Interpretable groups, stably embedded sets, and Vaughtian pairs

Bernhard Herwig,
Institüt für mathematische Logik,
Universität Freiburg,
D-79104 Freiburg,
Germany,
herwig@ruf.uni-freiburg.de

Ehud Hrushovski,
Department of Mathematics,
The Hebrew University,
Jerusalem,
Israel
ehud@math.huji.ac.il

Dugald Macpherson,
Department of Pure Mathematics,
University of Leeds,
Leeds LS2 9JT,
England
h.d.macpherson@leeds.ac.uk

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Abstract

The paper concerns sufficiently saturated structures M over a countable language with a unary predicate P. It is shown that if $P(M)$ is stably embedded and there are no Vaughtian pairs with respect to P, then an infinite group is interpretable over M (in an infinitary sense of 'interpretable'). Also, it is shown that if M is ω-categorical, $f : D \to P$ is a 0-definable map with finite fibres, and $P(M)$ is stably embedded but D is not, then some infinite group is interpretable over M.

1 Introduction

We consider (sufficiently saturated) first order structures M with a unary predicate P. We suppose that the structure is controlled in a rather weak sense by P; namely, that P does not give rise to a Vaughtian Pair, and is stably embedded. The most simple examples of such a situation arise as one of the following.

A) The structure is algebraic over P.
B) P has the structure of a group (say a vector space) and the rest of the structure is an affine space for P.

We will show that in general, either Case A holds or there is an infinite group living in M (in a sense which needs to be made precise). Under additional assumptions this group will live in P.

The proof was first visualised on a visit by Hrushovski to London in 1988. The original assumptions were stronger (relative categoricity – see Section 4) but no stability or simplicity were assumed. The main ingredients – stable embeddedness, co-analysability, and the definition of the group, were as below. Some of the other details were different from those below, or perhaps missing altogether.

Meanwhile, and independently, Shami and Wagner considered related questions in the case of simple theories. Wagner earlier realised the relevance of a theorem due to Schlichting and to Bergman and Lenstra, given below. We follow Shami-Wagner [9] here in quoting this lemma, once we obtain its hypotheses.

In the stable and simple contexts, to find a group one deals with two definable sets D and P, with D internal to P (that is, for some ℓ there is a definable surjection $P^\ell \to D$). In the general situation considered here, a third definable set C arises: the elements of C are parameters defining the function $P^\ell \to D$. To analyse the relation of C and P, one would seem to need a new definable set C', and so on. It would be interesting to pursue this. We analyse the phenomenon in a certain minimal manifestation, and there we show just three sets are needed in order to obtain a group.

Suppose M is an ω_1-saturated model , over a countable language. The 0-definable set $P(M)$ is *stably embedded* in M if for all $n > 0$, every definable subset of P^n is definable with parameters from P. By ω_1-saturation of M, this is equivalent to assuming the uniform version for an arbitrary model N of the theory : namely, that for every formula $\phi(\bar{x}, \bar{y})$ there is a formula $\psi(\bar{x}, \bar{z})$ without parameters such that for every $\bar{c} \in N$ there is $\bar{b} \in N(P)$ such that $N \models \forall \bar{x} \in P(\phi(\bar{x}, \bar{c}) \leftrightarrow \psi(\bar{x}, \bar{b}))$.

Remark 1.1 Assume that M is an uncountable structure over a countable language with a unary predicate P, and is saturated.
(a) The following are equivalent.
(i) P is stably embedded.
(ii) The restriction map $\mathrm{Aut}(M) \to \mathrm{Aut}(P)$ is surjective (in general the image would just be a dense subset of $\mathrm{Aut}(P)$).
(iii) For any tuple $\bar{a} \in M$, $\mathrm{tp}(\bar{a}/P(M))$ is definable.
(b) If P is stably embedded, then any two tuples from M with the same type over $P(M)$ are $\mathrm{Aut}(M/P)$-conjugate.

One source for this is (5.6) of [3]; it was mentioned as an exercise in [7], but similar results were known much earlier (cf [10], [8]). In fact, the equivalence holds under assumptions weaker than saturation. It suffices to ensure M is ω_1-saturated and that for any countable $B \subset M$, and every B-definable subset D of M which is stably embedded over C, the structure M is strongly ω_1-homogeneous over $B \cup D$; the homogeneity here means that if $\bar{a}, \bar{b}$ are tuples from M, $E \subset M$ is countable, and $\mathrm{tp}(\bar{a}/B \cup D \cup E) = \mathrm{tp}(\bar{b}/B \cup D \cup E)$, then there is $\sigma \in \mathrm{Aut}(M)$ fixing $B \cup D \cup E$ pointwise with $\sigma(\bar{a}) = \bar{b}$. It can be checked that just in ZFC, every complete theory over a countable language has a model of size $2^{\aleph_0}$ with this property.

Conventions 1.2 We assume M is a model over a countable language with a unary predicate P, and is ω_1-saturated and has sufficient homogeneity to ensure that the equivalences of Remark 1.1 hold. We often write P for $P(M)$, the set of realisations of P in M. This setting is assumed to hold in Section 2 (from Corollary 2.7 onwards) and in Section 3.

A $\Sigma^1(B)$ (respectively, $\Pi^1(B)$) -definable relation is a union (respectively, intersection) of countably many B-definable relations. A Σ^1 (respectively, Π^1) - definable relation is a $\Sigma^1(B)$-definable relation, for some countable $B \subset M$. In general, 'definable' means 'definable with parameters'.

A Π^1- *interpretable group* is a Π^1-definable set G, a Π^1-definable equivalence relation E on G, and a definable ternary relation R on a definable set containing G, which induces a function $* : (G/E)^2 \to (G/E)$ which is a group law. By *induces*, we mean that for $g_1, g_2, g_3 \in G$, we have $g_1/E * g_2/E = g_3/E$ if and only if there are $h_1, h_2, h_3 \in G$ with $h_i E g_i$ and $R(h_1, h_2, h_3)$.

When D', D'' are subsets of $M^{\rm eq}$, we write $\mathrm{Aut}(D'/D'')$ for the group of permutations of D' which are elementary over D''. This is a closed permutation group, and in general properly contains the group induced on D' by its setwise stabiliser in $\mathrm{Aut}(M/D'')$. If $D \subset M$ and $a \in \mathrm{acl}(D)$, we write $\mathrm{Mult}(a/D)$ for the number of realisations of $\mathrm{tp}(a/D)$. We sometimes use the notation AB for $A \cup B$, and Ac for $A \cup \{c\}$, where $A, B, \{c\}$ are sets of parameters.

We shall at various times be assuming some of the following hypotheses. Note that H2+ strengthens H2, and H2* strengthens H2+.

Hypotheses
H1 There is no uncountable $N \equiv M$ with $P(N)$ countable.
H2 P is stably embedded.
H2+ For any tuple a from M, $\mathrm{tp}(a/P(M))$ is isolated.
H2* For any tuple a from M, $\mathrm{tp}(a/P(M))$ is algebraic.

The main theorem of the paper now follows. Some corollaries are given in Section 4.

Theorem 1.3 *Assume Convention 1.2 for M, and also H1, H2, but not H2*. Then there exists an infinite group, Π^1-interpretable over M. If H1, H2+ hold but not H2*, there exists an infinite group interpretable over M.*

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2 Preliminaries

We first give a useful equivalent to the condition H1, implicit in Baldwin and Lachlan [1]. We include a proof of equivalence (Proposition 2.4 below).

Definition 2.1 *Let N be a structure with a subset defined by a unary predicate P, and let Q be a definable subset of N^m (where $n > 0$). Then*

(i) Q is P-co-analysable in 0 steps if Q is finite;

(ii) Q is P-co-analysable in $k+1$ steps if for some ℓ there is a set $R \subset Q \times P^\ell$, definable with parameters in N, such that

(a) the projection $R \to Q$ is surjective, and

(b) if $a \in P^\ell$, then the fibre $\{x \in Q : R(x, a)\}$ is P-co-analysable in k steps uniformly in a; that is, the formulas witnessing co-analysability of the fibres have a uniform form, but the parameters in these formulas may be chosen freely for each fibre.

(iii) Q is P-co-analysable if for some natural number k it is co-analysable in k steps.

Remark 2.2 The condition P-co-analysability is a property of the theory; that is, if Q is N-definable then Q is P-co-analysable in k steps in N if and only if it is P-co-analysable in k steps in any elementary extension of N. The uniformity condition in (ii)(b) could be omitted if we require that the ambient structure, over a language L, is $|L^+|$-saturated.

Lemma 2.3 *Suppose that for $i = 1, 2$, Q_i is P-co-analysable in k_i steps. Then $Q_1 \vee Q_2$ is P-co-analysable in $k := \max\{k_1, k_2\}$ steps.*

Proof. Suppose Q_i is k_i-step co-analysable witnessed by $R_i \in Q \times P^{\ell_i}$, for $i = 1, 2$. We may suppose $k_1 = k_2$, and by adding dummy parameters, that $\ell_1 = \ell_2$. Now $R_1 \cup R_2$ witnesses the result. $\square$

Proposition 2.4 *Let T be a complete theory in a countable language L, and P, Q be unary predicates. The following are equivalent.*

(i) T has no model N with $P(N)$ countable and $Q(N)$ uncountable.

(ii) Q is P-co-analysable.

Proof. $(i) \Rightarrow (ii)$ We shall suppose that Q is not co-analysable. Choose countable $N' \models T$, and suppose N' is an elementary substructure of a large sufficiently saturated $\mathbf{C} \models T$. By taking a countable elementary extension if necessary we may suppose that for any ℓ and any definable relation $R(x, y)$, if each set $R(N', b)$ ($b \in P^n$ for some n) is k-step co-analysable, then the $R(x, b)$ are *uniformly* (as the parameter b varies) co-analysable in k steps. It suffices to show there is an elementary extension N of N' with $P(N) = P(N')$ and $Q(N) \neq Q(N')$. We must find a non-algebraic complete type $q(x)$ over N' containing the formula $Q(x)$, such that if $a \in \mathbf{C}$ realises q then the partial type $p := \{P(y)\} \cup \{y \neq m : m \in P\}$ is not isolated over $N'a$. For then, if T' is the theory of $\mathbf{C}$ in $L(N'a)$, then we may take N to be the L-reduct of any model of T' omitting p.

We build a sequence $(Q_n(x) : n \in \omega)$ of formulas over N' which are not P-co-analysable, such that $Q_0 := Q$ and $N' \models \forall x(Q_{n+1} \to Q_n)$. Let $(\psi_i(x, y) : i < \omega)$ list all formulas over N' in the two (single) free variables x, y. Assume that $Q_0, \ldots, Q_n$ have been found as above, and that in addition, if $a \in \mathbf{C}$ with $\models Q_n(a)$, then for any $i < n$, if $\models \exists y(\psi_i(a, y) \wedge P(y))$, then there is $b \in P(N')$ with $\models \psi_i(a, b)$.

Case (i). $Q_n(x) \wedge \neg\exists y(\psi_n(x, y) \wedge P(y))$ is not P-co-analysable. In this case, put $Q_{n+1}(x) := Q_n(x) \wedge \neg\exists y(\psi_n(x, y) \wedge P(y))$.

Case (ii). $Q_n(x) \wedge \neg\exists y(\psi_n(x, y) \wedge P(y))$ is P-co-analysable. Now, by Lemma 2.3, $Q'(x) := Q_n(x) \wedge \exists y(\psi_n(x, y) \wedge P(y))$ is not P-co-analysable. Let R be the relation $Q'(x) \wedge \psi_n(x, y) \wedge P(y)$ on $Q' \times P$. Since R projects onto Q', there is some $b \in P$ such that $Q'(x) \wedge \psi_n(x, b)$ is not P-co-analysable. Now put $Q_{n+1}(x) := Q_n(x) \wedge \psi_n(x, b)$.

Finally, choose q to be any complete type over N' containing $\{Q_n : n \in \omega\}$. By the condition on Q_{n+1}, if a realises q and p is as above, then $\psi_n(a, y)$ does not isolate p.

$(ii) \Rightarrow (i)$ Suppose this is false, and, with k as small as possible, find $N \models T$, and k-step co-analysable Q such that $Q(N)$ is uncountable and $P(N)$ is countable. Clearly $k > 0$, so there is definable $R \subset Q \times P^\ell$ projecting surjectively to Q, with $(k-1)$-step co-analysable fibres over P^ℓ. By induction, each fibre $R(N, a)$ (for $a \in P^\ell$) is countable, so Q is countable. $\square$

Remark 2.5 The above proof just uses a version of co-analysability where the parameter l in Definition 2.1 (ii) takes value 1. It follows that such a version is equivalent to that in 2.1.

We shall use a group-theoretic result of Schlichting [5], re-discovered by Bergman and Lenstra [2]. In fact, P. M. Neumann (Theorem 5 of [2]) has shown that the function $f(n)$ below may be taken to be n^{n^n}.

Theorem 2.6 ([2] Theorem 4) *There is a function $f : \omega \to \omega$ such that the following holds. Let G be a group and H be a subgroup, and suppose there is $n \in \omega$ such that for all $g \in G$, $|H : H \cap H^g| \leq n$. Then there is $N \trianglelefteq G$ such that $|N : H \cap N| \leq n$ and $|H : H \cap N| \leq f(n)$.*

In the next corollary, when G is a permutation group on X and $x \in X$ and $A \subset X$ we write G_x for the stabiliser in G of x, and $G_{(A)}$ for the pointwise stabiliser in G of A. If G has an induced action on some invariant set B (possibly a quotient), we write G^B for the permutation group induced on B. The argument is very close to some in [9]. For the rest of Sections 2 and 3, Convention 1.2 regarding M applies.

Corollary 2.7 *Assume H2, let C be the set of realisations in M of a partial type over $P(M)$, and put $G := \mathrm{Aut}(C/P)$. Suppose there is $n \in \omega$ such that for all $a, b \in C$, $|G_a : G_a \cap G_b| \leq n$.*

(i) There is a G-invariant equivalence relation τ on C with finite classes, and if $\bar{C} := C/\tau$ then there is finite $F \subset \bar{C}$ such that $G_{(F)}$ fixes $\bar{C}$ pointwise.

(ii) If C is a G-orbit, then any τ as in (i) is the intersection with C^2 of a P-definable set.

Proof. (i) For each $a \in C$, we have $|G_a : G_a \cap G_a^g| \leq n$ for all $g \in G$. Hence there is $N \trianglelefteq G$ such that $|N : G_a \cap N| \leq n$ and $|G_a : G_a \cap N| \leq f(n)$ (where f is as in Theorem 2.6). Furthermore, for any $b \in C$, $|G_a : G_a \cap G_b| \leq n$ and $|G_b : G_a \cap G_b| \leq n$. It follows that $|G_b : G_b \cap N| \leq |G_b : G_b \cap G_a \cap N| = |G_b : G_b \cap G_a| \times |G_b \cap G_a : G_b \cap G_a \cap N| \leq n|G_a : G_a \cap N| \leq nf(n)$ and likewise $|N : G_b \cap N| \leq n^2$.

Let τ be the G-invariant equivalence relation on C whose classes are the N-orbits, and put $\bar{C} := C/\tau$. Let $b \in C$. Since $|N : G_b \cap N| \leq n^2$, the τ-class of b has size at most n^2. Also, since $|G_b : G_b \cap N| \leq nf(n)$, also $|G_b^{\bar{C}} : G_b^{\bar{C}} \cap N^{\bar{C}}| \leq nf(n)$. Thus, as $N^{\bar{C}} = 1$, $|G_b^{\bar{C}}| \leq nf(n)$. The existence of F follows.

(ii) In this case all τ-classes have the same size, ℓ, say. Now τ is is a union of finitely many (at most ℓ) complete types on C^2 over P; namely, for fixed $a \in C$, the types $\{\mathrm{tp}(ab/P) : b\tau a\}$. Thus, τ is a type and by compactness there is definable $\tau' \supset \tau$ such that for each $c \in C$ there are at most ℓ elements $c' \in C$ with $\tau'(c, c')$. Then $\tau' \cap C^2 = \tau \cap C^2$. $\Box$

The next lemma is essentially in [7] (stability is assumed there, but not used).

Lemma 2.8 *Assume H2, and that C is the set of realisations in M of a complete type over $P(M)$, with $C \subset \mathrm{dcl}(Pc)$ for any $c \in C$. Then*

(i) $\mathrm{Aut}(C/P)$ is Π^1-interpretable in M;

(ii) if H2+ holds, then $\mathrm{Aut}(C/P)$ is interpretable in M;

(iii) if H2+ holds and P carries a structure whose 0-definable relations are those induced from 0-definable relations of M, then $\mathrm{Aut}(C/P)$ is interpretable in P.

Proof. (i) An automorphism σ of C over P is coded by the pair $(c, \sigma(c))$. Since $C \subset \mathrm{dcl}(Pc)$, $\mathrm{Aut}(C/P)$ acts regularly on C, so σ is uniquely determined by $(c, \sigma(c))$. Furthermore (c, c') and (d, d') code the same automorphism if and only if $\mathrm{tp}(cd/P) = \mathrm{tp}(c'd'/P)$. For suppose $\mathrm{tp}(cd/P) = \mathrm{tp}(c'd'/P)$ and $\sigma(c) = c'$. There is a formula $\phi(x, y)$ over P such that $\phi(c, d)$ and ϕ witnesses interdefinability of c and d over P. Then $\phi(c', d')$ also holds, and, as $\sigma \in \mathrm{Aut}(C/P)$, $\phi(c', \sigma(d))$ holds, and it follows that $\sigma(d) = d'$. The relation $\mathrm{tp}(e/P) = \mathrm{tp}(e'/P)$ is Π^1. Hence there is a Π^1-definable equivalence relation E on C^2, where $E(cc', dd')$ if and only if $\mathrm{tp}(cd/P) = \mathrm{tp}(c'd'/P)$. Then $E(cc', dd')$ if and only if (c, c') and (d, d') code the same automorphism. By H2, C (and hence C^2) are also Π^1-definable.

Write $[cc']$ for the E-class of (c, c'), and $*$ for the group operation. Then $[cc'] * [dd'] = [ee']$ if and only if there is $f \in C$ with $E(dd', c'f)$ and $E(cf, ee')$. Let R be the definable relation given by $R(xx', yy', zz')$ if and only if $x' = y$, $z = x$, and $z' = y'$. R induces the group operation.

(ii) Under H2+, C is definable over P. Also, E is Σ^1-definable: $\mathrm{tp}(cd/P) = \mathrm{tp}(c'd'/P)$ if and only if there exists a formula $\phi(x_1, x_2, \bar{y})$ with no parameters such that for every $\bar{b} \in P$ and $a_1 \in C$ there exists at most one $a_2 \in C$ with $\phi(a_1, a_2, \bar{b})$ and such that $\exists \bar{y} \in P(\phi(c, d, \bar{y}) \wedge \phi(c', d', \bar{y}))$ holds. Taking a disjunction over all such ϕ, we see that E is Σ^1-definable, and hence is definable by (i).

(iii) Formally, we interpret on P the group K of permutations of C which commute with $\mathrm{Aut}(C/P)$. Since $\mathrm{Aut}(C/P)$ acts regularly on C, this is definably isomorphic to $\mathrm{Aut}(C/P)$ (it is the *opposite* group). For each $(c, c') \in C^2$ there is a unique

$g \in K$ with $g(c) = c'$: if $d \in C$ there is unique $\sigma \in \mathrm{Aut}(C/P)$ with $\sigma(c) = d$, and $g(d) = g\sigma(c) = \sigma(c')$. Furthermore, (c, c') and (d, d') determine the same element of K if and only if $\mathrm{tp}(cc'/P) = tp(dd'/P)$, a definable equivalence relation over P (as in (ii)).

For each $c \in C$, since $C \subset \mathrm{dcl}(Pc)$ there is a c-definable injection $t_c : C \to P^{\mathrm{eq}}$, with range $Q = Q(c)$. As C is a complete type over P, the set Q is independent of the choice of c. For each $h \in K$, let $c \in C$ and let σ_h denote the definable permutation $t_{h(c)}t_c^{-1}$ of Q. Since h commutes with elements of $\mathrm{Aut}(C/P)$, this definition is independent of the choice of c. The induced subgroup of $\mathrm{Sym}(Q)$ is isomorphic to K. Furthermore, the elements σ_h are uniformly definable from the pairs $(c, h(c))$, and hence by H2 are uniformly definable in the induced structure on P. The group operation is likewise definable over P, and the resulting group lives in P^{eq}. □

3 Proof of Theorem 1.3

Convention 1.2 on M is assumed throughout this section.

Lemma 3.1 *The hypotheses H1, H2, H2+ are preserved under adding finitely many parameters.*

Proof. For H1 there is nothing to prove. For H2 (respectively H2+) observe that if $\mathrm{tp}(ab/P)$ is definable (respectively, isolated) then so is $\mathrm{tp}(a/Pb)$. □

Lemma 3.2 *Assume H1 and not H2*. Then for some finite $B \subset M$, there is a B-definable set $D \subset M$ such that $D \not\subset \mathrm{acl}(PB)$ but $D \subset \mathrm{acl}(Pc)$ for some tuple c.*

Proof. Assume this is not the case. Then whenever D is B-definable and $D \subset \mathrm{acl}(PBc)$, we have that $D \subset \mathrm{acl}(PB)$. We claim that whenever D is B-definable and co-analysable over P in k steps, then $D \subset \mathrm{acl}(PB)$. This is proved by induction on k, over all D, B. For let $R \subset D \times P^{\ell}$ be a Bc-definable relation witnessing k-step co-analysability of D. Each fibre $R(D, a)$ is $(k-1)$-step co-analysable, and Bca-definable, so by induction is a subset of $\mathrm{acl}(PBca) = \mathrm{acl}(PBc)$ (as $a \in P^{\ell}$). Hence $D \subset \mathrm{acl}(PBc)$, so by our assumption, $D \subset \mathrm{acl}(PB)$, proving the claim. Finally, observe that by H1 and Proposition 2.4, $x = x$ is P-co-analysable and 0-definable, so $M \subset \mathrm{acl}(P(M))$, a contradiction. □

By Lemma 3.2 there is a tuple c and a positive integer m such that for any $d \in D$, $\mathrm{Mult}(d/PBc) \leq m$, and such that $|\mathrm{Aut}(D/PB) : \mathrm{Aut}(D/PBc)|$ is infinite. Now minimise m over all choices of $D' \subset D, B, c$, and let $C := \mathrm{tp}(c/PB)$. If H2+ is assumed, then C will be definable over P. By Lemma 3.1, we may suppose the elements of B are named by constants.

Lemma 3.3 *For any tuple e from M there exists $d \in D$ with $\mathrm{Mult}(d/Pe) \geq m$.*

Proof. Otherwise, fixing D, and replacing c by (c, e), we could lower the value of m. □

We have now proved the following.

Lemma 3.4 *Given H1 and not H2*, the following holds.*
H3 There is a 0-definable set D, and some complete type C over P, such that:
(i) for any $c \in C$ and $d \in D$, $\mathrm{Mult}(d/Pc) \leq m$;
(ii) for any e, there is $d \in D$, $\mathrm{Mult}(d/Pe) \geq m$;
(iii) $\mathrm{Aut}(D/Pc)$ does not have finite index in $\mathrm{Aut}(D/P)$;

(iv) m is minimal over all definable $D' \subset D$ and C satisfying (i)-(iii); that is, there does not exist a parameter set A, some A–definable $D' \subset D$, and some complete type C' over $P \cup A$, and some $m' < m$ such that (i)-(iii) holds with A incorporated as constants and with (D', C', m') replacing (D, C, m).

Let $I = I_m$ be the ideal (in the boolean algebra of definable subsets of D) consisting of the definable subsets D' of D, such that for some finite tuple e from M, for all $d' \in D'$, $\mathrm{Mult}(d'/Pe) < m$. By Lemma 3.4, I_m is a proper ideal. Let I^* be the filter dual to I_m. Elements of I^* will be called *large sets*. The following lemma takes a little argument, and is left as an exercise.

Lemma 3.5 *Under just Convention 1.2, assume that $D \subset \mathrm{acl}(Pc)$ is 0-definable and that for every $d \in D$, $\mathrm{Mult}(d/Pc) \leq m$. Then there exists a c-definable function $f : D \to P^{\mathrm{eq}}$ whose fibres have size at most m.*

Let C be as in Lemma 3.4. Then by Lemma 3.5, for every $c \in C$ there is a c-definable function $f_c : D \to P^{\mathrm{eq}}$, such that some large set $D' \subset D$ is a union of fibres of f_c of size precisely m. Let

$$P_1 = \{a : |f_c^{-1}(a)| = m\}$$

Then since C is a complete type over P, the definition of P_1 is independent of the choice of c. Also, $f_c^{-1}(P_1)$ is large (but does depend on c). Let $P_2 = \{a : 0 < |f_c^{-1}(a)| < m\}$. Then since C is a complete type, again P_2 does not depend on c. However, possibly $f_c^{-1}(P_2)$ does. In any event, the partial function $f_c : D \to P_1 \cup P_2$ has all fibres finite, of size $\leq m$. We shall assume, in addition to H3, by moving to imaginaries:

H3(v) If $c \neq c' \in C$ then $f_c \neq f_{c'}$.

Lemma 3.6 *Assume H2 and H3(i)-(v). Then $C \subset \mathrm{acl}(Pc_0)$ for any $c_0 \in C$.*

Proof. Fix $c_0 \in C$. It suffices to show that any $c \in C$ has only finitely many conjugates over Pc_0; equivalently that f_c has only finitely many conjugates.

Let $E(c, c') := P_2 \cup \{a \in P_1 : \neg\exists a' \in P_1(f_c^{-1}(a) = f_{c'}^{-1}(a'))\}$ and $G(c, c') := P_1 \setminus E(c, c')$. Then

$$f_c^{-1}(G(c, c')) = f_{c'}^{-1}(G(c', c)).$$

Now $G(c, c')$ and $G(c', c)$ are P-definable by stable embeddedness of P. Thus by the above equation, $f_c^{-1}(G(c, c'))$ is Pc'-definable.

Let $X(c) := D \setminus f_c^{-1}(G(c, c_0))$. So $X(c)$ is Pc_0-definable. Note that each point of $X = X(c)$ has multiplicity less than m over cc_0P. So by minimality of m (H3(iv)), $\mathrm{Aut}(X/cc_0P)$ has finite index in $\mathrm{Aut}(X/c_0P)$.

Since the graph of $f_c|_X$ is contained in $X \times P$, and is left invariant by $\mathrm{Aut}(X/cc_0P)$, it has finitely many conjugates over c_0P. It remains to show that $f_c|_{D\setminus X}$ also has finitely many conjugates over c_0P. In fact, if c' is conjugate to c over Pc_0 then $f_c|_{D\setminus X} = f_{c'}|_{D\setminus X}$. For let $d \in D \setminus X$. Then $f_c(d) \in G(c, c_0)$. So $a := f_c(d) \in P_1$, and for some $a' \in P_1$, $f_c^{-1}(a) = f_{c_0}^{-1}(a')$. Thus $f_{c'}^{-1}(a) = f_{c_0}^{-1}(a')$. So $d \in f_{c'}^{-1}(a)$; and $f_{c'}(d) = a = f_c(d)$. □

Proof of Theorem 1.3. Obtain first C as in Lemma 3.4 and the remarks before it. By Lemma 3.6, $C \subset \mathrm{acl}(Pc)$ for $c \in C$. Since $\mathrm{Aut}(D/Pc)$ does not have finite index in $\mathrm{Aut}(D/P)$, we cannot have $c \in \mathrm{acl}(P)$.

Let $\bar{C}$ be the quotient of C by an equivalence relation τ with finite classes, as in Corollary 2.7 above, so that $\bar{C} \subset \mathrm{dcl}(PF)$ for some finite $F \subset \bar{C}$. Since $C \not\subset \mathrm{acl}(P)$, also $\bar{C} \not\subset \mathrm{acl}(P)$. So $\mathrm{Aut}(\bar{C}/P)$ is infinite. Now let $\bar{d}$ be an enumeration of F, and $C' := \mathrm{tp}(\bar{d}/P)$. Then C' satisfies the assumptions of Lemma 2.8, and the result follows from that lemma. □

Remark 3.7 The proof of Theorem 1.3 shows that $\mathrm{Aut}(DC'/P)$ has a profinite normal subgroup, namely $\mathrm{Aut}(D/PC')$, such that the quotient group is isomorphic to a Π^1-interpretable group. For $\mathrm{Aut}(D/PC')$ is closed with all orbits finite, so profinite. The restriction homomorphism $\tau : \mathrm{Aut}(DC'/P) \to \mathrm{Aut}(C'/P)$ is surjective (this uses homogeneity, and the fact that $C' := \mathrm{dcl}(Pc)$ for any $c \in C'$). Since σ has kernel $\mathrm{Aut}(D/PC')$, the Π^1-interpretable group $\mathrm{Aut}(C'/P)$ is isomorphic to $\mathrm{Aut}(DC'/P)/\mathrm{Aut}(D/PC')$.

4 Further Observations

The original motivation for this paper lay in the notion of relative categoricity [6], and we recall the setting of that paper. There are countable first order languages $L^- \subset L$, with a unary predicate $P \in L \setminus L^-$. We consider L-theories T such that if $M \models T$ then the reduct of $P(M)$ to L^- is an infinite L^--structure. We say that T is *relatively ω-categorical* if for any models M, M' of T with $P(M) = P(M')$, both countable, there is an isomorphism $M \to M'$ over $P(M)$. If T is relatively ω-categorical, then it is *reduced*; that is, for each ℓ, and every $M \models T$, every subset of $P(M)^\ell$ which is 0-definable in the L-structure M is 0-definable in the L^--structure $P(M)$.

It is easy to see that if T is reduced and M is algebraic over $P(M)$ in a uniform way (that is, via finitely many formulas) then T is relatively ω-categorical. Much of the work in [6] concerned conditions under which the converse holds (for reduced theories T). An example where the converse fails is the group $(\mathbf{Z}/4\mathbf{Z})^\omega$, with P picking out the socle. More generally, by Theorem 6.1 of [6], if N is a countable L^--structure in which there is an infinite interpretable group, then there is a language $L^* \supset L^-$ with unary $P \in L^* \setminus L^-$, and a relatively ω-categorical L^*-structure M such that $M \not\subset \mathrm{acl}(P(M))$, and such that the reduct of $P(M)$ to L^- is isomorphic to N. It was suggested in [6] that if M is relatively ω-categorical but not algebraic over $P(M)$, then relative ω-categoricity must arise in something like this way; that is, that an infinite group is interpretable in $P(M)$. This follows immediately from Theorem 1.3.

Corollary 4.1 *If T is relatively ω-categorical over P, and for some countable $M \models T$, $M \not\subset \mathrm{acl}(P(M))$, then $P(M)$ interprets an infinite group (in the language L^-).*

Proof. By Section 3 of [6], H1 and H2+ hold, and by assumption, H2* fails. Now apply Theorem 1.3 (to some model satisfying Convention 1.2). □

Corollary 4.2 *Assume T is ω-categorical over a language with a unary predicate P, and satisfies H1, H2, but not H2*. Then there is an infinite group interpretable in the induced structure on P.*

Proof. By Theorem 1.3 (applied to a sufficiently saturated model), it suffices to check that if T is ω-categorical then H2 implies H2+. Let C be a complete type over P lying in a 0-definable set C'. For each m there is a 0-definable equivalence relation E_m on C', where $E_m xy$ hold if for all $\bar{a} \in P^m$, $\mathrm{tp}(\bar{a}x) = \mathrm{tp}(\bar{a}y)$. By ω-categoricity, there is $m_0 \in \omega$ such that $\bigcap_{i \in \omega} E_i = E_{m_0}$. Then C is an E_{m_0}-class. By H2, C is definable over P, as required for H2+. □

Our final result describes a situation which is realised, for example, by the 'affine duality' phenomemon of quasi-finite structures (Chapter 4 of [4]).

Proposition 4.3 *Assume that M satisfies Convention 1.2. Suppose also that $f : D \to P$ is a 0-definable map with finite fibres, and that P is stably embedded, but D is not. Then there exists an infinite torsion group, Π^1- interpretable over M.*

Proof. Since $\mathrm{Aut}(D/P)$ is a closed permutation group with all orbits finite, it is a profinite group with the natural topology.

Claim. For some tuple c, $\mathrm{Aut}(D/Pc)$ has infinite index in $\mathrm{Aut}(D/P)$.

Proof of Claim. Suppose this is false, and let c be a tuple in M. Then $\mathrm{Aut}(D/Pc)$ is a closed subgroup of finite index in $\mathrm{Aut}(D/P)$, so is open, and hence there is finite $F \subset D$ with $\mathrm{Aut}(D/PF) \leq \mathrm{Aut}(D/Pc)$. By adding to F a countable subset of P, we may also ensure that $\mathrm{Aut}(P/F) = \mathrm{Aut}(P/Fc)$, since P is stably embedded. It follows that $\mathrm{Aut}(PD/Fc) = \mathrm{Aut}(PD/F)$. (For let $g \in \mathrm{Aut}(PD/F)$; then there is $h \in \mathrm{Aut}(P/Fc)$ agreeing with g on P; now h extends to $h' \in \mathrm{Aut}(PD/Fc)$ and now gh'^{-1} induces on D an element of $\mathrm{Aut}(D/PFc)$, so g is elementary over Fc.) Thus every c-definable subset of PD is invariant under $\mathrm{Aut}(M/F)$, and is therefore F-definable. Hence PD is stably embedded. It follows that D also is stably embedded, as $PD \subset \mathrm{dcl}(D)$. This is a contradiction.

Now choose c as in the claim, and put $C := \mathrm{tp}(c/P)$. Adjusting C if necessary, we obtain m as in Lemma 3.4 with H3 (i)–(iv) holding, and we may suppose H3(v) also holds. We apply Lemma 3.6, and the proof of Theorem 1.3. There is an $\mathrm{Aut}(C/P)$-invariant equivalence relation τ on C with finite classes provided by Lemma 2.7, and an orbit C' of tuples from $\bar{C} := C/\tau$, such that $\mathrm{Aut}(C'/P)$ is infinite and Π^1-interpretable.

It remains to check that $\mathrm{Aut}(C'/P)$ is a torsion group. Let H be the image of $\mathrm{Aut}(DC\bar{C}C'/P)$ in $\mathrm{Aut}(\bar{C}/P)$, and let $h \in H$. There is $g \in \mathrm{Aut}(DC\bar{C}C'/P)$ extending h. Now for some finite m (equal say to $t!$, where t is the least common multiple of the sizes of the fibres of f), g^m fixes D pointwise. Since elements of C are canonical parameters for functions $D \to P^{\mathrm{eq}}$, g^m also fixes C pointwise, so $g^m = 1$. Since H is dense in $\mathrm{Aut}(C'/P)$, the latter group also has exponent m. □

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