}}\,\mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr)\Bigr)\big|_{M\times\{s\}}\,ds \;.$$

Let $\mathcal{L}$ denote the Lie derivative, then $\mathcal{L}_X = [d,\iota_X] := d\circ\iota_X + \iota_X\circ d$ by Cartan's formula. Because $\mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr)$ is closed, we find

$$\begin{aligned} d\widetilde{\mathrm{ch}}\bigl(V,\nabla^{V,0},\nabla^{V,1}\bigr) &= \int_0^1\Bigl(\bigl(d_{M\times\{s\}}\circ\iota_{\frac{\partial}{\partial s}}\bigr)\;\mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr)\Bigr)\big|_{M\times\{s\}}\,ds \\ &= \int_0^1\Bigl(\mathcal{L}_{\frac{\partial}{\partial s}}\,\mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr)\Bigr)\big|_{M\times\{s\}}\,ds \\ &= \mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr)|_{M\times\{1\}} - \mathrm{ch}\bigl(\overline{V},\nabla^{\overline{V}}\bigr)|_{M\times\{0\}} \;, \end{aligned}$$

which proves (2).

It remains to show that $\widetilde{\mathrm{ch}}(V,\nabla^{V,0},\nabla^{V,1})$ is (modulo exact forms) independent of the choice of $\nabla^{\overline{V}}$. Here, we proceed as above by regarding a connection on $\overline{\overline{V}}=V\times[0,1]^2\to M\times[0,1]^2$, which restricts to ∇^i over $M\times\{i\}\times[0,1]$ for $i=0$, 1, and to the two chosen extensions $\nabla^{\overline{V},i}$ over $M\times[0,1]\times\{i\}$ for $i=0$, 1. By a similar argument as above,

$$\begin{aligned}\int_{\overline{M}/M}&\mathrm{ch}\big(\overline{V},\nabla^{\overline{V},1}\big)-\int_{\overline{M}/M}\mathrm{ch}\big(\overline{V},\nabla^{\overline{V},0}\big)\\&=d\int_{M\times[0,1]^2/M}\mathrm{ch}\big(\overline{\overline{V}},\nabla^{\overline{\overline{V}}}\big)=d\int_0^1\int_0^1\Big(\iota_{\frac{\partial}{\partial s}}\iota_{\frac{\partial}{\partial t}}\mathrm{ch}\big(\overline{\overline{V}},\nabla^{\overline{\overline{V}}}\big)\Big)\big|_{M\times\{(s,t)\}}\,dt\,ds\,.\qquad\square\end{aligned}$$

From now on we assume that $(F,\nabla^F)\to M$ is a flat complex vector bundle of rank r. Obviously

$$\mathrm{ch}(F,\nabla^F)=\mathrm{tr}\big(e^0\big)=r\ ,$$

and Chern-Weil theory gives only trivial information on F. Suppose that g^F is a metric on F, then there exists another connection $\nabla^{F,*}$ defined by

$$d\big(g^F(v,w)\big)=g^F\big(\nabla^Fv,w\big)+g^F\big(v,\nabla^{F,*}w\big)\ . \tag{2.2}$$

It is easy to check in local coordinates that $\nabla^{F,*}$ is given by

$$\nabla^{F,*}=\nabla^F+\omega^F\ ,\qquad\text{where}\qquad\omega^F=\big(g^F\big)^{-1}\big[\nabla^F,g^F\big]\in\Omega^1(M;\mathrm{End}\,F)\ , \tag{2.3}$$

because

$$d\big(g^F(v,w)\big)=g^F\big(\nabla^Fv,w\big)+g^F\big(v,\nabla^Fw\big)+\big[\nabla^F,g^F\big](v,w)\ .$$

2.4. Remark.

(1) We have $\omega^F=0$ iff the metric g^F is parallel with respect to ∇^F, i.e., iff (2.2) holds with $\nabla^{F,*}$ replaced by ∇^F.

(2) The connection $\nabla^{F,*}$ is again flat because

$$0=d^2\big(g^F(v,w)\big)=g^F\big((\nabla^F)^2v,w\big)+g^F\big(v,(\nabla^{F,*})^2w\big)\ .$$

By Proposition 2.1 (2), we have

$$d\widetilde{\mathrm{ch}}\big(V,\nabla^F,\nabla^{F,*}\big)=0\ , \tag{2.5}$$

so $\pi i\,\widetilde{\mathrm{ch}}(V,\nabla^F,\nabla^{F,*})$ defines a cohomology class on M, the *odd Chern character* of (F,∇^F). This class is a Kamber-Tondeur class ([KT]) and as such related to the Borel regular classes.

We want to contruct a specific differential form, not just a cohomology class. Therefore, we consider the connection

$$\nabla^{\overline{F}}=(1-s)\,\nabla^F+s\,\nabla^{F,*}+\frac{\partial}{\partial s}\,ds$$

on $\overline{F}=F\times[0,1]\to\overline{M}$. Because both ∇^F and $\nabla^{F,*}$ are flat,

$$\big(\nabla^{\overline{F}}\big)^2=s(1-s)\,\big(\nabla^F\nabla^{F,*}+\nabla^{F,*}\nabla^F\big)+ds\,\big(\nabla^{F,*}-\nabla^F\big)=-s(1-s)\,\big(\omega^F\big)^2-\omega^F\,ds$$

using (2.3). The Chern character form on $\overline{M}$ is thus given by

$$\operatorname{ch}\left(\overline{F}, \nabla^{\overline{F}}\right) = \operatorname{tr}_F\left(e^{\frac{s(1-s)}{2\pi i}\,(\omega^F)^2}\right) + \frac{1}{2\pi i}\,\operatorname{tr}_F\left(\omega^F\, e^{\frac{s(1-s)}{2\pi i}\,(\omega^F)^2}\right) ds\,.$$

Integrating over $s \in [0,1]$ gives our preferred representative

$$\begin{aligned}(2.6)\qquad \operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g^F\right) &= \pi i \int_{\overline{M}/M} \operatorname{ch}\left(\overline{F}, \nabla^{\overline{F}}\right)\\ &:= \frac{1}{2}\int_0^1 \operatorname{tr}_F\left(\omega^F\, e^{\frac{s(1-s)}{2\pi i}\,(\omega^F)^2}\right) ds \in \Omega^{\mathrm{odd}}(M)\,.\end{aligned}$$

The components of $\operatorname{ch}^{\mathrm{o}}(\nabla^F, g^F)$ are given by

$$\operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g^F\right)^{[2k+1]} = \frac{1}{2\,(2\pi i)^k}\,\operatorname{tr}_F\left((\omega^F)^{2k+1}\right)\int_0^1 \frac{\left(s(1-s)\right)^k}{k!}\,ds\,,$$

so we might write as well

$$(2.7)\qquad \operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g^F\right) = \frac{1}{2}\int_0^1 \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{1-N^B}{2}}\operatorname{tr}\left(\omega^F\, e^{(\omega^F)^2}\right) ds\,.$$

2.8. Proposition (Bismut-Lott [BL], Bismut-Goette [BG]). *The form* $\operatorname{ch}^{\mathrm{o}}(\nabla^F, g^F) \in \Omega^{\mathrm{odd}}(M)$ *is real. We have*

$$(1)\qquad d\operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g^F\right) = 0\,.$$

Given two metrics g_0^F, g_1^F, *there is a natural class* $\widetilde{\operatorname{ch}}^{\mathrm{o}}(\nabla^F, g_0^F, g_1^F) \in \Omega^{\mathrm{even}}(M)/d\Omega^{\mathrm{odd}}(M)$ *such that*

$$(2)\qquad d\widetilde{\operatorname{ch}}^{\mathrm{o}}\left(\nabla^F, g_0^F, g_1^F\right) = \operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g_1^F\right) - \operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g_0^F\right)\,.$$

Proof. To show that $\operatorname{ch}^{\mathrm{o}}(\nabla^F, g^F)$ is real, we use that ω_X^F is symmetric with respect to g^F for all vector fields $X \in \mathfrak{X}(M)$. Then we have

$$\overline{(\omega^F)^{2k+1}}^{\,*} = (-1)^k\left(\omega^F\right)^{2k+1}$$

because the $(2k+1)$ arguments in $\mathfrak{X}(M)$ have to be reordered. The powers of i in our definition guarantee that we obtain an even form. Because $\operatorname{ch}^{\mathrm{o}}(\nabla^F, g^F)$ represents $\pi i\,\widetilde{\operatorname{ch}}(V, \nabla^F, \nabla^{F,*})$, it is automatically closed by (2.5), which proves (1).

To prove (2), we extend ∇^F to a flat connection on $\overline{F}$ and choose a metric $g^{\overline{F}}$ with $g_i^F = g^{\overline{F}}|_{M\times\{i\}}$ for $i = 0$, 1. We put

$$\widetilde{\operatorname{ch}}^{\mathrm{o}}\left(\nabla^F, g_0^F, g_1^F\right) = \int_{\overline{M}/M} \operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g^{\overline{F}}\right)\,.$$

Our claim follows from exactness of $\operatorname{ch}^{\mathrm{o}}\left(\nabla^F, g^{\overline{F}}\right)$ precisely as in the proof of Proposition 2.1 (2). The arguments there also show that the definition of $\widetilde{\operatorname{ch}}^{\mathrm{o}}(\nabla^F, g_0^F, g_1^F)$ is independent of the particular choice of the extension $g^{\overline{F}}$ modulo exact forms. □

2.b. The superconnection formalism. We recall that Franz-Reidemeister torsion was defined as an invariant of finite-dimensional chain complexes. We will give a generalisation of this invariant for families of finite-dimensional chain complexes. More precisely, we consider a $\mathbb{Z}$-graded vector bundle $V^\bullet$ with a connection ∇^V and a fibrewise map $v\colon V^\bullet \to V^{\bullet+1}$ of degree 1. We obtain a complex

$$\big(\Omega^\bullet(M;V^\bullet), \nabla^V + v\big)$$

iff $(\nabla^V + v)^2 = 0$. The total operator $A' = \nabla^V + v$ is a superconnection, and it describes a toy version of the de Rham complex of a fibre bundle. Let us introduce the superformalism, which is basically a formalisation of Koszul's sign rule.

A *superspace* is a $\mathbb{Z}_2$-graded (complex) vector space

$$V = V^{\text{even}} \oplus V^{\text{odd}} .$$

We decompose elements $v \in V$ into $v = v^{\text{even}} + v^{\text{odd}}$ with $v^{\text{even}} \in V^{\text{even}}$ and $v^{\text{odd}} \in V^{\text{odd}}$. We will sometimes write $V^0 = V^{\text{even}}$ and $V^1 = V^{\text{odd}}$ if no confusion is possible. For example, each $\mathbb{Z}$-graded vector space $V^\bullet$ becomes a superspace by putting

$$V^{\text{even}} = \bigoplus_{i \text{ even}} V^i \qquad \text{and} \qquad V^{\text{odd}} = \bigoplus_{i \text{ odd}} V^i .$$

Each ungraded vector space W becomes a superspace by setting

$$W^{\text{even}} = W \qquad \text{and} \qquad W^{\text{odd}} = 0 .$$

The tensor product of superspaces V and W is given by

$$(V \otimes W)^i = \bigoplus_{j+k=i} V^j \otimes W^k$$

with $i,\ j,\ k \in \{0,1\}$. The superspaces $\operatorname{Hom}(V,W)$ and $\operatorname{End}(V)$ are defined similarly. Note that $\operatorname{End} V$ is a *superalgebra*, i.e., a $\mathbb{Z}_2$-graded algebra.

If A is a superalgebra, we define the supercommutator of $a \in A^i$, $b \in A^j$ by

$$[a,b] = a \circ b - (-1)^{ij} b \circ a .$$

A *superalgebra* is called *supercommutative* iff all supercommutators vanish. For example, $\Lambda^\bullet W$ is supercommutative for any (ungraded) vector space W.

On the superalgebra $\operatorname{End} V$, we define the *supertrace* $\operatorname{str}_V\colon \operatorname{End} V \to \mathbb{C}$ by

$$\operatorname{str}(A) = \operatorname{tr}\big(A|_{V^{\text{even}}}\big) - \operatorname{tr}\big(A|_{V^{\text{odd}}}\big) .$$

The following fact is crucial.

2.9. Proposition. *Let V be a superspace, and let $A,\ B \in \operatorname{End} V$, then*

$$\operatorname{str}_V([A,B]) = 0 . \qquad \square$$

We define the *supertensorproduct* $A \mathbin{\widehat{\otimes}} B$ of two superalgebras A and B as the superspace $A \otimes B$ with the product

$$(2.10) \qquad \big(a_0 \mathbin{\widehat{\otimes}} b_0\big) \cdot \big(a_1 \mathbin{\widehat{\otimes}} b_1\big) = (-1)^{ij} \big(a_0 \cdot a_1\big) \mathbin{\widehat{\otimes}} \big(b_0 \cdot b_1\big) \qquad \text{if } b_0 \in B^i \text{ and } a_1 \in A^j .$$

The supertensorproduct of two supercommutative superalgebras is again supercommutative. For example, for two (ungraded) vector spaces V and W, we have a natural isomorphism of superalgebras

$$\Lambda^\bullet(V \oplus W) = \Lambda^\bullet V \mathbin{\widehat{\otimes}} \Lambda^\bullet W \ .$$

A superbundle is a $\mathbb{Z}_2$-graded vector bundle. If $V \to M$ is a superbundle, then so is $\Lambda^\bullet T^*M \otimes V$, and the $\Omega^\bullet(M;V^\bullet)$ is a superspace. We have a natural superalgebra

$$\Omega^\bullet(M;\operatorname{End} V^\bullet) = \Gamma\bigl(\Lambda^\bullet T^*M \mathbin{\widehat{\otimes}} \operatorname{End} V\bigr) \ .$$

Any additional structure that preserves the $\mathbb{Z}_2$-grading will be called *graded*, so a graded metric g^V and a graded connection ∇^V are of the form

$$g^V = g^{V^{\text{even}}} \oplus g^{V^{\text{odd}}} \qquad \text{and} \qquad \nabla^V = \nabla^{V^{\text{even}}} \oplus \nabla^{V^{\text{odd}}} \ .$$

2.11. Definition. Let V be a superbundle. A *superconnection* on a vector bundle V is an odd $\mathbb{C}$-linear operator A on $\Omega^\bullet(M;V)$ satisfying

$$A(\alpha \wedge s) = d\alpha \wedge s + (-1)^i \alpha \wedge As$$

for each $\alpha \in \Omega^i(M)$ and each $s \in \Omega^\bullet(M;V)$.

In particular, each connection on a vector bundle W is a superconnection if we put $W^{\text{even}} = W$, $W^{\text{odd}} = 0$.

2.12. Proposition. *Let A be a superconnection on a superbundle V, then for any graded connection ∇^V, we have*

$$A = \nabla^V + \sum_i a_i \qquad \textit{with} \qquad a_i \in \begin{cases} \Omega^i(M, \operatorname{End}^{\text{odd}} V) & i \textit{ even, and} \\ \Omega^i(M, \operatorname{End}^{\text{even}} V) & i \textit{ odd.} \end{cases} \tag{1}$$

Conversely, every A of the form (1) is a superconnection.

For every superconnection A on a superbundle V, we have

$$A^2 \in \Omega^\bullet(M;\operatorname{End} V)^{\text{even}} \ .$$

Motivated by (1.8), we call A^2 the *curvature* of the superconnection A.

We will refer to $\nabla^V + a_1$ and the $(a_i)_{i\neq 1}$ as the *components* of A.

Proof. Clearly, $a := A - \nabla^V$ is an $\Omega^\bullet(M)$-linear operator on $\Omega^\bullet(M;V)$ and can thus be represented by an element of $\Omega^\bullet(M;\operatorname{End} V)$. The precise form of a follows because a is odd. The converse is also easy to check.

To prove (2), we check that for all $\alpha \in \Omega^\bullet(M)$ and all $s \in \Omega^\bullet(M;V)$, we have

$$A^2(\alpha \wedge s) = \alpha \wedge A^2 s \ ,$$

so A^2 is $\Omega^\bullet(M)$-linear and thus $A^2 \in \Omega^\bullet(M;\operatorname{End} V)$. It is clear that A^2 is even. $\square$

To give some meaning to Proposition 2.12, let us see how a superconnection acts. For $\alpha \in \Omega^j(M)$ and $v \in \Gamma(V^k)$, we have

$$\Bigl(\nabla^V + \sum a_i\Bigr)(\alpha \otimes v) = (-1)^j \alpha \wedge \nabla^V v + \sum (-1)^{ij} \alpha \wedge a_i v \ .$$

This is compatible with the curvature formula

$$\Big(\nabla^V + \sum a_i\Big)^2 = (\nabla^V)^2 + \sum_i [\nabla^V, a_i] + \sum_{i,j} a_i \cdot a_j \;, \tag{2.13}$$

where the products in the last term have to be taken in the sense of (2.10). Note that $[\nabla^V, a_i]$ is an anticommutator because both ∇^V and a_i are odd. Still, $[\nabla^V, a_i]$ is a derivative of a_i by Exercise 2.36.

In many respects, superconnections are as well behaved as connections. For example, let N^M denote the *number operator* on $\Omega^\bullet(M)$, which acts as multiplication by k on $\Omega^k(M)$. Similarly, λ^{N^M} acts as multiplication by λ^k on $\Omega^k(M)$. For a superbundle V with superconnection A, we put

$$\operatorname{ch}(V, A) = (2\pi i)^{-\frac{N^M}{2}} \operatorname{str}\big(e^{-A^2}\big) \;.$$

Then the analogue of Proposition 2.1 holds, in particular

$$[\operatorname{ch}(V, A)] = \big[\operatorname{ch}\big(V, \nabla^V\big)\big] = \big[\operatorname{ch}\big(V^{\mathrm{even}}, \nabla^{V^{\mathrm{even}}}\big)\big] - \big[\operatorname{ch}\big(V^{\mathrm{odd}}, \nabla^{V^{\mathrm{odd}}}\big)\big] \;. \tag{2.14}$$

To prove this, one proceeds as in the proof of Proposition 2.1, but uses supercommutators instead of commutators. We will see in Exercise 2.37 that some bundles admit flat superconnections even though they do not admit a flat connection.

2.c. Adjoint Superconnections and odd Chern forms. We now want to transfer the definition of the odd Chern character form $\operatorname{ch}^{\mathrm{o}}(\nabla^F, g^F)$ to the case of a flat superconnection. We therefore need the notion of an adjoint superconnection. We will use the description in [G1], which is equivalent to the original definition in [BL].

A graded metric $g^V = g^{V^{\mathrm{even}}} \oplus g^{V^{\mathrm{odd}}}$ on V induces a pairing

$$\begin{aligned} & g^V\colon\ \Omega^\bullet(M, V) \otimes \Omega^\bullet(M; V) \longrightarrow \Omega^\bullet(M) \\ \text{with}\quad & \big(\alpha \mathbin{\widehat{\otimes}} v\big) \otimes \big(\beta \mathbin{\widehat{\otimes}} w\big) \longmapsto \langle \alpha \mathbin{\widehat{\otimes}} v, \beta \mathbin{\widehat{\otimes}} w\rangle = (-1)^{\left[\frac{\deg \beta}{2}\right]}\, \alpha\overline{\beta}\, \langle v, w\rangle \end{aligned} \tag{2.15}$$

for α, $\beta \in \Omega^\bullet(M)$ and v, $w \in \Gamma(V)$. The factor $(-1)^{[\frac{\deg \beta}{2}]}$ comes from reversing the order of the one-form components of $\overline{\beta}$. Note the absence of a sign factor $(-1)^{\deg V \cdot \deg \beta}$; in particular, the pairing g^V violates our sign convention. The fact that g^V is Hermitian is equivalent to the relation

$$\langle \beta \mathbin{\widehat{\otimes}} w, \alpha \mathbin{\widehat{\otimes}} v\rangle = (-1)^{\left[\frac{\deg(\alpha\beta)}{2}\right]}\, \overline{\langle \alpha \mathbin{\widehat{\otimes}} v, \beta \mathbin{\widehat{\otimes}} w\rangle} \;, \tag{2.16}$$

where the sign factor is once more equivalent to a reversal of the order of the one-form factors.

2.17. Definition. Define a $\mathbb{C}$-anti-linear operator on $\Omega^\bullet(M; \mathbb{C})$ by

$$\overline{\alpha}^* = (-1)^{\left[\frac{N^B+1}{2}\right]}\, \overline{\alpha} \;. \tag{2.18}$$

2.19. Remark. This operator has the following properties:

$$\begin{aligned} \overline{f}^* &= \overline{f} && \text{for all } f \in \Omega^0(B; \mathbb{C}), \\ \overline{\alpha}^* &= -\overline{\alpha} && \text{for all } \alpha \in \Omega^1(B; \mathbb{C}), \\ \text{and}\qquad \overline{(\alpha \wedge \beta)}^* &= \overline{\beta}^* \wedge \overline{\alpha}^* \;. && \text{for all } \alpha,\ \beta \in \Omega^\bullet(B; \mathbb{C}). \end{aligned}$$

Note that this operator differs from the operator $(-1)^{[\frac{\deg(\,\cdot\,)}{2}]}\overline{\,\cdot\,}$ encountered in (2.15) and (2.16) by a factor of $(-1)^{\deg(\,\cdot\,)}$.

We let E^* denote the classical (ungraded) adjoint of an endomorphism $E \in \operatorname{End} V$ with respect to g^V, and we let $\nabla^{V,*}$ denote the adjoint of a connection ∇^V.

2.20. Definition. The adjoint of a differential form $\alpha \mathbin{\widehat{\otimes}} E \in \Omega^\bullet(B; \operatorname{End} V)$ is defined as

$$(\alpha \mathbin{\widehat{\otimes}} E)^* = (-1)^{\deg \alpha \cdot \deg E} \, \overline{\alpha}^* \, E^* \, .$$

The adjoint of a superconnection $A = \nabla^V + a$ with $a \in \Omega^\bullet(M; \operatorname{End} V)$ is given by

$$A^* = \nabla^{V,*} + a^* \, .$$

We call E or A *self-adjoint* (*skew-adjoint*) iff $E^* = E$ or $A^* = A$ ($E^* = -E$ or $A^* = -A$) respectively.

2.21. Remark. The definition of A^* is independent of the splitting $A = \nabla^V + a$. For odd elements $E \in \Omega^\bullet(M; \operatorname{End} V)^{\mathrm{odd}}$ and for superconnections A on V, Definition 2.20 is compatible with the sign convention of (2.15) in the sense that

$$\begin{aligned} 0 &= \langle E^* v, w\rangle + (-1)^{\deg E \, (\deg v + \deg w)} \, \langle v, Ew\rangle \\ \text{and} \qquad d\langle v, w\rangle &= \langle A^* v, w\rangle + (-1)^{\deg E \, (\deg v + \deg w)} \, \langle v, Aw\rangle \end{aligned}$$

for all v, $w \in \Omega^\bullet(M; V)$.

Symmetry of g^V together with Definition 2.20 implies that $(E^*)^* = E$ and $(A^*)^* = A$ for all superconnections A and all $E \in \Omega^\bullet(M; \operatorname{End} V)$.

It is easy to see from Remark 2.21 that A^* is flat iff A is. Now we can define $\mathrm{ch}^{\mathrm{o}}(A, g^V)$ as in (2.7) by

$$\mathrm{ch}^{\mathrm{o}}\bigl(A, g^V\bigr) = \frac{1}{2} \int_0^1 \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^M - 1}{2}} ds \; \mathrm{str}_V\Bigl((A^* - A) \, e^{(A^* - A)^2}\Bigr) \, .$$

The analogue of Proposition 2.8 still holds and can easily be proved using the fact that $\frac{1}{2}(A + A^*)$ is a superconnection for which

$$\frac{1}{2} \, [A + A^*, A - A^*] = 0$$

by flatness of A and A^*. But in contrary to (2.14), there is in general no easy formula for $\mathrm{ch}^{\mathrm{o}}(A, g^V)$ in terms of usual connections and metrics only.

We therefore consider a very special case. Assume that $A' = \nabla^V + \partial$ is a superconnection on a $\mathbb{Z}$-graded vector bundle V, and that $\partial \in \operatorname{End} V$ is of degree 1, so

$$\partial \in \bigoplus_i \operatorname{Hom}\bigl(V^i, V^{i+1}\bigr) \, .$$

Then A' is flat iff the following three conditions hold.

$$\begin{aligned} \partial^2 &= 0 && \text{i.e., } (V, \partial) \text{ is a family of complexes,} \\ [\nabla^V, \partial] &= 0 && \text{i.e., } \partial \text{ is parallel with respect to } \nabla^V, \\ \text{and} \qquad (\nabla^V)^2 &= 0 && \text{i.e., each } \bigl(V^i, \nabla^{V^i}\bigr) \longrightarrow M \text{ is a flat bundle.} \end{aligned}$$

Let $N^V \in \operatorname{End} V$ denote the number operator of V which acts on V^i as multiplication by i. We fix a $\mathbb{Z}$-graded metric g^V and regard the family

$$g_t^V(\,\cdot\,, \,\cdot\,) = g^V(t^{N^V} \,\cdot\,, \,\cdot\,) \, . \tag{2.22}$$

Let A''_t denote the adjoint of A' with respect to g^V_t, then

$$A''_t = \nabla^{V,*} + t\,\partial^* \ .$$

Put

$$\begin{aligned}\widetilde{A}'_t &= t^{\frac{N^V}{2}}\, A'\, t^{-\frac{N^V}{2}} = \nabla^V + \sqrt{t}\,\partial \\ \text{and} \qquad \widetilde{A}''_t &= t^{\frac{N^V}{2}}\, A''_t\, t^{-\frac{N^V}{2}} = \nabla^{V,*} + \sqrt{t}\,\partial^* \ ,\end{aligned} \tag{2.23}$$

then $\widetilde{A}''_t$ is adjoint to $\widetilde{A}'$ with respect to g^V, and since trace is invariant by conjugation, we clearly have

$$\mathrm{ch}^\circ\bigl(\widetilde{A}'_t, g^V\bigr) = \mathrm{ch}^\circ\bigl(A', g^V_t\bigr) \ .$$

Then clearly

$$\lim_{t\to 0} \mathrm{ch}^\circ\bigl(A', g^V_t\bigr) = \lim_{t\to 0} \mathrm{ch}^\circ\bigl(\widetilde{A}'_t, g^V\bigr) = \mathrm{ch}^\circ\bigl(\nabla^V, g^V\bigr)$$

is cohomologous to $\mathrm{ch}^\circ(A', g^V_t)$ for all t.

For $t \to \infty$, the limit is even more interesting. We note that

$$(\widetilde{A}''_t - \widetilde{A}'_t)^2 = -(\widetilde{A}''_t + \widetilde{A}'_t)^2 = -t\,(\partial + \partial^*)^2 - \sqrt{t}\,\bigl([\nabla^V, \partial^*] + [\nabla^{V,*}, \partial]\bigr) - [\nabla^V, \nabla^{V,*}] \ .$$

Moreover, the operator $(\partial + \partial^*)^2$ is nonnegative, and clearly $\lim_{t\to\infty} e^{-t(\partial+\partial^*)^2}$ is the projection onto the kernel of $\partial + \partial^*$.

By finite-dimensional Hodge theory, we know that

$$V = \operatorname{im}\partial \oplus \operatorname{im}\partial^* \oplus \ker(\partial + \partial^*) \ ,$$

and

$$\ker(\partial + \partial^*) \cong H^\bullet(V^\bullet, \partial) =: \mathcal{H} \longrightarrow M \ . \tag{2.24}$$

Moreover, since ∂ is parallel with respect to the flat connection ∇^V, the bundle $\mathcal{H}$ carries a flat connection $\nabla^{\mathcal{H}}$. Also, the isomorphism (2.24) induces a metric $g^{\mathcal{H}}$ on $\mathcal{H}$ by pulling back g^V. Since $\lim_{t\to\infty} e^{-t(\widetilde{A}'_t + \widetilde{A}''_t)^2}$ is projection onto $\ker(\partial + \partial^*)$ in degree 0 with respect to $\Omega^\bullet(M)$, this at least motivates the following result from [BL].

2.25. Theorem ([BL]). *We have*

$$\lim_{t\to\infty} \mathrm{ch}^\circ\bigl(A', g^V_t\bigr) = \lim_{t\to\infty} \mathrm{ch}^\circ\bigl(\widetilde{A}'_t, g^V\bigr) = \mathrm{ch}^\circ\bigl(\nabla^{\mathcal{H}}, g^{\mathcal{H}}\bigr) \ ;$$

in particular, $\mathrm{ch}^\circ(\nabla^V, g^V)$ *and* $\mathrm{ch}^\circ(\nabla^{\mathcal{H}}, g^{\mathcal{H}})$ *are cohomologous.*

I do not know of a proof of this result that entirely avoids the superconnection formalism, even though no superconnection is visible in the final formula.

Proof. With $f(z) = z\, e^{z^2}$, we may write

$$\mathrm{ch}^\circ\bigl(\widetilde{A}'_t, g^V\bigr) = \frac{1}{2} \int_0^1 \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^M - 1}{2}} \mathrm{str}_V\Bigl(f\bigl(\widetilde{A}''_t - \widetilde{A}'_t\bigr)\Bigr)\, ds$$

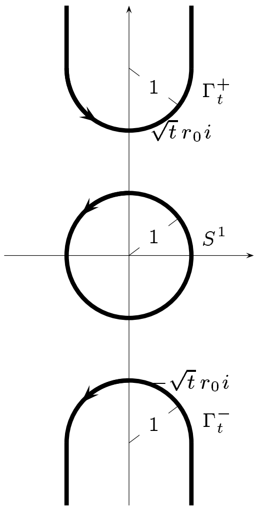

Figure 2.27. The contour $\Gamma_t = S^1 \cup \Gamma_t^+ \cup \Gamma_t^-$.

We want to compute $f(\widetilde{A}_t'' - \widetilde{A}_t')$ using holomorphic functional calculus. First of all, for each k there exist C and $c > 0$ such that

$$\left| \frac{\partial^k f(z)}{\partial z^k} \right| \leq C\, e^{-c|z|} \qquad \text{for all } z \text{ with } |\operatorname{Im} z| < r\ . \tag{2.26}$$

Next, the spectrum of $(\partial^* - \partial)$ is purely imaginary and contained in $\big((-\infty, -r_0) \cup \{0\} \cup (r_0, \infty)\big)\, i$ for some $r_0 > 0$. In particular, the spectrum of $\sqrt{t}(\partial^* - \partial)$ is contained within the contour $\Gamma_t = S^1 \cup \Gamma_t^+ \cup \Gamma_t^-$ depicted in Figure 2.27.

By holomorphic functional calculus,

$$\begin{aligned} f\big(\widetilde{A}_t'' - \widetilde{A}_t'\big) &= \frac{1}{2\pi i} \int_{\Gamma_t} \frac{f(z)\, dz}{z - \big(\widetilde{A}_t'' - \widetilde{A}_t'\big)} \\ &= \frac{1}{2\pi i} \int_{\Gamma_t} \left(\sum_{k=0}^{\infty} \left(\frac{1}{z - \sqrt{t}\,(\partial^* - \partial)}\, \omega^V \right)^k \right) \frac{f(z)\, dz}{z - \sqrt{t}\,(\partial^* - \partial)} \end{aligned}$$

Note that the sum is finite with $k \leq \dim M$. We can estimate $\left| \frac{1}{z - \sqrt{t}\,(\partial^* - \partial)} \right|$ by 1 on the whole contuor Γ_t, and $\left|\omega^V\right|$ by a constant. Because of (2.26), we find that

$$\left| \frac{1}{2\pi i} \int_{\Gamma_t^+ \cup \Gamma_t^-} \frac{f(z)\, dz}{z - \big(\widetilde{A}_t'' - \widetilde{A}_t'\big)} \right| = O\Big(e^{-\sqrt{t}c}\Big)\ .$$

Let $P\colon V \to \ker(\partial + \partial^*)$ denote the orthogonal projection with respect to g^V. On the contour S^1, we estimate

$$\left| \frac{1}{z - \sqrt{t}\,(\partial^* - \partial)} - \frac{P}{z} \right| \leq \frac{1}{\sqrt{t}\, r_0 - 1}$$

for t sufficiently large. Thus

$$\left| \int_{S^1} \frac{f(z) dz}{z - \big(\widetilde{A}_t'' - \widetilde{A}_t'\big)} - \int_{S^1} \left(\sum_{k=0}^{\infty} \left(\frac{P}{z}\, \omega^V \right)^k \right) \frac{P}{z}\, f(z)\, dz \right| = O\Big(t^{-\frac{1}{2}}\Big)\ .$$

It remains to check that

$$P\,\omega^V\,P = P\,\omega^{\mathcal{H}}\,P = P\left(\nabla^{\mathcal{H}} - \nabla^{\mathcal{H},*}\right)P$$

with respect to the metric $g^{\mathcal{H}}$ induced by identifying $\mathcal{H}$ with $\ker(\partial + \partial^*)$. Then apparently

$$\lim_{t\to\infty} f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big) = f'\big(P\,\omega^{\mathcal{H}}\,P\big)$$

and thus

$$\lim_{t\to\infty} \mathrm{ch}^{\mathrm{o}}\big(A', g^V_t\big) = \mathrm{ch}^{\mathrm{o}}\big(\nabla^{\mathcal{H}}, g^{\mathcal{H}}\big)\,. \qquad \square$$

2.d. Torsion forms for families of finite dimensional complexes. We refine Theorem 2.25 to an equality of differential forms. We will find that as a correction term, we need the exterior differential of a form that starts with the Franz-Reidemeister torsion in degree 0, and therefore will be called a *torsion form*. It is of course the toy model for the higher analytic torsion.

Recall that by Proposition 2.8 (2), there is a differential form $\widetilde{\mathrm{ch}}^{\mathrm{o}}(A', g^V_\tau, g^V_T)$ such that

$$d\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(A', g^V_\tau, g^V_T\big) = \mathrm{ch}^{\mathrm{o}}\big(A', g^V_T\big) - \mathrm{ch}^{\mathrm{o}}\big(A', g^V_\tau\big)\,.$$

We want to compute this form. Up to a correction term, the analytic torsion will equal the limit $\widetilde{\mathrm{ch}}^{\mathrm{o}}(A', g^V_\tau, g^V_T)$ as $\tau \to 0$ and $T \to \infty$.

We extend A' trivially to a superconnection $\overline{A}'$ on the bundle

$$V \times (0,\infty) \longrightarrow B \times (0,\infty)\,,$$

define $\bar{g}^V$ on this bundle such that

$$\bar{g}^V|_{(p,t)} = g^V_t|_p = g^V\big(t^{N^V}\cdot\,,\,\cdot\big)$$

and let $\overline{A}''$ denote the adjoint of $\overline{A}'$ with respect to $\bar{g}^V$. Then

$$\overline{A}' = \nabla^V + \partial + \frac{\partial}{\partial t}\,dt = A' + \frac{\partial}{\partial t}\,dt$$

$$\text{and}\qquad \overline{A}'' = \nabla^{V,*} + t\,\partial^* + \left(\frac{\partial}{\partial t} + \frac{N^V}{t}\right)dt = A''_t + \left(\frac{\partial}{\partial t} + \frac{N^V}{t}\right)dt\,.$$

The form $\widetilde{\mathrm{ch}}^{\mathrm{o}}(A', g^V_\tau, g^V_T)$ can be defined as

$$\mathrm{ch}^{\mathrm{o}}\big(A', g^V_T\big) = \frac{1}{2}\int_0^1 \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} ds \int_{B\times(0,\infty)/B} \mathrm{str}_V\Big(\big(\overline{A}'' - \overline{A}'\big)e^{(\overline{A}''-\overline{A}')^2}\Big)\,. \tag{2.28}$$

If we write $f(z) = ze^{z^2}$, then the integrand can be written as

$$\begin{aligned}\mathrm{str}_V\Big(f\big(\overline{A}'' - \overline{A}'\big)\Big) &= \mathrm{str}_V\left(\frac{N^V}{t}\,dt\,f'\big(A''_t - A'\big)\right)\\ &= \mathrm{str}_V\left(\frac{N^V}{t}\,f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\right)dt\,,\end{aligned}$$

where we have conjugated with $t^{\frac{N^V}{2}}$ in the last step. Thus

$$\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(A', g^V_\tau, g^V_T\big) = \int_0^1\int_\tau^T \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \mathrm{str}_V\Big(N^V\,f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big)\,\frac{dt}{2t}\,ds\,. \tag{2.29}$$

Unfortunately, this integral diverges as $\tau \to 0$ and $T \to \infty$. Let us write

$$\chi'(V) = \sum_i (-1)^i i\,\mathrm{rk}\,V^i \qquad \text{and} \qquad \chi'(\mathcal{H}) = \sum_i (-1)^i i\,\mathrm{rk}\,\mathcal{H}^i\,.$$

2.30. Proposition ([BL]). *As $t \to 0$,*

$$\operatorname{str}_V\Big(N^V\, f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big) = \chi'(V) + O(t)\ . \tag{1}$$

As $t \to \infty$,

$$\operatorname{str}_V\Big(N^V\, f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big) = \chi'(\mathcal{H}) + O\Big(t^{-\frac{1}{2}}\Big)\ . \tag{2}$$

Proof. Clearly,

$$\lim_{t\to 0} \operatorname{str}_V\Big(N^V\, f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big) = \operatorname{str}_V\Big(N^V\, f'\big(\nabla^{V^*} - \nabla^V\big)\Big) = \operatorname{str}_V\Big(N^V\, f'\big(\omega^V\big)\Big)\ .$$

We note that ω^V preserves the $\mathbb{Z}$-grading and thus commutes with N^V. Hence, since $f'(z) = (1+2z^2)e^{z^2}$ is an even function with $f'(0)=1$, we compute

$$\begin{aligned} \operatorname{str}_V\Big(N^V\, f'\big(\omega^V\big)\Big) &= f'(0)\ \operatorname{str}_V\big(N^V\big) + \frac{1}{2}\operatorname{str}_V\bigg(\Big[N^V\omega^V, \frac{f'(\omega^V) - f'(0)}{\omega^V}\Big]\bigg)\\ &= \operatorname{str}_V\big(N^V\big) = \chi'(V)\ , \end{aligned}$$

because the supertrace of a supercommutator vanishes. Equation (1) follows because

$$\operatorname{str}_V\Big(N^V\, f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big) = \operatorname{str}_V\Big(N^V\, f'\big(A''_t - A'\big)\Big)$$

is a polynomial in t.

By a similar argument as in the proof of Theorem 2.25,

$$\lim_{t\to\infty} \operatorname{str}_V\Big(N^V\, f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big) = \operatorname{str}_{\mathcal{H}}\Big(N^{\mathcal{H}}\, f'\big(\nabla^{\mathcal{H},*} - \nabla^{\mathcal{H}}\big)\big)\Big) = \operatorname{str}_{\mathcal{H}}\big(N^{\mathcal{H}}\big) = \chi'(\mathcal{H})\ .$$

More precisely, even (2) holds. $\square$

2.31. Theorem ([BL]). *The form*

$$\begin{aligned} T\big(A', g^V\big) = -\int_0^1\int_0^\infty \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \bigg(&\operatorname{str}_V\Big(N^V\, f'\big(\widetilde{A}''_t - \widetilde{A}'_t\big)\Big) - \chi'(\mathcal{H})\\ &- \big(\chi'(V) - \chi'(\mathcal{H})\big)\, f'\big(\sqrt{-t}\big)\bigg)\,\frac{dt}{2t}\, ds \end{aligned} \tag{1}$$

satisfies

$$dT\big(A', g^V\big) = \operatorname{ch}^\circ\big(\nabla^V, g^V\big) - \operatorname{ch}^\circ\big(\nabla^{\mathcal{H}}, g^{\mathcal{H}}\big)\ . \tag{2}$$

Moreover, its component in degree 0 is the Franz-Reidemeister torsion of the fibrewise complex (V, ∂).

Proof. It follows from Proposition 2.30 that the integral (1) converges. Moreover,

$$- d\int_0^1\int_\tau^T\left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}}\Biggl(\operatorname{str}_V\Bigl(N^V\,f'\bigl(\widetilde{A}''_t-\widetilde{A}'_t\bigr)\Bigr)-\chi'(H)$$
$$-\bigl(\chi'(V)-\chi'(H)\bigr)\,f'\bigl(\sqrt{-t}\bigr)\Biggr)\,\frac{dt}{2t}\,ds=\operatorname{ch}^o\bigl(A',g^V_\tau\bigr)-\operatorname{ch}^o\bigl(A',g^V_T\bigr)\;.$$

Equation (2) follows by taking the limits $\tau\to 0$ and $T\to\infty$.

For the component in degree 0, the integral over s acts as 1. Let us assume that $(\partial+\partial^*)^2$ has a single eigenvalue λ. If $\lambda=0$, then $\chi'(V)=\chi'(H)$ and

$$\int_0^\infty\Bigl(\operatorname{str}_V\bigl(N^V\,f'(0)\bigr)-\chi'(H)\Bigr)\,\frac{dt}{2t}=0\;.$$

If $\lambda>0$, then $\chi'(H)=0$ and

$$\int_\tau^T\Bigl(\operatorname{str}_V\Bigl(N^V\,f'\bigl(\sqrt{-t\lambda}\bigr)\Bigr)-\chi'(V)\,f'\bigl(\sqrt{-t}\bigr)\Bigr)\,\frac{dt}{2t}$$
$$=\chi'(V)\int_{\frac{\tau}{\lambda}}^{\tau}f'\bigl(\sqrt{-t}\bigr)\,\frac{dt}{2t}-\chi'(V)\int_{\frac{T}{\lambda}}^{T}f'\bigl(\sqrt{-t}\bigr)\,\frac{dt}{2t}$$

Because $\lambda>0$, by (2.26) clearly

$$\lim_{T\to\infty}\int_{\frac{T}{\lambda}}^{T}f'\bigl(\sqrt{-t}\bigr)\,\frac{dt}{2t}=0$$

and

$$\lim_{\tau\to 0}\int_{\frac{\tau}{\lambda}}^{\tau}f'\bigl(\sqrt{-t}\bigr)\,\frac{dt}{2t}=\lim_{\tau\to 0}\int_{\frac{1}{\lambda}}^{1}f'\bigl(\sqrt{-\tau t}\bigr)\,\frac{dt}{2t}$$
$$=f'(0)\int_{\frac{1}{\lambda}}^{1}\frac{dt}{2t}=\frac{1}{2}\,\log\lambda\;.$$

Combining these limits, we have

$$-\int_\tau^T\Bigl(\operatorname{str}_V\Bigl(N^V,f'\bigl(\sqrt{-t\lambda}\bigr)\Bigr)-\chi'(V)\,f'\bigl(\sqrt{-t}\bigr)\Bigr)\,\frac{dt}{2t}=-\frac{\chi'(V)}{2}\,\log\lambda\;.$$

Decomposing V fibrewise into eigenspaces of $(\partial+\partial^*)^2$, we finally find that

$$T\bigl(A',g^V\bigr)^{[0]}=-\frac{1}{2}\operatorname{str}_{\ker(\partial+\partial^*)^\perp}\Bigl(N^V\,\log\bigl((\partial+\partial^*)^2\bigr)\Bigr)=\tau(V,\partial)\;.\qquad\square$$

2.32. Proposition. *Assume that the metric g^V on V is parallel with respect to ∇^V. Then*

$$T\big(\nabla^V + \partial, g^V\big) = \tau(V,\partial) \in \mathbb{R} \subset C^\infty(B) \subset \Omega^\bullet(B) .$$

Proof. In this case, $\omega^V = \nabla^{V,*} - \nabla^V = 0$, so

$$\widetilde{A}_t'' - \widetilde{A}_t' = \sqrt{t}\,\big(\partial^* - \partial\big)$$

is of degree 0 with respect to $\Omega^\bullet(B)$. □

As an application, assume that we are given a fibre bundle $p\colon M \to B$ with compact fibres and a function $h\colon M \to \mathbb{R}$ such that $h|_{p^{-1}b}$ is a Morse function for each fibre of p. Let $C \subset M$ be the set of fibre-wise critical points, then $\hat{p} = p|_C\colon C \to B$ is a finite covering of B. Along C, the vertical tangent bundle $TX = \ker p_* \subset TM$ splits (uniquely up to homotopy) as

$$TX|_C = T^sX \oplus T^uX \;\longrightarrow\; C$$

such that d^2h is positive on T^sX and negative on T^uX. We have $\operatorname{rk} T^uX = \operatorname{ind} h$, and we let $o(T^uX) = \Lambda^{\operatorname{ind} h} T^uX$ denote the orientation bundle of T^uX, which has a natural flat connection $\nabla^{o(T^uX)}$. Let $(F, \nabla^F) \to M$ be a flat bundle and define

$$V = \hat{p}_* F \otimes o(T^uX) \;\longrightarrow\; B ,$$

then V inherits a flat connection ∇^V from ∇^F and $\nabla^{o(T^uX)}$. The bundle V is $\mathbb{Z}$-graded by the Morse index $\operatorname{ind} h$.

Assume moreover that there exists a vertical metric g^{TX}. such that the vertical gradient $\nabla^{TX} h \in \Gamma(TX)$ satisfies the Smale transversality condition on every fibre of p. Then over each $b \in B$, there exists a natural differential $\partial \in \operatorname{End}^1 V$ such that the bundle

$$\mathcal{H} = H^\bullet(V^\bullet, \partial) \;\longrightarrow\; B$$

is naturally isomorphic to the fibrewise cohomology $H^\bullet(M/B; F)$ of $M \to B$ with coefficients in F. Moreover, the operator ∂ is parallel with respect to ∇^V, so we can compute

$$T\big(\nabla^V + \partial, g^V\big) \in \Omega^{\text{even}}(B)$$

for any choice of metric g^V on V. Unfortunately, our assumptions on $M \to B$ are so strong that $T(\nabla^V + \partial, g^V)$ does not give an interesting invariant.

If we drop the fibrewise Smale condition, we already get interesting examples. We have to replace $A' = \nabla^V + \partial$ by a more general flat superconnection

$$A' = \nabla^V + \sum_i a_i ,$$

such that at each $b \in B$, $a_i|_b \in \Lambda^i T^*B \otimes N_b$, where $N_b \subset \operatorname{End} V$ is a certain nilpotent subalgebra. In this situation, we have to change the definition of the torsion to ensure convergence as $t \to 0$, see [G1]. In [G2], a relation to Igusa's definition of higher Franz-Reidemeister torsion in [I] is established.

2.33. Exercise. Prove Proposition 2.1 (2) by writing

$$\operatorname{ch}\big(\overline{V}, \nabla^{\overline{V}}\big) = \alpha + \beta\, ds$$

such that $\iota_{\frac{\partial}{\partial s}}\alpha = \iota_{\frac{\partial}{\partial s}}\beta = 0$. Then check that

$$d\int_0^1 \beta|_{M\times\{s\}}\,ds = \alpha|_{M\times\{1\}} - \alpha|_{M\times\{0\}}\ .$$

2.34. Exercise. Check (2.3) and prove Remark 2.4 (2) using (2.3).

2.35. Exercise. Prove Proposition 2.9.

2.36. Exercise. Check that with our sign rules, for $a \in \operatorname{End}^{\mathrm{odd}} V$, $v \in V$ and $X \in \mathfrak{X}(M)$, we have

$$\nabla^V_X(a\cdot v) = \iota_X\big([\nabla^V, a](v) - a\cdot\nabla^V v\big)\ .$$

Then check that the terms $[\nabla^V, a_i]$ in (2.13) are indeed derivatives of the components a_i.

2.37. Exercise. Construct a flat superconnection $A = \nabla^V + a_0 + a_1 + a_2$ on the superbundle V with $V^{\mathrm{even}} \cong V^{\mathrm{odd}} \cong W$ an ordinary vector bundle, such that

$$a_0 = \mathrm{id}_W\colon V^{\mathrm{even}} \longrightarrow V^{\mathrm{odd}} \qquad\text{and}\qquad \nabla^V + a_1 = \nabla^W \oplus \nabla^W\ .$$

In general, V does not admit a flat connection.

3. The higher analytic torsion of Bismut and Lott

We copy the construction of $T(A', g^V)$ by replacing the bundle V by the fibrewise de Rham complex of a fibre bundle $M \to B$.

3.a. The de Rham differential as a superconnection. Let $M \to B$ be a smooth fibre bundle, then the exterior differential d on $\Omega^\bullet(M)$ may naturally be regarded as an example of a superconnection.

Let $TX = \ker p_* \subset TM \to M$ denote the vertical tangent bundle. The bundle $\Lambda^\bullet T^*M$ admits a natural increasing filtration

$$0 = \mathcal{F}^{-1}\Lambda^\bullet T^*M \subset \mathcal{F}^0\Lambda^\bullet T^*M \subset \cdots \subset \mathcal{F}^{\dim B}\Lambda^\bullet T^*M = \Lambda^\bullet T^*M$$

given by

$$\mathcal{F}^j\Lambda^{j+k}T^*_xM = \big\{\,\alpha\in\Lambda^{j+k}T^*_xM \;\big|\; \iota_{X_0}\cdots\iota_{X_k}\alpha = 0 \text{ for all } X_0,\,\dots\,,\,X_k \in T_xX\,\big\}$$

at each point $x \in M$. These vector spaces clearly form bundles, and we have

$$\mathcal{F}^j\Lambda^\bullet T^*M/\mathcal{F}^{j-1}\Lambda^\bullet T^*M \cong p^*\Lambda^jT^*B \mathbin{\widehat{\otimes}} \Lambda^\bullet T^*X\ ,$$

where T^*X is the dual of TX.

Let us choose a horizontal subbundle $T^HM \to M$, such that

$$TM = T^HM \oplus TX\ ,$$

then $T^HM \cong TM/TX = p^*TB$, and we obtain a splitting

$$\Lambda^\bullet T^*M \cong \bigoplus_{j,k} p^*\Lambda^jT^*B \mathbin{\widehat{\otimes}} \Lambda^kT^*X$$

that depends on the choice of T^HM.

We can define an infinite-dimensional vector bundle

$$\Omega^\bullet(M/B) = p_*\Lambda^\bullet T^*X \longrightarrow B \qquad \text{by} \qquad \Omega^\bullet(M/B)_b = \Omega^\bullet(X_b)\ .$$

Then we have

$$\Omega^\bullet(M) = \Omega^\bullet\big(B;\Omega^\bullet(M/B)\big) = \Gamma\big(\Lambda^\bullet T^*B \mathbin{\widehat{\otimes}} \Omega^\bullet(M/B)\big)\ . \tag{3.1}$$

Then the fibrewise de Rham operator d on $\Omega(X_b)_{b\in B}$ extends to a global operator

$$d_X = \operatorname{id}\otimes d_X\ . \tag{3.2}$$

For a vector field $V \in \mathfrak{X}(B)$, there is a vector field $\overline{V} \in \mathfrak{X}(M)$ satisfying

$$\overline{V} \in \Gamma(T^HM) \qquad \text{and} \qquad p_*\overline{V}_x = V_{p(x)} \qquad \text{for all } x \in M\ ,$$

the *horizontal lift of* V. Since vector fields act on forms by the Lie derivative

$$\big(\mathcal{L}_X\alpha\big)(Y_1,\dots,Y_k) = X\big(\alpha(Y_1,\dots,Y_k)\big) + \sum_{i=1}^k(-1)^i\alpha\big([X,Y_i],Y_1,\dots,\widehat{X}_i,\dots,X_k\big)\ ,$$

we have a connection $\nabla^{M/B}$ on $\Omega^\bullet(M/B)\to B$ given by

$$\nabla^{M/B}_V\alpha = \mathcal{L}_{\overline{V}}\alpha \tag{3.3}$$

for all $\alpha \in \Gamma(\Omega^\bullet(M/B)) \subset \Omega^\bullet(M)$ and all $V \in \mathfrak{X}(B)$.

In general, the bundle T^HM is not integrable, i.e., there are no $\dim B$-dimensional local submanifolds tangent to T^HM. Even though over some small subset of B, one may choose T^HM integrable, there are certain obstructions against the existence of a global integrable complement to TX. The local nonintegrability is measured by the tensor

$$\Omega \in \Omega^2(B;p_*TX) \qquad \text{with} \qquad \Omega(V,W) = -\big[\overline{V},\overline{W}\big]^\perp \in \Gamma(TX)$$

for all V, $W \in \mathfrak{X}(B)$. To see that Ω is indeed a tensor, note that

$$\big[f\overline{V},g\overline{W}\big] - fg\,\big[\overline{V},\overline{W}\big] = fV(g)\,\overline{W} - gW(f)\,\overline{V} \in \Gamma\big(T^HM\big)\ .$$

3.4. Proposition ([BGV]). *With respect to the splitting (3.1), the exterior differential d on $\Omega^\bullet(M)$ becomes a flat superconnection on $\Omega^\bullet(M/B) = p_*\Lambda^\bullet T^*X$ with*

$$d = d_X + \nabla^{M/B} + \iota_\Omega\ .$$

Let us note that even though d_X is uniquely defined on a single fibre, its action on $\Omega^*(M)$ depends on the choice of T^HM. Of course, a similar formula as above applies to a flat connection ∇^F on a vector bundle $F\to M$. We define

$$\mathbb{A}' = \nabla^F = d_X^F + \nabla^{M/B,F} + \iota_\Omega \qquad \text{on} \qquad \Omega^\bullet(M;F) = \Omega^\bullet\big(B;\Omega^\bullet(M/B;F)\big)\ . \tag{3.5}$$

Proof. The operator d is clearly odd. Naturality of d implies that for $\alpha \in \Omega^k(B)$, $\beta \in \Omega^\bullet(M)$, one has

$$d(p^*\alpha \wedge \beta) = \bigl(p^* d\alpha\bigr) \wedge \beta + (-1)^k p^*\alpha \wedge d\beta \,.$$

By (3.1), d is thus a superconnection. It is flat because $d^2 = 0$. The formula for d follows by a careful study of (1.5). $\square$

Since we want to study the class $\mathrm{ch}^o(\mathbb{A}')$ of the flat superconnection $\mathbb{A}'$, we have to define a suitable family of metrics on $\overline{\Omega}^\bullet(M/B;F) \to \overline{B} = B \times \mathbb{R}_>$ as in (2.22). We start by choosing metrics g^{TX} on TX and g^F on F. For $t > 0$, the metric $\frac{1}{t}\, g^{TX}$ induces a fibrewise volume form $t^{-\frac{n}{2}}\, d\,\mathrm{vol}_X$ and together with g^F, we get a metric $g_t^{T^*X,F}$ on $\Lambda^\bullet T^*X \otimes F$. Together, these metrics define an L_2-metric $\bar g^{M/B;F}$ on $\overline{\Omega}^\bullet(M/B;F) \to \overline{B}$ with

$$\bar g^{M/B;F}|_{(b,t)} = \int_{X_b} g^{T^*X,F}\Bigl(t^{N^X - \frac{n}{2}}\,\cdot\,,\,\cdot\Bigr)\, d\,\mathrm{vol}_X \,.$$

Let $\overline{\mathbb{A}}' = \mathbb{A}' + \frac{\partial}{\partial t}\, dt$ be the trivial extension of $\mathbb{A}'$ to $\overline{\Omega}^\bullet(M/B;F)$. We compute the adjoint $\overline{\mathbb{A}}''$ of $\overline{\mathbb{A}}'$ with respect to $\bar g^{M/B;F}|_{(b,t)}$.

The adjoint of d_X^F clearly is $t\, d_X^{F,*}$ as in (1.11). Because ε is adjoint to ι, the adjoint of ι_Ω is $-\frac{1}{t}\, \varepsilon_\Omega$ because of the signs in Definition 2.17 and Definition 2.20.

The adjoint of $\nabla^{M/B,F} + \frac{\partial}{\partial t}\, dt$ involves the shape operator $S \in \Omega^1(B; p_* \operatorname{End} TX)$ given by

$$S(V) = -\bigl(g^{TX}\bigr)^{-1}\, \mathcal{L}_{\overline{V}} g^{TX} \in \Gamma(\operatorname{End} TX)$$

for all $V \in \mathfrak{X}(B)$, where we regard $g^{TX}\colon TX \to T^*X$. The operator $S(V)$ is selfadjoint with respect to g^{TX} for all V, and $H(V) = \mathrm{tr}_{TX}(S(V))$ is the mean curvature of the fibres of $M \to B$ in the corresponding horizontal direction—this can be defined solely in terms of a vertical metric.

We let S act as a formal derivative on $\Omega^\bullet(M/B)$ by

$$\bigl(S(V)\,\alpha\bigr)(X_1,\dots,X_k) = \alpha\bigl(S(V)\,X_1, X_2,\dots,X_k\bigr) + \dots + \alpha\bigl(X_1,\dots,X_{k-1}, S(V)\,X_k\bigr) \,.$$

The adjoint of $\overline{\nabla}^{M/B,F} = \nabla^{M/B} + \frac{\partial}{\partial t}\, dt$ with respect to $\bar g^{M/B,F}$ is then given by

$$\overline{\nabla}^{M/B,*} = \nabla^{M/B} + \omega^F(\overline{\cdot}) + S - \frac{1}{2}\, H + \Bigl(N^X - \frac{n}{2}\Bigr)\, \frac{dt}{t} \,, \tag{3.6}$$

because

$$\begin{aligned} d\bar g^{M/B,F}(\alpha,\beta) &= \int_X \Bigl(\mathcal{L}_{\overline{\cdot}} + \frac{\partial}{\partial t}\, dt\Bigr) \Bigl(g^{T^*X,F}\Bigl(t^{N^X - \frac{n}{2}}\alpha,\beta\Bigr)\, d\,\mathrm{vol}_X\Bigr) \\ &= \int_X g^{T^*X,F}\bigl(t^{N^X - \frac{n}{2}}\mathcal{L}_{\overline{\cdot}}\alpha,\beta\bigr)\, d\,\mathrm{vol}_X \\ &\quad + \int_X g^{T^*X,F}\Bigl(t^{N^X - \frac{n}{2}}\alpha, \mathcal{L}_{\overline{\cdot}}\beta + \omega^F(\overline{\cdot})\beta + \bigl(g^{T^*X,F}\bigr)^{-1}\bigl(\mathcal{L}_{\overline{\cdot}} g^{T^*X,F}\bigr)\beta \\ &\qquad + \Bigl(N^X - \frac{n}{2}\Bigr)\frac{dt}{t}\,\beta + d\,\mathrm{vol}_X^{-1}\,\bigl(\mathcal{L}_{\overline{\cdot}} d\,\mathrm{vol}_X\bigr)\cdot\beta\Bigr)\, d\,\mathrm{vol}_X \\ &= \int_X \Bigl(g^{T^*X,F}\bigl(\overline{\nabla}^{M/B}\alpha,\beta\bigr) + g^{T^*X,F}\bigl(\alpha,\overline{\nabla}^{M/B,*}\beta\bigr)\Bigr)\, d\,\mathrm{vol}_X \,. \end{aligned}$$

In particular,

$$\overline{\mathbb{A}}'' - \overline{\mathbb{A}}' = \left(t\, d_X^{F,*} - d_X^F\right) + \left(S - \frac{1}{2}\, H + \omega^F(\overline{\cdot})\right) - \left(\frac{1}{t}\,\varepsilon_\Omega + \iota_\Omega\right) + \left(N^X - \frac{n}{2}\right)\frac{dt}{t}$$
$$\in \Omega^\bullet\left(\overline{B}; \operatorname{End}\Omega^\bullet(M/B;F)\right) .$$

As a difference of two superconnections, the formula above should not contain any differentials. This is in fact true if we view the differential operator $d_X^{F,*} - d_X^F$ in degree 0 as an endomorphism of $\Omega^\bullet(M/B;F)$. As in (2.23), we conjugate with $t^{\frac{N^X}{2}}$ and put

$$\begin{aligned}(3.7)\qquad \widetilde{\mathbb{X}} &= t^{\frac{N^X}{2}}\left(\overline{\mathbb{A}}'' - \overline{\mathbb{A}}'\right) t^{-\frac{N^X}{2}} \\ &= \sqrt{t}\left(d_X^{F,*} - d_X^F\right) + \left(S - \frac{1}{2}\, H + \omega^F(\overline{\cdot})\right) - \frac{1}{\sqrt{t}}\left(\frac{1}{t}\,\varepsilon + \iota\right)_\Omega + \frac{dt}{t}\left(N^X - \frac{n}{2}\right) .\end{aligned}$$

3.b. Generalised Dirac and Laplace operators. Our formula for odd Chern classes forces us to consider $\operatorname{str}_{\Omega(M/B;F)}\left((\mathbb{A}'' - \mathbb{A}')\, e^{(\mathbb{A}''-\mathbb{A}')^2}\right)$, so we have to check that $e^{(\mathbb{A}''-\mathbb{A}')^2}$ and $(\mathbb{A}''-\mathbb{A}')\, e^{(\mathbb{A}''-\mathbb{A}')^2}$ are trace class operators. We do this by showing that these operators are generalised (even and odd) heat operators.

For $v \in TX \cong T^*X$, we define

$$c(v) = \varepsilon_v - \iota_v \qquad \text{and} \qquad \hat{c}(v) = \varepsilon_v + \iota_v \ .$$

Then c defines a Clifford action of T^*M on $\Lambda^\bullet T^*M$ with respect to the degenerate metric g^{T^*X}, and $\hat{c}$ defines an anti-Clifford action, and these actions anticommute, so

$$\begin{aligned}0 &= c(v)\, c(w) + c(w)\, c(v) + 2g^{T^*X}(v,w) \\ &= \hat{c}(v)\, \hat{c}(w) + \hat{c}(w)\, \hat{c}(v) - 2g^{TX}(v,w) \\ &= c(v)\, \hat{c}(w) + \hat{c}(w)\, c(v) \ .\end{aligned}$$

We fix a local orthonormal basis $e^1, \dots, e^n$ of $TX \cong T^*X$ and a basis $e^{n+1}, \dots, e^m$ of $TX^\perp \cong (T^HM)^*$. Let $e_1, \dots, e_m$ denote the dual basis. We abbreviate $c_k = c(e_k)$ and $\hat{c}_k = \hat{c}(e_k)$ for $1 \leq k \leq n$.

With this notation, we have

$$\left(d_X^{F,*} - d_X^F\right) = -\sum_{k=1}^n \left(\iota_k\, \nabla_{e_k}^{T^*X,F,*} + \varepsilon_k\, \nabla_{e_k}^{T^*X,F}\right) = -\sum_{k=1}^n \left(\hat{c}_k\, \nabla_{e_k}^{T^*X,F,u} - \frac{1}{2}\, c_k\, \omega^F(e_k)\right) ,$$

where

$$\nabla^{T^*X,F,u} = \frac{1}{2}\left(\nabla_{e_k}^{T^*X,F} + \nabla_{e_k}^{T^*X,F,*}\right) = \nabla_{e_k}^{T^*X,F} + \frac{1}{2}\,\omega^F \ .$$

We denote the coefficients of $S - \frac{1}{2}H$ by

$$s_{\alpha kl} = g^{TX}\left(S(e_\alpha)e_k, e_l\right)$$

for $1 \le i, j \le n$ and $n < \alpha \le m$, then $s_{\alpha kl} = s_{\alpha lk}$. The tensor $S - \frac{1}{2}H$ can be expressed as

$$\begin{aligned} S - \frac{1}{2}H &= \sum_{\alpha=n+1}^{m} \sum_{k,l=1}^{n} s_{\alpha kl}\, \varepsilon^\alpha \left(\varepsilon^k \iota_l - \frac{1}{2}\, \delta_{kl} \right) \\ &= \frac{1}{4} \sum_{\alpha=n+1}^{m} \sum_{k,l=1}^{n} s_{\alpha kl}\, \varepsilon^\alpha \left(\left(\hat{c}_k + c_k \right) \left(\hat{c}_l - c_l \right) - 2\delta_{kl} \right) \\ &= \frac{1}{2} \sum_{\alpha=n+1}^{m} \sum_{k,l=1}^{n} s_{\alpha kl}\, \varepsilon^\alpha c_k \hat{c}_l \ . \end{aligned}$$

Finally, with

$$\Omega(e_\alpha, e_\beta) = \sum_{k=1}^{n} \omega_{\alpha\beta k} e_k \ ,$$

we get

$$-\iota_\Omega - \varepsilon_\Omega = -\frac{1}{2} \sum_{\alpha,\beta} \omega_{\alpha\beta k}\, \varepsilon^\alpha \varepsilon^\beta \hat{c}_k \ .$$

To understand the operator $\widetilde{\mathbb{X}}$ better, we regard it on a single fibre for fixed t, but with coefficients in the exterior algebra of the base,

$$\widetilde{\mathbb{X}}|_{(b,t)} \in \Lambda^\bullet T^*_{(b,t)} \overline{B} \mathbin{\widehat{\otimes}} \operatorname{End} \Omega^\bullet \left(X_{(b,t)}; F \right) \subset \operatorname{End} \Omega^\bullet \left(X_{(b,t)}; \Lambda^\bullet T^*_{(b,t)} \overline{B} \otimes F \right) \ .$$

Then we rearrange the terms of (3.7) as in

(3.8)

$$\begin{aligned} \widetilde{\mathbb{X}}|_{(b,t)} &= -\sqrt{t} \sum_{k=1}^{n} \hat{c}_k \underbrace{\left(\nabla_{e_k}^{T^*X,F,u} + \frac{1}{\sqrt{t}} \sum_{\alpha=n+1}^{m} \sum_{l=1}^{n} s_{\alpha kl}\, \varepsilon_\alpha c_l + \frac{1}{2t} \sum_{\alpha,\beta=n+1}^{m} \omega_{\alpha\beta k}\, \varepsilon^\alpha \varepsilon^\beta \right)}_{=: \overline{\nabla}_{e_k}^{T^*X,F,t}} \\ &\quad + \frac{\sqrt{t}}{2} \sum_{k=1}^{n} \omega^F(e_k)\, c_k + \sum_{\alpha=n+1}^{m} \omega^F(e_\alpha)\, \varepsilon^\alpha \\ &= -\sqrt{t} \sum_{k=1}^{n} \hat{c}_k\, \overline{\nabla}_{e_k}^{T^*X,F,t} + \frac{\sqrt{t}}{2} \sum_{i=1}^{m} \mu_t^i\, \omega^F(e_i) \ , \end{aligned}$$

where $\overline{\nabla}^{T^*X,F,t}$ is a connection on the bundle $\Lambda^\bullet T^*M \otimes F|_{X_{(b,t)}}$, and where we write

$$\mu_t^i = \begin{cases} c_i & \text{if } e_i \in TX, \text{ and} \\ \frac{2}{\sqrt{t}}\, \varepsilon^i & \text{if } e_i \in T^H M \ . \end{cases} \tag{3.9}$$

Let $\overline{\Delta}^{T^*X,F,t}$ denote the Laplace operator associated to the connection $\overline{\nabla}^{T^*X,F,t}$, so

$$\overline{\Delta}^{T^*X,F,t} = -\sum_{i=1}^{k} \left(\overline{\nabla}_{e_k}^{T^*X,F,t} \overline{\nabla}_{e_k}^{T^*X,F,t} - \overline{\nabla}_{\nabla_{e_k}^{TX} e_k}^{T^*X,F,t} \right) \ .$$

The square of $\widetilde{\mathbb{X}}_t$ can be calculated as

$$\widetilde{\mathbb{X}}_t^2 = -t\,\overline{\Delta}^{T^*X,F,t} + \frac{t}{2}\sum_{k,l=1}^{n} \hat{c}_k\hat{c}_l\big(\overline{\nabla}^{T^*X,F,t}\big)^2_{e_k,e_l} - \frac{t}{2}\sum_{i=1}^{m}\sum_{k=1}^{n}\hat{c}_k\left[\overline{\nabla}^{T^*X,F,t}_{e_k}, \mu_t^i\omega^F(e_i)\right]$$
$$- \frac{t}{4}\sum_{i=1}^{m}\omega^F(e_i)^2 + \frac{t}{8}\sum_{i,j=1}^{m}\mu_t^i\mu_t^j\left[\omega^F(e_i),\omega^F(e_j)\right] .$$

Usually, the second term on the right is worked out in detail, where it gives scalar curvature and other interesting details, but this is not important here, we only need to know that this term contains no more derivatives.

The most important thing to notice is that $\widetilde{\mathbb{X}}_t^2$ has the principal symbol of t-times a Laplace operator. This implies that $\widetilde{\mathbb{X}}_t^2$ has discrete spectrum on the compact fibre X, and that the operators $e^{\widetilde{\mathbb{X}}_t^2}$, $\widetilde{\mathbb{X}}_t e^{\widetilde{\mathbb{X}}_t^2}$ and $\widetilde{\mathbb{X}}_t^2 e^{\widetilde{\mathbb{X}}_t^2}$ are trace class. Notice here that since the ε^α for $n < \alpha \leq m$ act as nilpotent operators, they have no influence on spectrum and tracability of $\widetilde{\mathbb{X}}_t^2$, only on the precise value of the trace in $\Lambda^\bullet T^*B$. In particular, it makes sense to define

$$\mathrm{ch}^{\mathrm{o}}\big(\mathbb{A}', g_t^{M/B,F}\big) = \int_0^1 \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \mathrm{str}\Big(\widetilde{\mathbb{X}}_t\, e^{\widetilde{\mathbb{X}}_t^2}\Big)\, ds \in \Omega^\bullet(B)\ . \tag{3.10}$$

As before, we will investigate the limits of $\mathrm{ch}^{\mathrm{o}}(\mathbb{A}', g_t^{M/B,F})$ as $t \to 0$ and $t \to \infty$.

3.c. The asymptotics of $\mathrm{ch}^{\mathrm{o}}(\mathbb{A}', g_t^{M/B,F})$**.** Since we want to compute the odd heat kernel of $\widetilde{\mathbb{X}}$ associated to the operator $\widetilde{\mathbb{X}}\, e^{\widetilde{\mathbb{X}}^2}$, we introduce an extra exterior variable z that anticommutes with all exterior and Clifford variables so far, and define

$$\mathrm{str}_z(A + zB) = \mathrm{str}(B)$$

if both A and B are trace class operators not containing the variable z. Then we can describe the supertrace of the odd heat kernel as the z-supertrace of an even heat kernel by

$$\mathrm{str}_z\big(e^{A^2+zA}\big) = \mathrm{str}_z\big((1+zA)\, e^{A^2}\big) = \mathrm{str}\big(A\, e^{A^2}\big)\ .$$

If we define a connection

$$\overline{\nabla}^{T^*X,F,t,z}_{e_k} = \overline{\nabla}^{T^*X,F,t}_{e_k} - \frac{1}{2\sqrt{t}}\, z\hat{c}_k \tag{3.11}$$

on the bundle $\Lambda^\bullet T^*M \mathbin{\widehat{\otimes}} (\mathbb{R}\oplus z\mathbb{R}) \otimes F|_{X_{(b,t)}}$ and define $\overline{\Delta}^{T^*X,F,t,z}_{e_k}$ as before, then

(3.12)
$$\widetilde{\mathbb{X}}_t^2 + z\,\widetilde{\mathbb{X}}_t = -t\,\overline{\Delta}^{T^*X,F,t,z} + \frac{t}{2}\sum_{k,l=1}^{n}\hat{c}_k\hat{c}_l\big(\overline{\nabla}^{T^*X,F,t,z}\big)^2_{e_k,e_l} - \frac{t}{2}\sum_{i=1}^{m}\sum_{k=1}^{n}\hat{c}_k\left[\overline{\nabla}^{T^*X,F,t}_{e_k},\mu_t^i\omega^F(e_i)\right]$$
$$- \frac{t}{4}\sum_{i=1}^{m}\omega^F(e_i)^2 + \frac{t}{8}\sum_{i,j=1}^{m}\mu_t^i\mu_t^j\left[\omega^F(e_i),\omega^F(e_j)\right] + \frac{\sqrt{t}\,z}{2}\sum_{i=1}^{m}\mu_t^i\,\omega^F(e_i)\ .$$

In particular, $\widetilde{\mathbb{X}}_t^2 + z\,\widetilde{\mathbb{X}}_t$ still has the pricipal symbol of a Laplace operator. So without extra effort, we find that the odd heat kernel is indeed of trace class.

From this point, we follow a standard procedure in local index theory, see e.g. [BGV]. First, we locally trivialise $\Lambda^\bullet T^*M \,\widehat{\otimes}\, (\mathbb{R} \oplus z\mathbb{R}) \otimes F|_{X_{(b,t)}}$ by parallel transport along radial geodesics in X from a fixed point x with respect to the connection $\overline{\nabla}^{T^*X,F,t,z}$. We express $\widetilde{\mathbb{X}}^2 + z\widetilde{\mathbb{X}}$ in these new coordinates. Next, we show that as $t \to 0$, the heat kernel of $e^{\widetilde{\mathbb{X}}^2+z\widetilde{\mathbb{X}}}$ can be computed up to an error of order $o(e^{-ct})$ using the local describtion of $\widetilde{\mathbb{X}}^2 + z\widetilde{\mathbb{X}}$ obtained above. Finally, the z-supertrace of the heat kernel involves only those terms with the maximum number of vertical Clifford variables c_k and $\hat{c}_k$. This allows us to perform a Getzler rescaling of the variables c_k. We replace the c_k by vertical exterior variables $\boldsymbol{\epsilon}^k$ and at the same time eliminate the singular factor $t^{-\frac{n}{2}}$. Note that the new $\boldsymbol{\epsilon}^k$ has nothing to do with the ε^k in the definition of c_k and $\hat{c}_k$.

After these three steps, the supertrace of the odd heat kernel can be written as an integral over the fibres of $M \to B$, and the integrand can be computed from the heat kernel of the model operator. A similar procedure also computes a supertrace that is important later in the definition of the higher torsion.

3.13. Theorem ([BL]). *As $t \to 0$,*

$$\operatorname{str}\Big(\widetilde{\mathbb{X}}_t\, e^{\widetilde{\mathbb{X}}_t^2}\Big) = \int_{M/B} \chi\big(TX, \nabla^{TX}\big)\ \operatorname{tr}_F\big(\omega^F\, e^{(\omega^F)^2}\big) + O\big(\sqrt{t}\big)\ , \tag{1}$$

$$\text{and} \qquad \operatorname{str}\Big(N^X\,\big(1 + 2\widetilde{\mathbb{X}}_t^2\big)\, e^{\widetilde{\mathbb{X}}_t^2}\Big) = \frac{n}{2}\,\chi(\mathcal{H}) + O\big(\sqrt{t}\big)\ . \tag{2}$$

In particular

$$\lim_{t\to 0} \operatorname{ch}^{\mathrm{o}}\big(\mathbb{A}', g_t^{M/B,F}\big) = \int_{M/B} \chi\big(TX, \nabla^{TX}\big)\ \operatorname{ch}^{\mathrm{o}}\big(\nabla^F, g^F\big)\ . \tag{3}$$

The details are given in Section 3.e.

We now investigate the behaviour of $\operatorname{ch}^{\mathrm{o}}(\mathbb{A}', g_t^{M/B,F})$ as $t \to \infty$. The only difference with respect to the proof of Theorem 2.25 is that V is replaced by an infinite dimensional bundle. The crucial facts are

(1) the operator $\widetilde{\mathbb{X}}_t$ is skew-adjoint, has discrete spectrum, and is of trace class, and
(2) the kernel of $\widetilde{\mathbb{X}}_t$ forms a bundle $\mathcal{H} \to B$.

These facts ensure that for each sufficiently small subset of B, we have a constant $r_0 > 0$ such that $\sqrt{t}\, r_0$ separates 0 from the rest of the spectrum of $\widetilde{\mathbb{X}}$, and we can again employ holomorphic functional calculus, using the contour Γ_t of Figure 2.27.

3.14. Theorem ([BL]). *As $t \to \infty$,*

$$\operatorname{str}\Big(\widetilde{\mathbb{X}}_t\, e^{\widetilde{\mathbb{X}}_t^2}\Big) = \operatorname{str}_{\mathcal{H}}\big(\omega^{\mathcal{H}}\, e^{(\omega^{\mathcal{H}})^2}\big) + O\Big(t^{-\frac{1}{2}}\Big)\ , \tag{1}$$

$$\text{and} \qquad \operatorname{str}\Big(N^X\,\big(1 + 2\widetilde{\mathbb{X}}_t^2\big)\, e^{\widetilde{\mathbb{X}}_t^2}\Big) = \chi'(\mathcal{H}) + O\Big(t^{-\frac{1}{2}}\Big)\ . \tag{2}$$

In particular,

$$\lim_{t\to\infty} \operatorname{ch}^{\mathrm{o}}\big(\mathbb{A}', g_t^{M/B,F}\big) = \operatorname{ch}^{\mathrm{o}}\big(\nabla^{\mathcal{H}}, g_{L^2}^{\mathcal{H}}\big)\ . \tag{3}$$

This implies that

$$\operatorname{ch}^{\mathrm{o}}\big(\nabla^{\mathcal{H}}\big) = \int_{M/B} \chi(TX)\ \operatorname{ch}^{\mathrm{o}}\big(\nabla^F\big) \in H^\bullet(B)\ . \tag{4}$$

3.d. The higher analytic torsion form. As in Section 2.d, we refine Theorem 3.14 (4) to an equality of differential forms. By (3.7), we are led to consider the expression

$$\widetilde{\mathrm{ch}}^{\circ}\big(\mathbb{A}', g_\tau^{M/B;F}, g_T^{M/B;F}\big) = \int_0^1 \int_\tau^T \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \mathrm{str}\Big(\Big(N^X - \frac{n}{2}\Big)\, f'\big(\widetilde{\mathbb{X}}_t\big)\Big)\, \frac{dt}{2t}\, ds$$

with $f(z) = z\, e^{z^2}$ as before.

Let us first understand the simpler expression $\mathrm{str}\big(f'(\widetilde{\mathbb{X}}_t)\big)$. By the arguments of Section 3.e below, we know that

$$\lim_{t\to 0} \mathrm{str}\Big(f'\big(\widetilde{\mathbb{X}}_t\big)\Big) = \chi(X)\ \mathrm{rk}(F) = \chi(\mathcal{H})\ .$$

The expression $\mathrm{str}\big(f'(\widetilde{\mathbb{X}}_t)\big)$ is almost independent of t, because by (3.7),

$$\widetilde{\mathbb{X}}_t = \sqrt{t}\, t^{-\frac{N^B}{2}}\, \widetilde{\mathbb{X}}_1\, t^{\frac{N^B}{2}}\ ,$$

hence

$$\mathrm{str}\Big(f'\big(\widetilde{\mathbb{X}}_t\big)\Big) = t^{-\frac{N^B}{2}}\ \mathrm{str}\Big(f'\big(\sqrt{t}\widetilde{\mathbb{X}}_1\big)\Big)\ ,$$

and

$$\frac{\partial}{\partial t}\, \mathrm{str}\Big(f'\big(\sqrt{t}\widetilde{\mathbb{X}}_1\big)\Big) = \frac{1}{2\sqrt{t}}\ \mathrm{str}\Big(\widetilde{\mathbb{X}}_1\, f''\big(\sqrt{t}\widetilde{\mathbb{X}}_1\big)\Big) = \frac{1}{4\sqrt{t}}\ \mathrm{str}\Big(\Big[\widetilde{\mathbb{X}}_1, f''\big(\sqrt{t}\widetilde{\mathbb{X}}_1\big)\Big]\Big) = 0\ .$$

By the last two equations,

$$\mathrm{str}\Big(f'\big(\widetilde{\mathbb{X}}_t\big)\Big) = t^{-\frac{N^B}{2}}\ \mathrm{str}\Big(f'\big(\widetilde{\mathbb{X}}_1\big)\Big) = \chi(\mathcal{H}) \qquad \text{for all } t \geq 0\ . \tag{3.15}$$

So far,

$$\widetilde{\mathrm{ch}}^{\circ}\big(\mathbb{A}', g_\tau^{M/B;F}, g_T^{M/B;F}\big) = \int_0^1 \int_\tau^T \left(\frac{s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \mathrm{str}\Big(N^X\, f'\big(\widetilde{\mathbb{X}}_t\big)\Big)\, \frac{dt}{2t}\, ds - \frac{n}{4}\, \log \frac{T}{\tau}\, \chi(X)\ \mathrm{rk}\, F\ .$$

Since the second term is an overall constant and has no exterior derivative, we will drop it when we define the torsion. Once more, both limits $\tau \to 0$ and $T \to \infty$ diverge, see Theorem 3.13 and Theorem 3.14. As in Theorem 2.31, we can eliminate the divergences in the definition of the analytic torsion

3.16. Theorem ([BL]). *The form*

$$\text{(1)}\quad \mathcal{T}\big(T^H M, g^{TX}, \nabla^F, g^F\big) = -\int_0^1 \int_0^\infty \left(\frac{s(1-s)}{2\pi i}\right)^{N^B} \Big(\mathrm{str}\Big(N^X\, f'\big(\widetilde{\mathbb{X}}_t\big)\Big) - \chi'(\mathcal{H})\, f'(0)$$
$$+ \Big(\chi'(\mathcal{H}) - \frac{n}{2}\, \chi(X)\ \mathrm{rk}\, F\Big)\, f'\big(\sqrt{-t}\big)\Big)\, \frac{dt}{2t}\, ds$$

satisfies

$$\text{(2)}\qquad d\mathcal{T}\big(T^H M, g^{TX}, \nabla^F, g^F\big) = \int_{M/B} e\big(TX, \nabla^{TX}\big)\ \mathrm{ch}^{\circ}\big(\nabla^F, g^F\big) - \mathrm{ch}^{\circ}\big(\nabla^{\mathcal{H}}, g^{\mathcal{H}}_{L^2}\big)\ .$$

Moreover, its component in degree 0 is the Ray-Singer torsion of the corresponding fibre.

Proof. Convergence of (1) and equation (2) are shown as in the proof of Theorem 2.31. For the last statement, one has to work a little harder, see [BL]. □

In general, the form $\mathcal{T}(T^H M, g^{TX}, \nabla^F, g^F)$ is not closed, and thus not very helpful. In fact, often a small perturbation of $T^H M$, g^{TX} or g^F can alter $\mathcal{T}(T^H M, g^{TX}, \nabla^F, g^F)$ considerably. We have the following variation formula

$$\begin{aligned}(3.17)\quad \mathcal{T}\big(T^H M_1, g_1^{TX}, \nabla^F, g_1^F\big) - \mathcal{T}\big(T^H M_0, g_0^{TX}, \nabla^F, g_0^F\big) \\ = \int_{M/B} \Big(\tilde{e}\big(TX, \nabla_0^{TX}, \nabla_1^{TX}\big) \operatorname{ch}^{\mathrm{o}}\big(\nabla^F, g_0^F\big) + e\big(TX, \nabla_1^{TX}\big) \widetilde{\operatorname{ch}}^{\mathrm{o}}\big(\nabla^F, g_0^F, g_1^F\big)\Big) \\ - \widetilde{\operatorname{ch}}^{\mathrm{o}}\big(\nabla^{\mathcal{H}}, g_{L^2,0}^{\mathcal{H}}, g_{L^2,1}^{\mathcal{H}}\big) \qquad \text{modulo } d\Omega^\bullet(B)\ .\end{aligned}$$

Fortunately, in certain situations, we can get rid of the dependence on the secondary data $T^H M$, g^{TX} and g^F. Let $(\,\cdot\,)^{[>0]}$ denote the sum of the homogeneous parts of a form on B that are of degree > 0.

3.18. Theorem. *Assume that both $(F, \nabla^F) \to M$ and $(\mathcal{H}, \nabla^{\mathcal{H}})$ carry parallel metrics, say g^F and g^H. Then the form*

$$\Big(\mathcal{T}\big(T^H M, g^{TX}, \nabla^F, g^F\big) + \widetilde{\operatorname{ch}}^{\mathrm{o}}\big(\nabla^{\mathcal{H}}, g^{\mathcal{H}}, g_{L^2}^{\mathcal{H}}\big)\Big)^{[>0]}$$

is closed and its cohomology class is independent of g^F and g^H.

3.19. Definition. If $(F, \nabla^F) \to M$ and $(\mathcal{H}, \nabla^{\mathcal{H}})$ carry parallel metrics g^F and g^H, we define the higher analytic torsion of $M \to B$ with coefficients in F as

$$\mathcal{T}(M/B; F) = \Big(\mathcal{T}\big(T^H M, g^{TX}, \nabla^F, g^F\big) + \widetilde{\operatorname{ch}}^{\mathrm{o}}\big(\nabla^{\mathcal{H}}, g^{\mathcal{H}}, g_{L^2}^{\mathcal{H}}\big)\Big)^{[>0]}\ .$$

This is the class that can be compared against any of the topological definitions of torsion. Moreover, it can be used to detect that certain fibre bundles are homeomorphic but not diffeomorphic.

Proof of Theorem 3.18. Let g_0^F and g_1^F be two parallel metrics on F. Then the family $(g_t^F)_{t\in[0,1]}$ with

$$g_t^F = (1-t)\, g_0^F + t\, g_1^F$$

is a family of parallel metrics. Hence, for all t, the connection ∇^F is metric and thus $\omega_t^F = 0$. In particular, modulo exact forms,

$$\widetilde{\operatorname{ch}}^{\mathrm{o}}\big(\nabla^F, g_0^F, g_1^F\big) = \int_0^1 \int_0^1 \left(\frac{s(1-s)}{2\pi i}\right)^{N^B} \operatorname{tr}\Big(\big(g_t^F\big)^{-1} \frac{\partial g_t^F}{\partial t}\, f'(0)\Big)\, dt\, ds \in \Omega^0(B)\ .$$

Now, the theorem follows from (3.17). □

3.e. Small time limits. We sketch the main steps in the proof of Theorem 3.13 as outlined in Section 3.c. We start from equation (3.12).

First, we have to compute the curvature of the connection $\overline{\nabla}^{T^*X,F,t,z}$ on the vector bundle $\Lambda^\bullet T^*M \otimes F \otimes (\mathbb{R} \oplus z\mathbb{R})|_X$, which will appear a few times later on. Recall that

$$\overline{\nabla}_{e_k}^{T^*X,F,t,z} = \nabla_{e_k}^{T^*X,F,u} + \frac{1}{\sqrt{t}} \sum_{\alpha=n+1}^{m} \sum_{l=1}^{n} s_{\alpha k l}\, \varepsilon_\alpha c_l + \frac{1}{2t} \sum_{\alpha,\beta=n+1}^{m} \omega_{\alpha\beta k}\, \varepsilon^\alpha \varepsilon^\beta - \frac{1}{2\sqrt{t}} z \hat{c}_k$$

by (3.8) and (3.11), where $\nabla_{e_k}^{T^*X,F,u}$ is the tensor product connection of the Levi-Civita connection ∇^{T^*X} on $\Lambda^\bullet T^*X$, the trivial connection on $p^*\Lambda^\bullet T^*B$ and the unitary connection $\nabla^F + \frac{\omega^F}{2}$ on F. We compute

$$\left(\overline{\nabla}^{T^*X,F,t,z}\right)^2_{e_k,e_l} = (\nabla^{T^*X})^2_{e_k,e_l} + \left(\nabla^F + \frac{\omega^F}{2}\right)^2 + \frac{1}{\sqrt{t}} \sum_{\alpha=n+1}^{m} \varepsilon^\alpha\, c\Big(\big(\nabla^{TX}_{e_k} S(e_\alpha)\big)e_l\Big) - \frac{2}{t} \sum_{\alpha,\beta=m+1}^{n} \langle S(e_\alpha)e_k, S(e_\beta)e_l\rangle\, \varepsilon^\alpha\, \varepsilon^\beta + \frac{1}{2t} \sum_{\alpha,\beta=m+1}^{n} \langle \nabla^{TX}_{e_k} \Omega(e_\alpha, e_\beta), e_l\rangle\, \varepsilon^\alpha\, \varepsilon^\beta\ . \tag{3.20}$$

The first two terms can also be evaluated more explicitly. Because T^*X is isometric to TX, we have

$$\begin{aligned}(\nabla^{T^*X})^2_{e_k,e_l} &= \sum_{p,q=1}^{n} \langle R^{TX}_{e_k,e_l} e_p, e_q\rangle \varepsilon^q \iota_p \\ &= \frac{1}{4} \sum_{p,q=1}^{n} \langle R^{TX}_{e_k,e_l} e_p, e_q\rangle (\hat{c}_p + c_p)(\hat{c}_q - c_q) \\ &= \frac{1}{4} \sum_{p,q=1}^{n} \langle R^{TX}_{e_k,e_l} e_p, e_q\rangle\, (\hat{c}_p\hat{c}_q - c_p c_q)\end{aligned} \tag{3.21}$$

for symmetry reasons. Also, since both ∇^F and $\nabla^{F,*} = \nabla^F + \omega^F$ are flat, we find that

$$\left(\nabla^F + \frac{\omega^F}{2}\right)^2 = \frac{1}{4}\left(\nabla^F + \nabla^{F,*}\right)^2 = -\frac{1}{4}\left(\nabla^{F,*} - \nabla^F\right)^2 = -\frac{1}{4}\left(\omega^F\right)^2 .$$

Next, we need the following important observation by Bismut ([BGV]). The bundle $TX \to M$ carries a natural metric connection ∇^{TX} extending the Levi-Civita connection on $TX \to X$ that was used so far, depending on g^{TX} and T^HM, given for horizontal lifts $\overline{V}$ of vector fields V on B by

$$\nabla^{TX}_{\overline{V}} = \mathcal{L}_{\overline{V}} + \frac{1}{2}\left(g^{TX}\right)^{-1} \mathcal{L}_{\overline{V}} g^{TX} = \mathcal{L}_{\overline{V}} - \frac{1}{2} S(\overline{V})\ . \tag{3.22}$$

The nice fact about this connection is that its curvature R^{TX} collects some of the nasty terms in (3.20). This should be viewed as a generalisation of a well-known symmetry of the Levi-Civita connection, see e.g. [BGV]. Using the notation μ_t introduced in (3.9), we get

$$\left(\overline{\nabla}^{T^*X,F,t,z}\right)^2_{e_k,e_l} = \frac{1}{4} \sum_{p,q=1}^{n} \langle R^{TX}_{e_p,e_q} e_k, e_l\rangle\, \hat{c}_p\hat{c}_q - \frac{1}{4} \sum_{p,q=1}^{m} \langle R^{TX}_{e_p,e_q} e_k, e_l\rangle\, \mu_t^p \mu_t^q - \frac{1}{4}\left[\omega^F(e_k), \omega^F(e_l)\right] . \tag{3.23}$$

We insert the above into equation (3.12) and obtain

$$\begin{aligned}\widetilde{\mathbb{X}}_t^2 + z\, \widetilde{\mathbb{X}}_t = -t\overline{\Delta}^{T^*X,F,t,z} - \frac{t\kappa}{4} - \frac{t}{8} \sum_{p,q=1}^{m} \langle R^{TX}_{e_p,e_q} e_k, e_l\rangle\, \hat{c}_k\hat{c}_l\, \mu_t^p\mu_t^q \\ - \frac{t}{8}\, \hat{c}_k\hat{c}_l \left[\omega^F(e_j), \omega^F(e_k)\right] - \frac{t}{2} \sum_{i=1}^{m}\sum_{k=1}^{n} \hat{c}_k \mu_t^i \big(\overline{\nabla}^{T^*X,F,u}_{e_k} \omega^F\big)(e_i) \\ - \frac{t}{4} \sum_{i=1}^{m} \omega^F(e_i)^2 + \frac{t}{8} \sum_{i,j=1}^{m} \mu_t^i\mu_t^j \left[\omega^F(e_i), \omega^F(e_j)\right] + \frac{\sqrt{t}\, z}{2} \sum_{i=1}^{m} \mu_t^i\, \omega^F(e_i)\ ,\end{aligned} \tag{3.24}$$

where we have used that

$$\sum_{k,l,p,q=1}^{n} \langle R^{TX}_{e_p,e_q} e_k, e_l\rangle\, \hat c_k \hat c_l \hat c_p \hat c_q = -2\kappa\ .$$

In the next step, we fix $b \in B$ and $q \in M_b = p^{-1}(b)$. We choose geodesic normal coordinates x_1, $\dots$, x_n of $p^{-1}(b)$ around q and trivialize $\Lambda^\bullet T^*M \otimes \Lambda^\bullet T^*M_b \otimes F \otimes (\mathbb{R} + z\mathbb{R})$ by parallel translation along radial geodesics with respect to the connection $\overline{\nabla}^{TX,F,t,z}$.

Let us point out that the coordinate functions give rise to a basis $dx_1, \dots , dx_n$ of T^*X. At q, we fix forms $e^{i_1\dots i_k}|_q \in \Lambda^\bullet T^*_q M$ for $1 \le i_1 < \dots < i_k \le m$, with $e^{i_1\dots i_k}|_q = dx_{i_1} \wedge \dots \wedge dx_{i_k}|_q$ if $1 \le i_1 < \dots < i_k \le n$. These forms are extended $\overline{\nabla}^{TX,F,t,z}$-parallelly along radial geodesics in X. Away from q, we no longer have $dx_i = e^i$.

In these new coordinates, the connection $\overline{\nabla}^{TX,F,t,z}$ itself takes the form

$$\begin{aligned}\overline{\nabla}^{TX,F,t,z}_{e_k} &= \partial_k - \frac{1}{2}\Bigl(\overline{\nabla}^{TX,F,t,z}\Bigr)^2_{e_k,\mathcal{R}} + O\bigl(|\mathcal{R}|^2\bigr)\\ &= \partial_k - \frac{1}{8}\Biggl(\sum_{p,q=1}^{n} \langle R^{TX}_{e_p,e_q} e_k, \mathcal{R}\rangle\, \hat c_p \hat c_q - \sum_{p,q=1}^{m} \langle R^{TX}_{e_p,e_q} e_k, \mathcal{R}\rangle\, \mu_t^p \mu_t^q\\ &\qquad\qquad - \bigl[\omega^F(e_k), \omega^F(\mathcal{R})\bigr]\Biggr) + O\bigl(|\mathcal{R}|^2\bigr)\ ,\end{aligned}$$

where $\mathcal{R} = \sum_i x_i \partial_i$ denotes the radial vector field emanating from the point q. For later applications it is important that the correction term $O(|\mathcal{R}|^2)$ consists of terms with at most two variables of type μ_t^i, $\hat c_j$, or z. This is is due to the following two facts. First, the curvature R^{TX} takes its values in the Lie algebra $\mathfrak{so}(T^*X)$, which can be described by $c_i c_j$ or $\hat c_i \hat c_j$, depending on its action on $\Lambda^\bullet T^*M$. Second, equation (3.20) also involves only products of the type $\hat c_i \hat c_j$ or $\mu_t^i \mu_t^j$.

One also has to be careful with the Clifford multiplications $c(X)$ and $\hat c(X)$. With respect to the bases chosen above, they have to be replaced by

$$c_i \leadsto c_i - \frac{2}{\sqrt t} \sum_{\alpha=m+1}^{n} \varepsilon^\alpha\, S(e_\alpha)(\mathcal{R}, e_i) \qquad \text{and} \qquad \hat c_i \leadsto \hat c_i - \frac{1}{\sqrt t}\, z\, \langle \mathcal{R}, e_i\rangle\ .$$

Since the correction term is always of the order $O(|\mathcal{R}|)$, these terms will not do any harm later on.

Applying the above to (3.24) gives

(3.25)

$$\begin{aligned}\widetilde{\mathbb{X}}_t^2 + z\,\widetilde{\mathbb{X}}_t = t\sum_k \Biggl(\partial_k &- \frac{1}{8}\Biggl(\sum_{i,j=1}^{n} \langle R^{TX}_{e_i,e_j} e_k, \mathcal{R}\rangle\, \hat c_i \hat c_j - \sum_{i,j=1}^{m} \langle R^{TX}_{e_i,e_j} e_k, \mathcal{R}\rangle\, \mu_t^p \mu_t^q\\ &- \bigl[\omega^F(e_k), \omega^F(\mathcal{R})\bigr]\Biggr)\Biggr)^2 - \frac{t\kappa}{4} - \frac{t}{8} \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \langle R^{TX}_{e_i,e_j} e_k, e_l\rangle\, \hat c_k \hat c_l\, \mu_t^i \mu_t^j\\ - \frac{t}{8}\, \hat c_k \hat c_l \bigl[\omega^F(e_j), \omega^F(e_k)\bigr] &- \frac{t}{2} \sum_{i=1}^{m} \sum_{k=1}^{n} \hat c_k \mu_t^i \bigl(\overline{\nabla}^{T^*X,F,u}_{e_k} \omega^F\bigr)(e_i) - \frac{t}{4} \sum_{i=1}^{m} \omega^F(e_i)^2\\ + \frac{t}{8} \sum_{i,j=1}^{m} \mu_t^i \mu_t^j \bigl[\omega^F(e_i), \omega^F(e_j)\bigr] &+ \frac{\sqrt t\, z}{2} \sum_{i=1}^{m} \mu_t^i\, \omega^F(e_i) + O(|\mathcal{R}|)\end{aligned}$$

in our new coordinates. Once more, each term in the correction $O(|\mathcal{R}|)$ contains at most two of the variables μ_t^i. We find that in each such term, the power of $\sqrt{t}$ is not smaller than the number of variables of type μ_t^i.

It is a classical result that the Schwarz kernels $k_t(x,y)$ of heat operators like $e^{\widetilde{\mathbb{X}}_t^2+z\widetilde{\mathbb{X}}_t}$ have asymptotic expansion near the diagonal in X as $t\to 0$ with leading term of the order $t^{-\frac{n}{2}}$, whose coefficients can be constructed locally from the coefficients of the corresponding Laplace-type operator $\widetilde{\mathbb{X}}_t^2+z\widetilde{\mathbb{X}}_t$, see [BGV]. We describe the heat kernel

$$k_t(p,q)\colon \big(\Lambda^\bullet T^*M\otimes F\otimes(\mathbb{R}\oplus x\mathbb{R})\big)_q \longrightarrow \big(\Lambda^\bullet T^*M\otimes F\otimes(\mathbb{R}\oplus x\mathbb{R})\big)_p$$

near the diagonal $\Delta X=\{\,(q,q)\mid q\in M\,\}\subset M\times_B M$ by a sum of monomials of the form

$$z^\varepsilon\,\hat c_{j_1}\cdots\hat c_{j_l}\,\mu_t^{i_1}\cdots\mu_t^{i_k}\,A_{i_1\dots i_k,j_1\dots j_k}$$

with $1\le i_1,\dots<i_k\le m$, $1\le j_1<\dots<j_l\le n$, $\varepsilon\in\{0,1\}$ and $A_{i_1\dots i_k,j_1\dots j_k,\varepsilon}\in\operatorname{End}F$. The only monomials contributing to the z-supertrace are those containing $c_1\cdots c_n\,\hat c_1\cdots\hat c_n\,z$, and for $n+1\le\alpha_1<\dots<\alpha_k\le m$ and $A\in\operatorname{End}F$, we have

$$\operatorname{str}_z\big(z\,\hat c_1\cdots\hat c_n\,c_1\cdots c_n\,e^{\alpha_1}\cdots e^{\alpha_k}\,A\big)=(-1)^{\left[\frac{n}{2}\right]}\,2^n\,e^{\alpha_1}\cdots e^{\alpha_k}\,\operatorname{tr}_F(A)\;. \tag{3.26}$$

For this reason, no information is lost if we do the following Getzler rescaling. We represent c_i by $\boldsymbol{\epsilon}^i-\boldsymbol{\iota}_i$, where the exterior variables $\boldsymbol{\epsilon}^i$ are purely formal and have nothing to do with the original exterior variables that still appear in $\hat c_j=\varepsilon^j+\iota_j$. We then look for the term containing all variables $\boldsymbol{\epsilon}^1,\dots,\boldsymbol{\epsilon}^n$, $\hat c_1,\dots,\hat c_n$ and z. If we replace

$$t\rightsquigarrow rt\;,\quad x_i\rightsquigarrow\sqrt{r}\,x_i\;,\quad \partial_i\rightsquigarrow\frac{1}{\sqrt{r}}\,\partial_i\;,$$
$$c_i\rightsquigarrow\frac{1}{\sqrt{r}}\,\boldsymbol{\epsilon}^i-\sqrt{r}\,\boldsymbol{\iota}_i\;,\quad \varepsilon^\alpha\rightsquigarrow\boldsymbol{\epsilon}^\alpha\;,\quad\text{and}\quad \hat c_j\rightsquigarrow\hat c_j\;,$$

the supertrace changes by a factor of $r^{-\frac{n}{2}}$, which cancels with the $(rt)^{-\frac{n}{2}}$ in front of the leading term. This way, the supertrace, which would have been a term of high order in t in the asymptotic development of the original heat kernel, becomes a term of order r^0 in r and thus easy to compute. Moreover, due to the rescaling of the coordinates x_i, we automatically localise near our base point q.

Let us put

$$\boldsymbol{\epsilon}_t^p=\begin{cases}\boldsymbol{\epsilon}^p & \text{for } 1\le p\le n\text{, and}\\ \frac{2}{\sqrt{t}}\,\boldsymbol{\epsilon}^p & \text{for } n<p\le m.\end{cases} \tag{3.27}$$

We apply the Getzler rescaling above to (3.25) and obtain

$$\begin{aligned}\widetilde{\mathbb{X}}_t^2+z\,\widetilde{\mathbb{X}}_t\rightsquigarrow t\sum_k\Bigg(\partial_k+\frac18\sum_{p,q=1}^m\langle R^{TX}_{e_i,e_j}e_k,\mathcal{R}\rangle\,\boldsymbol{\epsilon}_t^i\boldsymbol{\epsilon}_t^j\Bigg)^2-\frac t8\sum_{i,j=1}^m\sum_{k,l=1}^n\langle R^{TX}_{e_i,e_j}e_k,e_l\rangle\,\hat c_k\hat c_l\,\boldsymbol{\epsilon}_t^i\boldsymbol{\epsilon}_t^j\\ +\frac t8\sum_{i,j=1}^m\boldsymbol{\epsilon}_t^i\boldsymbol{\epsilon}_t^j\,\big[\omega^F(e_i),\omega^F(e_j)\big]+\frac{\sqrt{t}\,z}{2}\sum_{i=1}^m\boldsymbol{\epsilon}_t^i\,\omega^F(e_i)+O\big(\sqrt{r}\big)\;.\end{aligned} \tag{3.28}$$

Here we have used that each term of the correction $O(|\mathcal{R}|)$ in (3.25) involves a power of $\sqrt{t}$ at least as large as the number of variables of type μ_t^i, so the $O(|\mathcal{R}|)$ of (3.25) is completely accounted for by the $O(\sqrt{r})$ above. The limit operator H_t as $r \to 0$ decomposes into several commuting operators, namely

$$-t\Delta + \frac{t}{64} \sum_{i,j,k,l=1}^{m} \big\langle R^{TX}_{e_i,e_j}\mathcal{R}, R^{TX}_{e_k,e_l}\mathcal{R}\big\rangle\, \boldsymbol{\epsilon}^i_t\boldsymbol{\epsilon}^j_t\,\boldsymbol{\epsilon}^k_t\boldsymbol{\epsilon}^l_t\;,$$

$$\frac{t}{4} \sum_{i,j=1}^{m}\sum_{k=1}^{n} \langle R^{TX}_{e_i,e_j}e_k, \mathcal{R}\rangle\, \boldsymbol{\epsilon}^i_t\boldsymbol{\epsilon}^j_t\,\partial_k\;,$$

$$-\frac{t}{8}\sum_{i,j=1}^{m}\sum_{k,l=1}^{n} \langle R^{TX}_{e_i,e_j}e_k, e_l\rangle\, \hat c_k\hat c_l\,\boldsymbol{\epsilon}^i_t\boldsymbol{\epsilon}^j_t + \frac{t}{8}\sum_{i,j=1}^{m} \boldsymbol{\epsilon}^i_t\boldsymbol{\epsilon}^j_t\,\big[\omega^F(e_i),\omega^F(e_j)\big] + \frac{\sqrt{t}\,z}{2}\sum_{i=1}^{m}\boldsymbol{\epsilon}^i_t\,\omega^F(e_i)\;.$$

If we replace $\frac14 \sum_{i,j} \boldsymbol{\epsilon}^i_t\boldsymbol{\epsilon}^j_t\,R^{TX}_{e_i,e_j}$ by a skew adjoint endomorphism A of $TX \cong \mathbb{R}^n$, then the first operator becomes a quantised harmonic oscillator $t(\Delta - \|A\mathcal{R}\|^2/4)$, the second an infinitesimal rotation $-t\partial_{A\mathcal{R}}$, and the third an endomorphism of $\Lambda^\bullet T^*X \otimes F$.

By Mehler's formula, the operator $e^{-t(\Delta+\|A\mathcal{R}\|^2/4)}$ has a Schwarz kernel of the form

$$p_t(x,y) = (4\pi t)^{-\frac n2}\,\det\Big(\frac{tA}{\sinh(tA)}\Big)^{\frac12}\, e^{-\frac{1}{4t}\left(\langle x, tA\coth(tA)\,x\rangle - 2\langle x, tA\sinh(tA)^{-1}\,y\rangle + \langle y, tA\coth(tA)\,y\rangle\right)}\;,$$

see [BGV]. The operator $e^{-t\partial_{A\mathcal{R}}}$ is a rotation around 0, which has no effect if we evaluate at $x = y = 0$. Exponentiating the third operator gives the endomorphism

$$e^{-\frac t2\sum_{k,l}\langle Ae_k, e_l\rangle\,\hat c_k\hat c_l}\,\Big(1 + \frac{\sqrt t}{2}\,z\,\boldsymbol{\omega}^F_t\Big)\,e^{\frac t4\,(\boldsymbol{\omega}^F_t)^2}\;,$$

where we write $\boldsymbol{\omega}^F_t$ for $\sum_i \boldsymbol{\epsilon}^i_t\,\omega^F(e_i)$.

To compute supertraces, we have to extract those terms that are saturated in the variables c_i, $\hat c_j$ and z. If we assume that our basis is adapted to the skew adjoint matrix A, so $Ae_{2i-1} = a_ie_{2i}$, we find that

$$e^{-\frac t2\sum_{k,l}\langle Ae_k, e_l\rangle\,\hat c_k\hat c_l} = \prod_{i=1}^{[n/2]}\big(\cos(ta_i) - \sin(ta_i)\,\hat c_{2i-1}\hat c_{2i}\big)\;,$$

so the relevant term is

$$\prod_{i=1}^{[n/2]}\big(-\sin(ta_i)\big)\,\hat c_1\cdots\hat c_n = \det\big(\sinh(tA)\big)^{\frac12}\,\hat c_1\cdots\hat c_n$$

if n is even and 0 otherwise. Thus if n is even, the relevant term in the heat kernel of e^{H_t} along the diagonal ΔX is simply

$$\begin{aligned}(4\pi t)^{-\frac n2}\,\det\Big(\frac{tA}{\sinh(tA)}\Big)^{\frac12}\,\det\big(\sinh(tA)\big)^{\frac12}\,\hat c_1\cdots\hat c_n\,\frac{\sqrt t}{2}\,z\,\boldsymbol{\omega}^F_t\,e^{\frac t4\,(\boldsymbol{\omega}^F_t)^2}\\ = (4\pi t)^{-\frac n2}\,\det(tA)^{\frac12}\,\hat c_1\cdots\hat c_n\,\frac{\sqrt t}{2}\,z\,\boldsymbol{\omega}^F_t\,e^{\frac t4\,(\boldsymbol{\omega}^F_t)^2}\;.\end{aligned}$$

This expression is independent of the choice of coordinates and remains valid if we replace A again by $\frac{1}{4}\sum_{i,j}\boldsymbol{\epsilon}_t^i\boldsymbol{\epsilon}_t^j\,R^{TX}_{e_i,e_j}$. The supertrace is given by the terms that are also saturated in the variables $\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_n$ that have taken the role of $c_1, \dots, c_n$. It follows that the supertrace is independent of t, and we put $t = 4$ which fits well with our definition of $\boldsymbol{\epsilon}_t^i$ in (3.27). Then we have $\boldsymbol{\epsilon}_4^i = \varepsilon^i$, $\boldsymbol{\omega}_4^F = \omega^F$ and

$$\sum_{i,j}\boldsymbol{\epsilon}_4^i\boldsymbol{\epsilon}_4^j\,R^{TX}_{e_i,e_j} = 2R^{TX}_{\cdot,\cdot}\ .$$

After all that has been said so far, we find that

$$\begin{aligned}\operatorname{str}_z\big(e^{H_4}\big) &= \int_{M/B} 2^n\,(-1)^{-\frac{n}{2}}\,(16\pi)^{-\frac{n}{2}}\,\det\big(2R^{TX}\big)^{\frac{1}{2}}\,\operatorname{tr}_F\big(\omega^F\,e^{(\omega^F)^2}\big)\\ &= \int_{M/B} e\big(TX,\nabla^{TX}\big)\,\operatorname{tr}_F\big(\omega^F\,e^{(\omega^F)^2}\big)\end{aligned}$$

if n is even, and $\operatorname{str}_z(e^{H_4}) = 0$ otherwise. The factor $2^n\,(-1)^{-\frac{n}{2}}$ appears because of (3.26), and integration over the fibre makes sure that only terms containing all of the variables $\varepsilon^1, \dots, \varepsilon^n$ enter. We have just proved Theorem 3.13 (1).

The proof of Theorem 3.13 (2) is similar. Because we do not have to consider the formal variable z, only even powers of ω^F appear, but since $\operatorname{tr}_F((\omega^F)^{2k}) = 0$ for $k > 0$, the bundle F enters only by its rank. An extra complication is the operator

$$N^X = \frac{1}{2}\bigg(n + \sum_{i=1}^n \hat{c}_i c_i\bigg)\ ,$$

which undergoes the same rescaling as $\widetilde{\mathbb{X}}_t$. Yet another complication is the factor $(1 + 2\widetilde{\mathbb{X}}_t^2)$; the easiest way to deal with it consists in differentiating the kernel of $e^{s\widetilde{\mathbb{X}}_t^2}$ with respect to s, see [BL]. In the formulation of Theorem 3.13 (2) we have also used the Gauß-Bonnet-Chern theorem, by which

$$\int_{M/B} e\big(TX,\nabla^{TX}\big)\,\operatorname{rk}F = \chi(X)\,\operatorname{rk}F = \chi(\mathcal{H})\ .$$

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