Preface

What is a mathematical proof? How can proofs be justified? Are there limitations to provability? To what extent can machines carry out mathematical proofs?

Only in the last century has there been success in obtaining substantial and satisfactory answers. The present book contains a systematic discussion of these results. The investigations are centered around first-order logic. Our first goal is Gödel's completeness theorem, which shows that the consequence relation coincides with formal provability: By means of a calculus consisting of simple formal inference rules, one can obtain all consequences of a given axiom system (and in particular, imitate all mathematical proofs).

A short digression into model theory will help to analyze the expressive power of first-order logic, and it will turn out that there are certain deficiencies. For example, first-order logic does not allow the formulation of an adequate axiom system for arithmetic or analysis. On the other hand, this difficulty can be overcome—even in the framework of first-order logic—by developing mathematics in set-theoretic terms. We explain the prerequisites from set theory necessary for this purpose and then treat the subtle relation between logic and set theory in a thorough manner.

Gödel's incompleteness theorems are presented in connection with several related results (such as Trakhtenbrot's theorem) which all exemplify the limitations of machine-oriented proof methods. The notions of computability theory that are relevant to this discussion are given in detail. The concept of computability is made precise by means of the register machine as a computer model.

We use the methods developed in the proof of Gödel's completeness theorem to discuss Herbrand's Theorem. This theorem is the starting point for a detailed description of the theoretical fundamentals of logic programming. The corresponding resolution method is first introduced on the level of propositional logic.

The deficiencies in expressive power of first-order logic are a motivation to look for stronger logical systems. In this context we introduce, among others, second-order logic and the infinitary logics. For each of them we prove that central facts which hold for first-order logic are no longer valid. Finally, this empirical fact is confirmed by Lindström's theorems, which show that there is no logical system that extends first-order logic and at the same time shares all its advantages.

The book does not require special mathematical knowledge; however, it presupposes an acquaintance with mathematical reasoning as acquired, for example, in the first year of a mathematics or computer science curriculum.

For the present third English edition the text has been carefully revised. Moreover, two important decidability results in arithmetic are now included, namely the decidability of Presburger arithmetic and the decidability of the weak monadic theory of the successor function. For the latter one, some facts of automata theory that are usually taught in a computer science curriculum are developed as far as needed.

The authors have done their best to avoid typos and errors, but almost surely the book will still contain some. Please let the authors know of any errors you find. Corresponding corrections will be accessible online via the Springer page of the book.

After the appearance of the first German edition of the book (1978), A. Ferebee saw to the translation for the first English edition (1984), and J. Ward assisted in preparing the final text of that edition. We are grateful to Margit Messmer who translated the materials added in the second edition, and assisted with polishing the English of the new sections in the present edition.

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Part A