VAFA-WITTEN ESTIMATES FOR COMPACT SYMMETRIC SPACES

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Abstract. We give an optimal upper bound for the first eigenvalue of the untwisted Dirac operator on a compact symmetric space G/H with $\operatorname{rk} G - \operatorname{rk} H \leq 1$ with respect to arbitrary Riemannian metrics. We also prove a rigidity statement.

Herzlich gave an optimal upper bound for the lowest eigenvalue of the Dirac operator on spheres with arbitrary Riemannian metrics in [9] using a method developed by Vafa and Witten in [14]. More precisely, he proved that for every metric $\bar{g}$ on S^n that is pointwise larger than the round metric g, the first eigenvalue $\lambda_1(\bar{D}^2)$ of the Dirac operator with respect to $\bar{g}$ is not larger than the first Dirac eigenvalue $\lambda_1(D^2)$ of the round sphere.

Herzlich asked if there are other Riemannian manifolds with optimal Vafa-Witten bounds, in particular if the Fubini-Study metric on $\mathbb{C}P^{2m-1}$ has this property. In the present note we give positive answers to both questions by generalising Herzlich's results to symmetric spaces G/H of compact type, where $\operatorname{rk} G - \operatorname{rk} H \leq 1$. In particular, we improve a recent estimate by Davaux and Min-Oo for complex projective spaces in [4], see Example 6.2 below.

1. **Theorem.** *Let $M = G/H$ be a simply connected symmetric space of compact type with $\operatorname{rk} G - \operatorname{rk} H \leq 1$ and assume that M is G-spin. Let g be a symmetric metric, and let D denote the corresponding Dirac operator on M. If $\bar{g}$ is another metric with $\bar{g} \geq g$ on TM and $\bar{D}$ is the corresponding Dirac operator, then*

$$\lambda_1(\bar{D}^2) \leq \lambda_1(D^2) .$$

In the case of equality, we have $\bar{g} = g$.

For an arbitrary Riemannian metric $\bar{g}$ such that $c^2\bar{g} \geq g$ for some suitable positive constant c^2, the theorem implies

$$\lambda_1(\bar{D}^2) \leq c^2\, \lambda_1(D^2) .$$

We combine the methods of [9] and [4] with related estimates in [7]. In particular, we compare $\bar{D}$ to an operator $\bar{D}_1$ with nonvanishing kernel acting on the same sections as $\bar{D}_0 = \bar{D} \otimes \mathrm{id}_{\mathbb{R}^k}$. We use Parthasarathy's formula to compute $\lambda_1(D^2)$, and we exhibit a similar formula to estimate $\left\|\bar{D}_1 - \bar{D}\right\|$. Both formulas give the same value for $g = \bar{g}$.

To prove that $\bar{D}_1$ has a kernel, we use the invariance of the Fredholm index if $\operatorname{rk} H = \operatorname{rk} G$. If $\operatorname{rk} H = \operatorname{rk} G - 1$ we use the invariance of the mod-2-index as in [7]. Unfortunately, both approaches fail if $\operatorname{rk} G - \operatorname{rk} H \geq 2$. Note that in [9], a spectral flow argument was used instead in the case $\operatorname{rk} H = \operatorname{rk} G - 1$.

In [2], Baum applied the Vafa-Witten approach to Lipschitz maps f of high degree from a closed Riemannian spin manfifold of dimension $2n$ to S^{2n}. We extend her result to Lipschitz maps of high $\hat{A}$-degree from higher dimensional closed Riemannian spin manifolds to S^{2n}. Recall that if N^n and M^m are closed oriented manifolds, $[N]$ is the fundamental class of N

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and ω is a generator of $H^m(M;\mathbb{Z})$, then the $\hat{A}$-degree is defined as

$$\deg_{\hat{A}} f = \big(\hat{A}(TN) \smile f^*\omega\big)[N] \ .$$

If $n = m$, then $\deg_{\hat{A}}$ is the usual degree.

Let D_N denote the untwisted Dirac operator on N, and let $0 \le \lambda_1(D_N^2) \le \lambda_2(D_N^2) \le \dots$ denote the eigenvalues of D_N^2, where each eigenvalue is repeated according to its multiplicity.

2. **Theorem.** *Let N be a closed Riemannian spin manifold and let $k \in \mathbb{N}$. Then there is no Lipschitz map $N \to S^{2m}$ of Lipschitz constant 1 with*

$$\left|\deg_{\hat{A}} f\right| > 2^{m-1}(k-1)$$

unless

$$\lambda_k(D_N^2) \le \lambda_1(D_M) \ .$$

If we replace the target manifold by a symmetric space G/H with $\operatorname{rk} G = \operatorname{rk} H$, then we can prove similar theorems where the degree condition on f is replaced by

$$\langle [N], \hat{A}(TN) \smile f^*\alpha \rangle > C(k-1) \ ,$$

with $\alpha \in H^*(M;\mathbb{Z})$ and $C > 0$ depending only on M. Alternatively, there is no 1-Lipschitz map $N \to M$ with

$$\left|\deg_{\hat{A}} f\right| > C(k-1)$$

for some constant C depending on M, unless $\lambda_k(D_N^2) \le c$ for some c depending only on M, with $c > \lambda_1(D_M)$ in general. This will be discussed in section 7. In the case $\operatorname{rk} G - \operatorname{rk} H = 1$ we need a K-theoretic condition on f instead of the cohomological $\hat{A}$-degree condition. We may thus ask if even-dimensional spheres are the only manifolds that admit optimal Vafa-Witten bounds for Lipschitz spin maps of sufficiently large $\hat{A}$-degree.

1. *Remark.* The actual estimate and the index theoretic considerations involved in the proof of Theorem 1 are very similar to those used for the scalar curvature comparison result in [7]. Nevertheless, we need a stronger metric condition ($\bar{g} \ge g$ on TM, not just on $\Lambda^2 TM$). This is due to the fact that the Vafa-Witten estimate is related to Gromov's K-length, whereas scalar curvature comparison is related to K-area, see [8].

2. *Remark.* Note that Theorem 1 still holds if we replace $\bar{D}$ by $\bar{D}' = \bar{D} + A$, where A is a symmetric endomorphism of the spinor bundle. This is because in the proof, we can then replace $\bar{D}_i$ by $\bar{D}'_i = \bar{D}_i + A \otimes \mathrm{id}$ for $i = 0, 1$. Then $\bar{D}_1$ still has a kernel, and of course $\left\|\bar{D}'_0 - \bar{D}'_1\right\| = \left\|\bar{D}_1 - \bar{D}_0\right\|$.

3. *Remark.* Remark 2 in particular implies that on a symmetric space,

$$\lambda_1(D + A) \le \lambda_1(D)$$

for all symmetric endomorphisms A of the spinor bundle. In Section 8 we will exhibit an isotropy irreducible quotient $G/H = \mathrm{SO}(5)/\mathrm{SO}(3)$ of compact Lie groups with a normal metric, for which the generalised estimate above does not hold. This shows in particular that the methods of the present paper do not readily generalise to normal homogeneous spaces.

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1. The Smallest Dirac Eigenvalue of a Symmetric Metric

Let $M = G/H$ be symmetric quotient of compact Lie groups of equal rank with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. We fix an Ad-invariant metric g on $\mathfrak{g}$ and let $\mathfrak{p} = \mathfrak{h}^\perp$. The tangent bundle of M can be written as

$$TM = G \times_H \mathfrak{p} \ ,$$

where H acts on $\mathfrak{p}$ by the restriction of the adjoint action Ad_G. The scalar product g induces a symmetric Riemannian metric on M that will also be denoted by g. Note that if M is symmetric, then the Levi-Civita connection on TM is precisely the reductive connection on $G \times_H \mathfrak{p}$.

Let Σ be a spinor module for the Clifford algebra of $\mathfrak{p}$. If we assume that M is G-spin, then the H-representation of H on $\mathfrak{p}$ induces an action $\sigma\colon H \to \mathrm{End}\Sigma$. The natural metric on Σ is σ-invariant, so we obtain a G-equivariant metric g on SM. Equipped with this metric and the reductive connection, the G-equivariant vector bundle

$$SM = G \times_H \Sigma \to M$$

can be identified with the spinor bundle of M, and the G-equivariant Dirac operator D acts as an essentially selfadjoint operator on $L^2(M; SM)$.

Let $\hat{G}$ denote the set of equivalence classes of irreducible G-representations. We write $\gamma\colon G \to \mathrm{End}V^\gamma$ for all $\gamma \in \hat{G}$. By Frobenius reciprocity and the Peter-Weyl theorem, the L^2-sections of SM can be decomposed G-equivariantly into a Hilbert sum

$$L^2(M; SM) = \overline{\bigoplus_{\gamma \in \hat{G}} V^\gamma \otimes \mathrm{Hom}_H(V^\gamma, \Sigma)} \ . \tag{1.1}$$

The Dirac operator preserves this decomposition, and we have

$$D|_{V^\gamma \otimes \mathrm{Hom}_H(V^\gamma,\Sigma)} = \mathrm{id}_{V^\gamma} \otimes {}^\gamma D$$

with

$${}^\gamma D = \sum_{i=1}^n \gamma^*_{e_i} \otimes c_i \quad \in \quad \mathrm{End}(\mathrm{Hom}_H(V^\gamma, \Sigma)) \ . \tag{1.2}$$

Here $e_1, \dots, e_n$ is a g-orthonormal base of $\mathfrak{p}$ and c_i denotes Clifford multiplication by e_i.

Let c_G^γ denote the Casimir operator of the G-action γ with respect to the metric g, which acts as a scalar on V^γ. If M is symmetric, then the Casimir operator c_H^σ of σ also acts as a scalar, even though σ is in general not irreducible. By Parthasarathy's formula [12],

$${}^\gamma D^2 = c_G^{\gamma^*} + c_H^\sigma \ .$$

1.3. **Proposition.** *The smallest eigenvalue of D^2 is given by*

$$\lambda_1(D^2) = \min\left\{ c_G^{\gamma^*} + c_H^\sigma \;\middle|\; \gamma \in \hat{G} \text{ with } \mathrm{Hom}_H(V^\gamma, \Sigma) \neq 0 \right\} . \qquad \square$$

A more explicit formula for the first eigenvalue in the case $\mathrm{rk}\, G = \mathrm{rk}\, H$ has recently been given in [11].

2. The Vafa-Witten Estimate

Let $M = G/H$ and g as before and consider an arbitrary Riemannian metric $\bar{g}$ on M with $\bar{g} > g$. The corresponding Dirac operator will be denoted $\bar{D}$. Let $\lambda_1(\bar{D}^2)$ denote the smallest eigenvalue of $\bar{D}^2$. As mentioned in the introduction, we will estimate $\lambda_1(\bar{D}^2)$ by comparing the related operator $\bar{D}_0 = \bar{D} \otimes \mathrm{id}_{\mathbb{C}^N}$ to an operator $\bar{D}_1$ acting on the same space of sections.

Assume that we are given vector bundle $W \subset V = M \times \mathbb{C}^N \to M$ such that the twisted Dirac operator on $SM \otimes W$ has nonvanishing index; these bundles will be constructed in steps below and in sections 4 and 5. Let ∇^0 be the trivial connection on V, and let ∇^1 be another connection for which $W \subset V$ is a parallel subbundle. Let $\bar{D}_0$, $\bar{D}_1$ denote the corresponding twisted Dirac operators on $SM \otimes \mathbb{C}^N$ for the metric $\bar{g}$. Then $\bar{D}_0$ is just the direct sum of N copies of $\bar{D}$, whereas $\bar{D}_1$ has nontrivial kernel. By a Rayleigh quotient argument, thus

$$\lambda_1(\bar{D}^2) = \lambda_1(\bar{D}_0^2) \leq \frac{\left\|(\bar{D}_1 - \bar{D}_0)s\right\|_{L^2}^2}{\|s\|_{L^2}^2} \leq \sup_{p \in M} \left\|(\bar{D}_1 - \bar{D}_0)_p^2\right\|_{\mathrm{op}} \tag{2.1}$$

for some $0 \neq s \in \ker(\bar{D}_1)$. Here $\|\cdot\|_{L^2}$ denotes the L^2-norm of sections, whereas $\|\cdot\|_{\mathrm{op}}$ denotes the pointwise operator norm of an endomorphism. The operator $\bar{D}_1 - \bar{D}_0$ is of order zero, so $\left\|(\bar{D}_1 - \bar{D}_0)_p^2\right\|_{\mathrm{op}}$ is well-defined and can be estimated pointwise on M.

In our current situation, let $\gamma \in \hat{G}$ be a G-representation such that

$$\mathrm{Hom}_H(V^\gamma, \sigma) \neq 0 \qquad \text{and} \qquad \lambda_1(D^2) = c_G^{\gamma^*} + c_H^\sigma\ , \tag{2.2}$$

cf. Proposition 1.3. Let γ^* denote the dual representation on $V^{\gamma^*} = (V^\gamma)^*$ and let

$$V = G \times_H (V^{\gamma^*}|_H)$$

denote the corresponding vector bundle over G, then V is trivialized by the map

$$V \to M \times V^{\gamma^*} \qquad \text{with} \qquad [g, v] \mapsto (gH, \gamma_g^* v)\ .$$

To a map $v\colon M \to V^\gamma$ corresponds the section

$$gH \mapsto [g, (\gamma^*)_g^{-1} v(g)] \in V_{gH}\ .$$

Now let ∇^0 denote the trivial connection and let ∇^1 denote the reductive connection on V. Then $\overline{D}_0$, $\overline{D}_1$ are the corresponding Dirac operators on $SM \otimes V$.

If we apply the reductive connection ∇^1 on V to a section that is constant in the given trivialisation, then

$$\nabla^1_{[g,X]} v = \left[g, \gamma_X^* (\gamma^*)_g^{-1} v\right] = (gH, \gamma_{*\,\mathrm{Ad}_g X} v)$$

where $X \in \mathfrak{p} = \mathfrak{h}^\perp$ and

$$[g, X] = \frac{\partial}{\partial t}\left(g e^{-tX} H\right) \in T_{gH} M\ .$$

Thus we can identify $T_p M$ with $\mathfrak{p}$ and choose a g-orthonormal frame $f_1, \ldots, f_m$ of $\mathfrak{h}$. We recall that the action σ_* of $\mathfrak{h}$ on SM can be described in terms of Clifford multiplication and Lie brackets by

$$\sigma_{*f_k} = \frac{1}{4} \sum_{i,j=1}^n \langle [e_i, e_j], f_k \rangle\, c_i c_j\ .$$

Let $\bar{e}_1, \ldots, \bar{e}_n$ be a $\bar{g}$-orthonormal base of $T_p M$. We may assume that there exist $0 < \mu_1$, $\ldots$, $\mu_n \leq 1$ such that $\bar{e}_i = \mu_i e_i$ for all $i \in \{1, \ldots, n\}$, where $e_1, \ldots, e_n$ is an orthonormal base with respect to g. We have to compute the operator norm of the operator

$$C = \bar{D}_1 - \bar{D}_0 = \sum_{i=1}^n c_i \gamma_{\bar{e}_i}^* = \sum_{i=1}^n \mu_i c_i \gamma_{e_i}^*\ ,$$

which is the difference of two Dirac operators on $SM \otimes V$ with respect to the metric $\bar{g}$. We note that this formula looks similar to (1.2) above. This is a special feature of symmetric spaces, which does not even generalise to normal homogeneous spaces, see Section 8 below.

Because C is selfadjoint, we have $\|Cv\|^2 = \langle C^2 v, v\rangle$, and it suffices to estimate the eigenvalues of C^2. We follow the proof of Parthasarathy's formula [12]. Using $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$ we compute

$$\begin{aligned}
C^2 &= -\sum_{i=1}^n \mu_i^2 (\gamma^*_{e_i})^2 + \frac{1}{2}\sum_{i,j=1}^n \mu_i\mu_j\, c_i c_j\, \gamma^*_{[e_i,e_j]} \\
&= -\sum_{i=1}^n \mu_i^2 (\gamma^*_{e_i})^2 + \sum_{k=1}^m \Bigl(\gamma^*_{f_k} + \frac{1}{4}\sum_{i,j=1}^n \mu_i\mu_j\, \langle [e_i,e_j], f_k\rangle\, c_i c_j\Bigr)^2 \\
&\qquad - \sum_{k=1}^m (\gamma^*_{f_k})^2 - \frac{1}{16}\sum_{k=1}^m \Bigl(\sum_{i,j} \mu_i\mu_j\, \langle [e_i,e_j], f_k\rangle\, c_i c_j\Bigr)^2 \\
&\le c_G^{\gamma^*} - \frac{1}{16}\sum_{i,j,k,l=1}^n \mu_i\mu_j\mu_k\mu_l\, \langle [e_i,e_j],[e_k,e_l]\rangle\, c_i c_j c_k c_l\ .
\end{aligned}$$

A term similar to the last one on the right hand side has already been estimated in [7], equation (1.11). Because $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$, we have

$$\begin{aligned}
-\frac{1}{16}\sum_{i,j,k,l=1}^n \mu_i\mu_j\mu_k\mu_l\, \langle [e_i,e_j],[e_k,e_l]\rangle\, c_i c_j c_k c_l &= \frac{1}{8}\sum_{i,j=1}^n \mu_i^2\mu_j^2\ \|[e_i,e_j]\|^2 \\
&\le \frac{1}{8}\sum_{i,j=1}^n \|[e_i,e_j]\|^2 = -\sum_{k=1}^m \Bigl(\sum_{i,j=1}^n \langle [e_i,e_j], f_k\rangle\, c_i c_j\Bigr)^2 = c_H^\sigma\ . \qquad (2.3)
\end{aligned}$$

Combining the calculations above, we have the estimate

$$C^2 = (\bar D_1 - \bar D_0)^2 \le c_G^{\gamma^*} + c_H^\sigma = \lambda_1(D^2)\ . \qquad (2.4)$$

3. Rigidity

We will now give a short of proof due to M. Listing that the equality $\lambda_1(\bar D^2) = \lambda_1(D^2)$ in Theorem 1 implies that $\bar g = g$.

Let $0 \ne s \in \ker(\bar D_1)$ as in (2.1). If $\lambda_1(\bar D^2) = \lambda_1(D^2)$, we conclude that

$$\bigl\|(\bar D_1 - \bar D_0)s\bigr\|_{L^2}^2 = \lambda_1(D^2)\ \|s\|_{L^2}^2\ .$$

by (2.1) and (2.4). This implies in particular that $\bigl\|(\bar D_1 - \bar D_0)_p^2\bigr\|_{\mathrm{op}} = \lambda_1(D^2)$ holds for all $p \in \operatorname{supp}(s)$. By [1], we know that s is nonzero on a dense open subset of M, so we have

$$\bigl\|(\bar D_1 - \bar D_0)_p^2\bigr\|_{\mathrm{op}} = \lambda_1(D^2) \qquad (3.1)$$

for all $p \in M$, and we must have equality in (2.3).

Since M is of compact type, $\mathfrak{g}$ has trivial center. In particular, for each $1 \le i \le n$ there exists $1 \le j \le n$ such that $[e_i,e_j] \ne 0$. Thus (3.1) implies by (2.3) that $\mu_1 = \dots = \mu_n = 1$. The rigidity statement follows once we have shown that $\ker(\bar D_1) \ne 0$.

4. The Equal Rank Case

We prove Theorem 1 for $\operatorname{rk} G = \operatorname{rk} H$. It remains to show that the operator $\bar D_1$ has a kernel.

4.1. **Proposition** ([12], [5]). *Let $M = G/K$ be a symmetric space with $\operatorname{rk} G = \operatorname{rk} K + k$. Then the complex spinor bundle Σ is locally induced by a representation σ of the Lie algebra $\mathfrak{k}$ of K, which splits as*

$$\sigma = 2^{\left[\frac{k}{2}\right]} \bigoplus_{i=1}^{q} \sigma_i \ ,$$

where $\sigma_1, \ldots, \sigma_q$ are certain pairwise non-isomorphic irreducible complex representations of $\mathfrak{k}$.

By our choice of γ in (2.2), we have $\operatorname{Hom}_H(V^\gamma, \Sigma) \neq 0$. By Schur's lemma, the H-representations $\gamma|_H$ and σ contain a common irreducible subrepresentation, say σ_1 acting on Σ_1. Consider the vector bundle

$$W = G \times_H \Sigma_1^* \ ,$$

then W is a parallel subbundle of V with respect to the reductive connection ∇^1. Let D_1 be the Dirac operator acting on $SM \otimes W$.

As above, we now have

$$L^2(M; SM \otimes W) = \bigoplus_{\gamma \in \hat{G}} V^\gamma \otimes \operatorname{Hom}_H(V^\gamma, \Sigma \otimes \Sigma_1^*) \ .$$

Again by Parthasarathy's formula ([12], [5]), the operator D_1^2 acts on $V^\gamma, \Sigma \otimes \Sigma_1$ as

$$^\gamma D_1^2 = c_G^\gamma + c_H^\sigma - c_H^\sigma = c_G^\gamma \ .$$

Because $c_G^\gamma = 0$ iff γ is the trivial representation, the kernel of D_1^2 is precisely

$$\ker D_1 = \ker D_1^2 = \operatorname{Hom}_H(\mathbb{C}, \Sigma \otimes \Sigma_1^*) \cong \operatorname{Hom}_H(\Sigma_1, \Sigma) = \operatorname{Hom}_H(\Sigma_1, \Sigma_1)$$

by Schur's Lemma. In particular $\dim \ker D_1 = 1$, whence $\operatorname{ind} D_1 = \pm 1 \neq 0$.

To complete the proof of Theorem 1 in this case, we note that the Dirac operator on $\overline{SM} \otimes W$ has nonzero index, and hence $\bar{D}_1$ has a kernel. The claim now follows from (2.1) and (2.4).

5. Mod-2-Indices and the Case $\operatorname{rk} H = \operatorname{rk} G - 1$

We now prove Theorem 1 for $\operatorname{rk} G = \operatorname{rk} H + 1$. We proceed similar as in [7], section 2.c.

Assume first that $n = \dim M \equiv 1 \bmod 8$. In this case, there exists a real vector bundle $S_\mathbb{R} M$ such that $SM = S_\mathbb{R} M \otimes_\mathbb{R} \mathbb{C}$, and the even part of the real Clifford algebra still acts on $S_\mathbb{R} M$. The bundle $S_\mathbb{R} M$ is induced by real representation $\sigma_\mathbb{R}$ of H. Let $\sigma_{\mathbb{R},i}$ denote its irreducible components, such that $\sigma_i = \sigma_{\mathbb{R},i} \otimes_\mathbb{R} \mathbb{C}$, and let $W_\mathbb{R} = G \times_H \Sigma_{\mathbb{R},1}^*$.

Let

$$\omega_\mathbb{R} = c_1 \cdots c_n$$

denote the real Clifford volume element, then $\omega_\mathbb{R}$ is parallel, anti-selfadjoint, and commutes with $D\omega_\mathbb{R}^* = \omega_\mathbb{R}$. The operator

$$D_{\mathbb{R},1} = \omega_\mathbb{R} \, D_1$$

is anti-selfadjoint and has coefficients in the even part of the real Clifford algebra, so it acts on $S_\mathbb{R} M$. By the same reasoning as above,

$$\ker(\omega_\mathbb{R} \, D_{\mathbb{R},1}) = \operatorname{Hom}_{\mathbb{R},H}(\mathbb{C}, \Sigma_{\mathbb{R},1} \otimes_\mathbb{R} \Sigma_{\mathbb{R},1}^*) = \operatorname{Hom}_{\mathbb{R},H}(\Sigma_{\mathbb{R},1}, \Sigma_{\mathbb{R},1})$$

is one-dimensional.

With respect to the metric $\bar{g}$, we similarly construct the operator $\bar{D}_{\mathbb{R},1} = \bar{\omega}_\mathbb{R} \, \bar{D}_1$ acting on $\bar{S}_\mathbb{R} M \otimes_\mathbb{R} W_\mathbb{R}$. Its restriction to $\overline{S}_\mathbb{R} M \otimes_\mathbb{R} V_\mathbb{R}$ can be deformed into $D_{\mathbb{R},1}$ through a family of elliptic, formally anti-selfadjoint operators. Since the parity of the dimension of the kernel is preserved by such a deformation (see [10]), the operator $\bar{\omega}_\mathbb{R} \, \bar{D}_{\mathbb{R},1}$ has nontrivial kernel. We regard $V \to M$ as a real vector bundle by forgetting the complex structure, then $W_{\mathbb{R}}$ is again a parallel subbundle of V with respect to ∇^1, so $\bar{D}_1$ has a nontrivial kernel, too. Once again, our theorem follows from (2.1) and (2.4).

Finally, if $\operatorname{rk} G - \operatorname{rk} H = 1$ and $\dim M \not\equiv 1 \bmod 8$, we consider $N = M \times S^{2m}$ with $\dim N \equiv 1 \bmod 8$. We fix a round metric g_0 on S^{2m} and regard N with the metrics $g \oplus g_0$ and $\bar{g} \oplus g_0$. Then our main theorem holds for N, and the result for M follows because

$$\lambda_1(\bar{D}_M^2) = \lambda_1(\bar{D}_N^2) - \lambda_1(D_{S^{2m}}^2) \le \lambda_1(D_N^2) - \lambda_1(D_{S^{2m}}^2) = \lambda_1(D_N^2) . \qquad \square$$

6. Examples

We consider spheres and projective spaces.

6.1. *Example.* For even-dimensional spheres $S^{2m} = \mathrm{Spin}(2m+1)/\mathrm{Spin}(2m)$, the restriction of the 2^m-dimensional spinor representation σ of $\mathrm{Spin}(2m+1)$ to H splits into two irreducible representations

$$\sigma|_{\mathrm{Spin}(2m)} = \sigma^+ \oplus \sigma^- .$$

In [9], the bundles

$$V = SM = G \times_H \Sigma \quad \supset \quad W = S^+M = G \times_H \Sigma^+$$

were used to prove the optimality of the Vafa-Witten estimate. Our method is a direct generalisation of this approach to other symmetric spaces.

6.2. *Example.* The complex projective space $\mathbb{C}P^n$ is spin iff n is odd, so we regard the odd-dimensional projective space $M = \mathbb{C}P^{2m-1} = G/H$ with $G = \mathrm{SU}(2m)$ and $H = \mathrm{U}(2m-1)$. The Dirac spectrum has been computed in [3] and [13]. To summarize, the spinor bundle splits as

$$SM = \bigoplus_{q=0}^{2m-1} \Lambda^{0,q}T^*M \otimes \tau^m ,$$

where $\Lambda^{0,q}T^*M$ denotes the bundle of anti-holomorphic differential forms of degree q, and τ denotes the tautological bundle. Note that τ^m is a square root of the canonical bundle.

The smallest eigenvalue of D^2 on $\Lambda^{0,q}T^*M \otimes \tau^m$ with respect to the Fubini-Study metric is given by

$$\lambda_1\big(D^2|_{\Lambda^{0,q}T^*M\otimes\tau^m}\big) = \begin{cases} 8m^2 - 4m(q+1) & \text{for } q < m, \text{ and} \\ 8m^2 - 4m(2m-q) & \text{for } q \ge m . \end{cases} \tag{6.3}$$

Thus the lowest eigenvalue is attained in the middle degrees $q = m-1$, m, and is given by $4m^2 = (n+1)^2$. In contrast, Davaux and Min-Oo used the bundle τ^m for $q = 0$ in [4], which explains their larger upper bound $8m^2 - 4m = 2n(n+1)$.

Let us give a geometric description of the bundles $\Lambda^{0,q}T^*M \otimes \tau^m$ for $q = m-1$, m. We have

$$T\mathbb{C}P^{2m-1} = \big\{\, [x,v] \in S^{2m-1} \otimes \mathbb{C}^{2m} \;\big|\; x \ne 0, x \perp v \,\big\}$$
$$\text{with} \qquad [z,v] = \big\{\, (zx, zv) \;\big|\; z \in S^1 \,\big\} .$$

For $\alpha \in \Lambda^{0,m}T^*M$, we note that

$$\alpha(zx, zv_1, \dots, zv_{m-1}) = \bar{z}^m\, \alpha(x, v_1, \dots, v_{m-1})$$
$$\text{and} \qquad \alpha(zv_1, \dots, zv_m) = \bar{z}^m\, \alpha(v_1, \dots, v_m) .$$

Thus inserting representatives of tangent vectors into m-forms gives a natural trivialisation

$$\Phi = \Phi^{m-1} \oplus \Phi^m \colon \mathbb{C}P^{2m-1} \times \Lambda^{0,m}\mathbb{C}^{2m} \longrightarrow (\Lambda^{0,m-1}T^*M \oplus \Lambda^{0,m}T^*M) \otimes \tau^m .$$

7. Lipschitz maps of High $\hat{A}$-Degree

We give a proof of Theorem 2. Let N be a closed Riemannian spin manifolds and $f\colon N \to M$ a Lipschitz map of Lipschitz constant 1. Then f can be approximated by smooth $(1+\varepsilon)$-Lipschitz maps for all $\varepsilon > 0$. We may therefore assume that f is smooth.

As in [2], [9], consider the spinor bundles $\Sigma^\pm \to S^{2n}$ equipped with the Dirac connection ∇^1 with respect to the round metric g. Let ∇^0 denote the trivial connection on $\Sigma^+ \oplus \Sigma^- \cong S^{2n} \otimes \Sigma$, where Σ is the spinor module of $\mathbb{R}^{2n+1}$.

The Chern character of the spinor bundles is given by

$$\operatorname{ch}(\Sigma^\pm) = 2^{m-1} \pm \omega \ ,$$

where $\omega \in H^{2m}(S^{2m};\mathbb{Z})$ is a generator. Let $D_{N,1,\pm}$ be the Dirac operator on N twisted by $f^*\Sigma^\pm$, and let $D_{N,1}$ denote their direct sum. By the Atiyah-Singer index theorem, we have

$$\operatorname{ind}(D_{N,1,\pm}) = \big(\hat{A}(TN) \smile f^* \operatorname{ch}(\Sigma^\pm)\big)[N] = 2^{m-1}\hat{A}(TN)[N] \pm \deg_{\hat{A}} f \ .$$

By combining both bundles, it is now easy to see that

$$\dim\ker(D_{N,1}) \geq \max\big(\big|2^m \hat{A}(TN)[N]\big|, 2\left|\deg_{\hat{A}} f\right|\big) > 2^m(k-1) \tag{7.1}$$

if $\left|\deg_{\hat{A}} f\right| > 2^{m-1}(k-1)$.

Let $p \in N$, then there exists an orthonormal frame $\bar{e}_1, \dots, \bar{e}_n$ of T_pN and an orthonormal frame $e_1, \dots, e_{2m}$ of $T_{f(p)}S^{2m}$ such that

$$d_pf(\bar{e}_k) = \begin{cases} \mu_k\, e_k & \text{if } k \leq 2m \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

If f is $(1+\varepsilon)$ Lipschitz, then $\mu_1, \dots, \mu_{2m} \leq 1+\varepsilon$.

Let $D_{N,0}$ denote the Dirac operator on N twisted by $f^*\Sigma$, but with respect to the trivial connection ∇^0. Then

$$D_{N,1} - D_{N,0} = \sum_{i=1}^{2m} \bar{c}_i\, \gamma_{*\mu_i e_i} \ .$$

Now a similar computation as in Section 2 gives

$$\|D_{N,1} - D_{N,0}\|^2 \leq (1+\varepsilon)\,(c^\gamma_G + c^\sigma_H) = (1+\varepsilon)\,\lambda_1(D^2) \ . \tag{7.2}$$

Combining (7.1) and (7.2) as before, we conclude at least $2^m(k-1)$ eigenvalues of $D_{N,0}$ are not larger than $(1+\varepsilon)\,\lambda_1(D^2)$. Because $D_{N,0}$ consists of 2^m copies of the Dirac operator D_N, we conclude that at least k eigenvalues of D_N^2 (counted with multiplicities) are not larger than $(1+\varepsilon)\,\lambda_1(D^2)$. Since we can choose $\varepsilon > 0$ arbitrarily small, Theorem 2 is proved. □

We remarked in the Introduction that Theorem 2 does not hold unchanged for other symmetric spaces. Regard $M = \mathbb{C}P^{2m-1}$ as in Example 6.2. By (6.3), the Vafa-Witten will be optimal only if we choose one of the bundles $W = \Lambda^{0,m-1}T^*M \otimes \tau^m$ or $W^* = \Lambda^{0,m}T^*M \otimes \tau^m$ as twist bundle.

To determine the Chern characters of these bundles, recall that

$$T'\mathbb{C}P^{2m-1} \oplus \mathbb{C} \cong 2m\,\tau^{-1} \ ,$$

and thus in K-theory,

$$[\Lambda^{0,q}T^*M] = \sum_{i=0}^{q} (-1)^{q-i}\,[\Lambda^i(2m\,\tau^{-1})] = \sum_{i=0}^{q} (-1)^{q-i} \binom{2m}{i} [\tau^{-i}] \ .$$

We know that $a = c_1(\tau)$ is a generator of $H^2(\mathbb{C}P^{2m-1})$, and by the above, we find

$$\mathrm{ch}(W) = \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{2m}{i} e^{(m-i)a} .$$

Already for $\mathbb{C}P^3$, the explicit classes are

$$\mathrm{ch}(W) = 3 + 2a - \frac{2}{3} a^3 \qquad \text{and} \qquad \mathrm{ch}(W^*) = 3 - 2a + \frac{2}{3} a^3 .$$

On the other hand, we have seen that

$$W \oplus W^* \cong \mathbb{C}^{\binom{2m}{m}} .$$

Proceeding as above, we can prove the following result.

7.3. **Proposition.** *Let N be a closed Riemannian spin manifold and let $k \in \mathbb{N}$. Then there is no Lipschitz map $N \to \mathbb{C}P^{2m-1}$ of Lipschitz constant 1 with*

$$\left| \hat{A}(TN) \smile f^* \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(e^{(m-i)a} - 1\right) \right| > \binom{2m-1}{m-1} (k-1) ,$$

where a is a generator of $H^2(\mathbb{C}P^{2m-1})$, unless

$$\lambda_k(D_N^2) \leq \lambda_1(D_M^2) . \qquad \square$$

8. A Homogeneous Counterexample

We now explain Remark 3. In general, the spectrum of a Dirac operator on a homogeneous space is hard to compute. On the Berger space $M = \mathrm{SO}(5)/\mathrm{SO}(3)$ however, most of the relevant geometric structure can be described using octonions, see [6]. We will see that a straightforward adaptation of the arguments of Sections 1–5 to the Berger space is not possible.

We embed $H = \mathrm{SO}(3)$ in $G = \mathrm{SO}(5)$ via the irreducible SO(3)-representation of dimension 5. On $\mathfrak{g} = \mathfrak{so}(5)$, we take the scalar product $\langle A, B \rangle = -\frac{1}{2} \operatorname{tr}(AB)$. As above, let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{h} = \mathfrak{so}(3)$. Then $\dim \mathfrak{p} = 7$, and the isotropy action of SO(3) on $\mathfrak{p}$ factors as

$$\mathrm{SO}(3) \longrightarrow \mathrm{G}_2 \longrightarrow \mathrm{Spin}(7) \longrightarrow \mathrm{SO}(7) . \tag{8.1}$$

Let $[\cdot,\cdot]_{\mathfrak{p}}$ denote the projection of the Lie bracket in $\mathfrak{g}$. Let $\mathbb{I} \subset \mathbb{O}$ denote the imaginary octonions, let $*$ denote octonion multiplication, and let $*_{\mathbb{I}}$ denote octonion multiplication followed by the projection onto $\mathbb{I}$.

8.2. **Lemma** ([6]). *With a suitable isometric, G_2-equivariant identification of $\mathfrak{p}$ with the imaginary octonions, one has*

$$[v, w]_{\mathfrak{p}} = \frac{1}{\sqrt{5}} v *_{\mathbb{I}} w \qquad \text{for all} \qquad v, w \in \mathfrak{p} . \qquad \square$$

Let $\Sigma_{\mathbb{R}}$ again denote the real spinor module of $\mathfrak{p}$, then G_2 also acts on $\Sigma_{\mathbb{R}}$ by (8.1).

8.3. **Lemma** ([6]). *We identify $\mathfrak{p} \cong \mathbb{I}$ as in Lemma 8.2. With respect to a suitable isometric, G_2-equivariant identification $\Sigma_{\mathbb{R}} \cong \mathbb{O}$ and a suitable orientation of $\mathfrak{p}$, Clifford multiplication $\mathfrak{p} \times \Sigma_{\mathbb{R}} \to \Sigma_{\mathbb{R}}$ equals Cayley multiplication $\mathbb{I} \times \mathbb{O} \to \mathbb{O}$ from the right.* $\square$

We fix an orthonormal base $e_1, \ldots, e_7$ of $\mathfrak{p} \cong \mathbb{I}$ such that

$$e_i * e_{i+1} = e_{i+3} \;,$$

where indices are taken mod 7. Clifford multiplication with e_i will be abbreviated as c_i. As in [5] and [6], let us introduce the notation

$$c_{ijk} = \langle [e_i, e_j]_{\mathfrak{p}}, e_k \rangle = -\frac{1}{\sqrt{5}} \Re\big((e_i * e_j) * e_k\big)$$

$$\text{and} \qquad \tilde{\mathrm{ad}}_{\mathfrak{p},X} = \frac{1}{4} \sum_{i,j=1}^{7} \langle [X, e_i], e_j \rangle \, c_i c_j \;.$$

Let us define

$$A = \frac{1}{3} \sum_{i=1}^{7} c_i \tilde{\mathrm{ad}}_{\mathfrak{p},e_i} = \frac{1}{12} \sum_{i,j,k=1}^{7} c_{ijk} \, c_i c_j c_k \;.$$

It is easy to see that A is a symmetric, G_2-invariant endomorphism of $\Sigma_{\mathbb{R}}$. Thus it induces a G-invariant symmetric endomorphism of $S_{\mathbb{R}}M$. One may compute that

$$A = \frac{1}{2\sqrt{5}} \begin{pmatrix} 7 & 0 \\ 0 & -1 \end{pmatrix} : \mathbb{R} \oplus \mathbb{I} \to \mathbb{R} \oplus \mathbb{I} \;.$$

Let D denote the real Dirac operator acting on $\Gamma(S_{\mathbb{R}}M)$. We want to consider the family of operators

$$D^\lambda = D + \left(3\lambda - \frac{3}{2}\right) A \;. \tag{8.4}$$

The operator $D^{\frac{1}{2}}$ is the Riemannian Dirac operator on M, whereas $\tilde{D} = D^{\frac{1}{3}}$ is called the *reductive* or *cubic* Dirac operator.

8.5. **Proposition.** *For λ sufficiently close to $\frac{1}{2}$, the smallest eigenvalue of $(D^\lambda)^2$ is $\frac{441}{20}\,\lambda^2$.*

In particular there exists $\varepsilon > 0$ sufficiently small such that

$$\lambda_1\big((D^{\frac{1}{2}+\varepsilon})^2\big) > \lambda_1(D^2) \;.$$

As stated in Remark 3, this implies that the Vafa-Witten technique cannot be used to prove sharp upper bounds for the first Dirac eigenvalue.

For certain questions, the reductive Dirac operator $\tilde{D}$ on a normal homogeneous space plays the same role as the geometric Dirac operator on a symmetric space, but here clearly $\lambda_1(\tilde{D}^2) < \lambda_1(D^2)$, so with respect to optimal Vafa-Witten estimates, the reductive Dirac operator is even worse than the geometric Dirac operator.

Proof. The space of sections $L^2(M; S_{\mathbb{R}}M)$ can be decomposed as in (1.1). As in [5], we see that D^λ acts on the isotypical components $V^\gamma \otimes \mathrm{Hom}_H(V^\gamma, \Sigma_{\mathbb{R}})$ by $\mathrm{id}_{V^\gamma} \otimes {}^\gamma D^\lambda$, where

$${}^\gamma D^\lambda = \sum_{i=1}^{7} \Big(\gamma^*_{e_i} \otimes c_i + \lambda \, \mathrm{id}_{V^\gamma} \otimes c_i \tilde{\mathrm{ad}}_{\mathfrak{p},e_i}\Big) = \sum_{i=1}^{7} \gamma^*_{e_i} \otimes c_i + \frac{\lambda}{4} \sum_{i,j,k=1}^{7} c_{ijk} \, c_i c_j c_k \;.$$

Let us write $\mu = 3\lambda - 1$, then

$$D^\lambda = \tilde{D} + \mu \, \mathrm{id}_{V^\gamma} \otimes A \;.$$

The square of $\tilde{D}$ has been computed in [5] as

$$({}^\gamma\tilde{D})^2 = \|\gamma + \rho_G\|^2 - \|\rho_H\|^2 \;.$$

If we use integers $p \geq q \geq 0$ to index the irreducible representations of SO(5), then by [6], we have the explicit formula

$$({}^{\gamma_{p,q}}\tilde{D})^2 = \|\gamma_{p,q} + \rho_G\|^2 - \|\rho_H\|^2 = p^2 + 3p + q^2 + q + \frac{49}{20} \quad \text{on } \mathrm{Hom}_H(V^{\gamma_{p,q}}, \Sigma_{\mathbb{R}}) .$$

Let us now analyze all irreducible representations of SO(5). For the trivial representation, clearly

$${}^{\gamma_{0,0}}D^\lambda = 3\lambda\, {}^{\gamma_{0,0}}\tilde{D} = 3\lambda\, A|_{\mathbb{R}} = \frac{21}{2\sqrt{5}}\,\lambda\ , \qquad \text{so} \qquad {}^{\gamma_{0,0}}D = {}^{\gamma_{0,0}}D^{\frac{1}{2}} = \frac{21}{4\sqrt{5}}$$

The standard representation $\gamma_{1,0}$ contains no SO(3)-irreducible subrepresentation isomorphic to $\mathbb{R}$ or $\mathbb{I}$, so it does not contribute to the spectrum of D^λ. The representation $\gamma_{1,1} = \Lambda^2\gamma_{1,0}$ contains no trivial SO(3)-subrepresentation, but an SO(3)-subrepresentation isomorphic to $\mathbb{I}$, and we find that

$$|\lambda_1({}^{\gamma_{1,1}}D)| \geq \left|{}^{\gamma_{1,1}}\tilde{D}\right| - \frac{1}{2}\left|A|_{\mathbb{I}}\right| = \frac{13}{2\sqrt{5}} - \frac{1}{4\sqrt{5}} > \frac{21}{4\sqrt{5}} = \left|{}^{\gamma_{0,0}}D\right| .$$

For all other representations, we have $p > 2$, thus

$$|\lambda_1({}^{\gamma_{p,q}}D)| \geq \left|{}^{\gamma_{2,0}}\tilde{D}\right| - \frac{1}{2}\,|A| = \frac{\sqrt{249}}{2\sqrt{5}} - \frac{7}{4\sqrt{5}} > \frac{21}{4\sqrt{5}} = \left|{}^{\gamma_{0,0}}D\right| .$$

In particular, the smallest eigenvalue of D^λ equals $\frac{21}{2\sqrt{5}}\,\lambda$ for λ sufficiently close to $\frac{1}{2}$, and is attained by a G-invariant spinor. □

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