Extending partial automorphisms and the profinite topology on free groups

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Abstract
A class of structures $\mathcal{C}$ is said to have the extension property for partial automorphisms (EPPA) if, whenever C_1 and C_2 are structures in $\mathcal{C}$, C_1 finite, $C_1 \subseteq C_2$, and p_1, p_2, …, p_n are partial automorphisms of C_1 extending to automorphisms of C_2, then there exist a finite structure C_3 in $\mathcal{C}$ and automorphisms α_1, α_2, …, α_n of C_3 extending the p_i. We will prove that some classes of structures have the EPPA and show the equivalence of these kinds of results with problems related with the profinite topology on free groups. In particular, we will give a generalisation of the theorem, due to Ribes and Zalesskiì stating that a finite product of finitely generated subgroups is closed for this topology.

1 Introduction

In this paper, we will consider and relate two kinds of results. We begin by giving the basic definitions that are needed to understand these relations.

On the one hand, there will be the theorems concerning the so called "profinite topology" on the free groups. Given a group G, the profinite topology on G is the topology for which a basis of open subsets is

$$\{gH; g \in G \text{ and } H \text{ is a subgroup of } G \text{ of finite index}\}$$

Let us recall a classical result, due to M. Hall [4]:

Theorem 1.1 *Let P be a finite set, and $F(P)$ the free group generated by P. Then every finitely generated subgroup of $F(P)$ is closed for the profinite topology.*

This result can be rephrased as follows: let H be a finitely generated subgroup of $F(P)$. Then

$$H = \bigcap \{K \ ; \ K \text{ is a subgroup of finite index of } F(P) \text{ and } H \subseteq K\}.$$

More recently, Ribes and Zalesskiì ([9]) proved:

Theorem 1.2 *Let H_1, H_2, …, H_n be finitely generated subgroups of $F(P)$. Then*

$$H_1 H_2 \dots H_n = \{h_1 h_2 \dots h_n \ ; \ h_1 \in H_1, h_2 \in H_2, \dots, h_n \in H_n\}$$

is closed for the profinite topology.

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On the other hand, we will consider some combinatorial results concerning the extension of partial automorphisms.

Definition 1.3 *Let M, M' be two structures in a given finite relational language $\mathcal{L}$. A* partial isomorphism *from M into M' is an isomorphism of a substructure of M onto a substructure of M'. We will denote by* $\mathrm{Part}(M, M')$ *the set of partial isomorphisms from M into M'. A* partial automorphism *of M is a partial isomorphism from M into M.*

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures (containing both finite and infinite structures), M_0 a finite structure in $\mathcal{C}$ and P a set of partial automorphisms of M_0. We consider the following problem (the $(M_0, P, \mathcal{C})$-*extension problem*): find a structure $M_1 \in \mathcal{C}$, which is an extension of the structure M_0 and for each $p \in P$ an automorphism α_p of M_1 extending p. We say $(M_1, \alpha_p)_{p \in P}$ is a solution of our problem, and we will say it is finite if M_1 is.

We say that $\mathcal{C}$ has the *extension property for partial automorphisms* (EPPA for short) if for all finite M_0 and $P \subseteq \mathrm{Part}(M_0, M_0)$, if the $(M_0, P, \mathcal{C})$-extension problem has a solution, then it has a finite solution.

An example of this family of results is the following theorem of Hrushovski ([8]):

Theorem 1.4 *Let Γ_0 be a finite graph. Then there exists a finite graph Γ_1 extending Γ_0 and such that every partial automorphism of Γ_0 can be extended to an automorphism of Γ_1.*

(Here, a graph means undirected loop free graph.)

Hrushovski's theorem just states that the class of all graphs has the EPPA (note that in the case where $\mathcal{C}$ is the class of all graphs, every extension problem has a solution, because every finite graph is embeddable in the random graph, which is homogeneous).

Herwig has generalised this result to the class of structures of a given finite relational language ([5]) and various other classes of graphs (see [6]). This kind of result is of importance for proving the small index property for the automorphism group of the corresponding generic structures (see [7] or [6] for more about this question).

This paper is organised as follows: in the next section, we show how to use the properties of the profinite topology to prove some EPPA-results. In particular, we give a proof of Hrushovski's theorem (Theorem 1.4) from the theorem of Ribes and Zalesskiì (Theorem 1.2). This proof is not simpler than the original one. It is only given here as an illustration.

Next, we go in the other direction. First, starting from the fact that the class of n-partitioned cycle-free graphs has the EPPA, we show the Ribes-Zalesskiì theorem. Then, using the most general extension result that we have been able to prove (Theorem 3.2), we prove a property of the profinite topology (Theorem 3.3) generalising the theorem of Ribes and Zalesskiì.

The next two sections are devoted to proving extension results. First, we give a proof of the EPPA for the class of graphs (that is the theorem of Hrushovski). This proof has the advantage of being short and of admitting natural generalisation to the class of all structures in a given finite relational language. This last result, which had already been obtained by the first author (see [5]), will be used later. We will also give a proof of the EPPA for the class of n-partitioned cycle-free graphs. This proof was not necessary, since it is just a particular case of Theorem 3.2, but we included it here because we think that some of our readers (if any) will be mainly interested in an alternative simple proof of the theorem of Ribes and Zalesskiì, and we wanted to spare them the complication of the proof of Theorem 3.2. Section 5, almost half of the paper, is devoted to this proof.

We will be dealing, throughout the paper with structures in some relational language. We assume that the reader understands these words, and also the notation $M \models Ra_1a_2 \ldots a_n$ (where

M is a structure in a language $\mathcal{L}$, R is a symbol of arity n in the language $\mathcal{L}$ and $a_1, a_2, \ldots, a_n$ are elements in M). If $\mathcal{L}'$ is a language included in $\mathcal{L}$ and M an $\mathcal{L}$-structure, $M_{|\mathcal{L}'}$ is the restriction of M to $\mathcal{L}'$, that is the $\mathcal{L}'$-structure obtained from M by just forgetting the interpretation of the symbols of $\mathcal{L}$ which are not in $\mathcal{L}'$. We say M_1 is an extension of M_0 (or M_0 is a substructure of M_1) if the underlying set of M_0 is contained in that of M_1 and for every symbol R in $\mathcal{L}$ and $a_1, a_2, \ldots a_n \in M_0$: $M_0 \models Ra_1a_2 \ldots a_n \Leftrightarrow M_1 \models Ra_1a_2 \ldots a_n$.

We will use the same letter, M, for example, for a structure and its underlying set. The sign $\cdot$ will denote the product operation in whatever group we are manipulating (but it will be often omitted, depending on the context), and $\circ$ will denote the composition of maps (which may be partial).

If I is a set which is totally ordered by the relation $<$, we may define the lexicographical order on the set $I^{<\omega}$ of finite sequences of elements of I: given two sequences $a = (i_1, i_2, \ldots, i_n)$ and $b = (j_1, j_2, \ldots, j_m)$, then $a < b$ if and only if one of the following cases occurs

- a is a proper initial segment of b;
- a is not an initial segment of b, and if k is the smallest integer such that $i_k \neq j_k$, then $i_k < j_k$.

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2 From the profinite topology to the extension properties

2.1 A sophisticated proof of a theorem of Hrushovski

We will give a proof (using Theorem 1.2) of the theorem of Hrushovski (Theorem 1.4).

Let Γ_0 be a finite graph and let P be the (finite) semi-group of partial automorphisms of Γ_0. Let us agree that, when we write $p(a) = p'(a')$ or $p(a) \neq p'(a')$ (where p, p' are elements of P and a, a' are elements of Γ_0) this means that both $p(a)$ and $p'(a')$ are defined and of course the equality or the inequality holds.

Choose an element a_0 of Γ_0 and let H_0 be the subgroup of $F(P)$ generated by X_0 where

$$X_0 = \{p^{-1} \cdot p' \ ; \ p, p' \in P, \, p(a_0) = p'(a_0)\} \cup \{p_3^{-1} \cdot p_1 \cdot p_2 \ ; \ p_1, p_2, p_3 \in P, \\ p_1 \circ p_2(a_0) = p_3(a_0)\}$$

(note that X_0 is finite). Let H be any subgroup of $F(P)$ such that

$$H_0 \subseteq H \tag{1}$$

For each $a \in \Gamma_0$, there is a partial automorphism $p \in P$ such that $p(a_0) = a$, and if $p' \in P$ is such that $p'(a_0) = a$, then $p \cdot H = p' \cdot H$; so we may define a map ϕ from Γ_0 into $F(P)/H$ by : for all $a \in \Gamma_0$, $\phi(a) = p \cdot H$ where p is any element of P such that $p(a_0) = a$.

If moreover we demand that

$$H \cap X_1 = \emptyset \tag{2}$$

where :

$$X_1 = \{p^{-1} \cdot p' \ ; \ p, p' \in A, \, p(a_0) \neq p'(a_0)\}$$

then this map ϕ is injective.

We assume that this condition is satisfied. For each $g \in F(P)$, define the permutation $\widetilde{g}$ of $F(P)/H$ by : for all $x \in F(P)$, $\widetilde{g}(xH) = gxH$. We remark that, for every $p \in P$ and $a \in \Gamma_0$, if $p(a)$

is defined, then $\phi(p(a)) = \widetilde{p}(\phi(a))$: indeed for some $q \in P$, we have $a = q(a_0)$ and $p(a) = p \circ q(a_0)$. Let $p' = p \circ q$. Thus $\phi(p(a)) = p'H$. On the other hand, $\phi(a) = qH$, and $\widetilde{p}(\phi(a)) = pqH$. But $p'H = pqH$, since $p'^{-1}pq \in H$, by (1).

We will consider Γ_0 as a subset of $F(P)/H$ by identifying each $a \in \Gamma_0$ with $\phi(a)$. Thus, for all $p \in P$, $\widetilde{p}$ extends p. It is also clear that the map $g \longmapsto \widetilde{g}$ is a group homomorphism from $F(P)$ into the group of permutations of $F(P)/H$.

We want to endow $F(P)/H$ with a graph structure extending Γ_0 and in such a way that for every $g \in F(P)$, $\widetilde{g}$ is a an automorphism of this graph. We do that by adding the minimal number of edges: given α and α' in $F(P)/H$, we decide that $F(P)/H \models R\alpha\alpha'$ if and only if there exist $a, a' \in \Gamma_0$ and $g \in F(P)$ such that $\Gamma_0 \models Raa'$ and $\widetilde{g}(a) = \alpha$ and $\widetilde{g}(a') = \alpha'$. We denote by Γ_1 the graph that we get this way.

So, by construction, every $g \in F(P)$ induces an automorphism of Γ_1. What is not clear is whether Γ_1 is an extension of Γ_0. We have to be careful not to add an edge between two elements of Γ_0. This will be true if and only the following condition is satisfied:

- For all $p_0, p_1, p_2, p_3 \in P$ such that $\Gamma_0 \models R\ p_0(a_0)\ p_1(a_0) \wedge \neg R\ p_2(a_0)\ p_3(a_0)$, there is no $g \in F(P)$ such that $gp_0H = p_2H$ and $gp_1H = p_3H$.

A straightforward calculation shows that this condition is equivalent to:

$$\text{For all } p_0, p_1, p_2, p_3 \in P \text{ if } \Gamma_0 \models R\ \ p_0(a_0)\ \ p_1(a_0) \wedge \neg R\ \ p_2(a_0)\ \ p_3(a_0), \text{ then} \qquad (3)$$
$$p_0p_2^{-1}p_3p_1^{-1} \notin p_0Hp_0^{-1}\,p_1Hp_1^{-1}$$

Let us sum up: for every subgroup H of $F(P)$ satisfying the conditions (1), (2) and (3), we have an extension Γ_1 of Γ_0 whose universe is $F(P)/H$ such that every partial automorphism of Γ_0 extends to an automorphism of Γ_1. So the problem is to find such a subgroup K of finite index.

We remark that if we drop the assumption that K is of finite index, then we can solve the problem. Indeed, we know that there exists a graph Γ (possibly infinite) extending Γ_0 and for all $p \in P$ an automorphism $\widetilde{p}$ of Γ extending p (for example the random graph). Let η be the homomorphism of $F(P)$ into $\mathrm{Aut}(\Gamma)$ such that $\eta(p) = \widetilde{p}$ for all $p \in P$, and write $\widetilde{h}$ instead of $\eta(h)$. Reversing all that we have just said, we see that, if we set

$$H = \left\{h \in F(P)\ ;\ \widetilde{h}(a_0) = a_0\right\}$$

then H satisfies the conditions (1), (2) and (3). Thus H_0 also satisfies the conditions (1), (2) and (3).

Since a finite intersection of subgroups of finite index is of finite index, it suffices to prove the following facts:

- For all $\alpha \in X_1$, there exists a subgroup K of $F(P)$ of finite index, containing H_0, but not containing α;

This is exactly the theorem of M. Hall (Theorem 1.1).

- For all $p_0, p_1, p_2, p_3 \in P$ such that $\Gamma_0 \models R\ p_0(a_0)\ p_1(a_0) \wedge \neg R\ p_2(a_0)\ p_3(a_0)$, there exists a subgroup K of $F(P)$ of finite index containing H_0 and such that $p_0p_2^{-1}p_3p_1^{-1} \notin p_0Kp_0^{-1} \cdot p_1Kp_1^{-1}$.

Here we apply the Theorem 1.2: since $p_0H_0p_0^{-1}p_1H_0p_1^{-1}$ is closed for the profinite topology and does not contains $p_0p_2^{-1}p_3p_1^{-1}$, there exist a subgroup N of $F(P)$ of finite index such that

$$p_0p_2^{-1}p_3p_1^{-1} \notin (p_0H_0p_0^{-1}p_1H_0p_1^{-1})N.$$

We may moreover choose N to be normal in $F(P)$. Then $K = H_0 \cdot N$ is a subgroup of $F(P)$ of finite index containing H_0 and $(p_0 H_0 p_0^{-1} p_1 H_0 p_1^{-1})N = p_0 K p_0^{-1} p_1 K p_1^{-1}$, so $p_0 p_2^{-1} p_3 p_1^{-1} \notin p_0 K p_0^{-1} p_1 K p_1^{-1}$.

♡

2.2 Generalisation

In fact the same argument can be used to prove many other results of the same kind. For example, let us prove that the class of triangle free graphs has the EPPA. We start from a finite triangle free graph. We construct a graph Γ_1 extending Γ_0 as above, using a subgroup K of $F(P)$ of finite index, and, as above, $F(P)$ acts on Γ_1. We demand in addition that Γ_1 is triangle free. For this, it is sufficient and necessary that the following condition is satisfied:

- For all $a_1, a_2', a_2, a_3', a_3, a_1' \in \Gamma_0$ if $\Gamma_0 \models Ra_1a_2' \wedge Ra_2a_3' \wedge Ra_3a_1'$, then, there is no $h_1, h_2, h_3 \in F(P)$ such that $\widetilde{h_3}(a_1) = \widetilde{h_2}(a_1')$, $\widetilde{h_1}(a_2) = \widetilde{h_3}(a_2')$, $\widetilde{h_2}(a_3) = \widetilde{h_1}(a_3')$.

For $i = 1, 2, 3$, let p_i and p_i' be elements of P such that $a_i = p_i(a_0)$ and $a_i' = p_i'(a_0)$. A calculation shows that the above condition is equivalent to

- For all $p_1, p_2', p_2, p_3', p_3, p_1' \in P$ if $\Gamma_0 \models R\ \ p_1(a_0)\ \ p_2'(a_0) \wedge R\ \ p_2(a_0)\ \ p_3'(a_0) \wedge R\ \ p_3(a_0)\ \ p_1'(a_0)$, then $1 \notin p_1' K p_1^{-1} p_2' K p_2^{-1} p_3' K p_3^{-1}$.

We finish the proof as above, using the Theorem 1.2.

The case of the $\mathcal{K}_4$-free graphs seems to be more difficult and, in fact, we have not been able to deduce it from the theorem of Ribes and Zalesskiĭ. But it has been proved by Herwig ([6]), and, as a matter of fact, is just a particular case of Theorem 3.2.

Let us phrase the above arguments in a systematic way: let X be finite set. We consider the set

$$\mathrm{Part}(X) = \{p; p \text{ is an injective map from a subset of } X \text{ into } X\}$$

with its natural monoid structure. Let P be a subset of $\mathrm{Part}(X)$. Consider the set Σ of words on the alphabet $P \cup P^{-1}$ (that is the free monoid generated by $P \cup P^{-1}$). To a given word w in Σ we may naturally associate a partial automorphism of Y. It is $\varsigma(w)$, where ς is the homomorphism from Σ into the monoid of partial automorphisms of Y defined by: for $p \in P \cup P^{-1}$, $\varsigma(p) = p$. Let $(X_i\,;\, i = 1, 2, \ldots n)$ be the partition of X into orbits relatively to P (that is two elements x and y of X lie in the same X_i if and only if there exists $w \in \Sigma$ such that $\varsigma(w)(x) = y$) and for each $i = 1, 2, \ldots n$, choose an element x_i in X_i. Furthermore choose for every $x \in X_i$ a word $w_x \in \Sigma$ such that $x = \varsigma(w_x)(x_i)$. Then there is a correspondence between

- the n-tuples $(H_1, H_2, \ldots, H_n)$ of subgroups of $F(P)$ such that for all p in P and $i = 1, 2, \ldots, n$:
  (1) $w_y^{-1} \cdot p \cdot w_x \in H_i$ if $y, x \in X_i$ and $p(x) = y$,
  (2) $w_y^{-1} \cdot w_x \notin H_i$ if $x, y \in X_i$, $x \neq y$.

on one hand, and

- the tuples $(Y, (\widetilde{p}\,;\, p \in P))$, where $X \subseteq Y$, and, for all $p \in P$, $\widetilde{p}$ is a permutation of Y extending p.

on the other hand.

Indeed, let $(H_1, H_2, \ldots, H_n)$ be a sequence subgroups of $F(P)$ satisfying the conditions (1) and (2). Let Y be the disjoint union of the sets $F(P)/H_i$. If $x \in X_i$, set $\phi(x) = w_x H_i$. Hereby we consider w_x in a natural way as an element of $F(P)$. Condition (2) insures that the map ϕ is injective. We will identify x with $\phi(x)$, so that X will be viewed as a subset of Y.

For each $p \in P$, the left multiplication defines a permutation $\widetilde{p}$ on Y which extends by condition (1) the map p.

In the reverse direction, assume that Y is a set including X, and for each $p \in A$, $\widetilde{p}$ a permutation of Y extending p. Let φ be the group homomorphism from $F(P)$ into $Perm(Y)$, the group of permutation of Y, defined by: for all $p \in P$, $\varphi(p) = \widetilde{p}$. Set, for $i = 1, 2, \ldots, n$,

$$H_i = \{h \in F(P); \varphi(h)(x_i) = x_i\}$$

Then, the sequence $(H_1, H_2, \ldots, H_n)$ satisfies the conditions (1) and (2).

Now assume that X is the universe of a structure, also denoted X, in some finite relational language $\mathcal{L}$. Assume moreover that the maps p in P are partial automorphisms of X. We want to find the solutions to the following problem (subsequently referred to as $\mathcal{P}$): find an $\mathcal{L}$-structure Y extending X and, for each $p \in P$, an automorphism $\widetilde{p}$ of Y extending p.

Suppose that $(Y, (\widetilde{p}\ ;\ p \in P))$ is such a solution of $\mathcal{P}$. Let again φ be the homomorphism from $F(P)$ into $\mathrm{Aut}(Y)$, which is defined by: $\varphi(p) = \widetilde{p}$, and, for $h \in F(P)$, write $\widetilde{h}$ instead of $\varphi(h)$. Now, if R is a symbol of the language $\mathcal{L}$ of arity k, and if $y_1, y_2, \ldots, y_k$ are elements in Y such that there exist $x_1, x_2, \ldots, x_k$ in X and $h \in F(P)$ such that, for all $j = 1, 2, \ldots, k$, $\widetilde{h}(x_j) = y_j$ and $X \models Rx_1x_2\ldots x_k$, then necessarily, $Y \models Ry_1y_2\ldots y_k$. This proves that the following condition is satisfied:

If R is a symbol of the language $\mathcal{L}$ of arity k, if $z_1, z_2, \ldots, z_k, t_1, t_2, \ldots, t_k$ are elements in X and if

$$X \models Rz_1z_2\ldots z_k \wedge \neg Rt_1t_2\ldots t_k$$

then, there is no $h \in F(P)$ such that, for all $i = 1, 2, \ldots, k$, $\widetilde{h}(z_i) = t_i$.

Setting $H_i = \{h \in F(P)\ ;\ \widetilde{h}(x_i) = x_i\}$ as above, an easy computation shows that this condition is equivalent to

(3) If R is a symbol of the language $\mathcal{L}$ of arity k, if $i_1, i_2, \ldots, i_k$ are elements of $\{1, 2, \ldots, n\}$, if $x_1, x'_1 \in X_{i_1}, x_2, x'_2 \in X_{i_2} \ldots, x_k, x'_k \in X_{i_k}$ and if $X \models Rx_1x_2\ldots x_k \wedge \neg Rx'_1x'_2\ldots x'_k$ then there is no $h \in F(P)$ such that, for all $j = 1, 2, \ldots, k$, $h \cdot w_{x_j} \equiv w_{x'_j} \bmod(H_{i_j})$

This condition (3) (taking for granted that the conditions (1) and (2) are satisfied) is sufficient: it suffices to define on the disjoint union, say Y, of the sets $F(P)/H_i$ considered as an extension of X, the $\mathcal{L}$-structure by setting: for all R, symbol of the language $\mathcal{L}$ of arity k, and for all $y_1, y_2, \ldots, y_k$ in Y,

$$Y \models Ry_1y_2\ldots y_k \text{ if and only if there exist } x_1, x_2, \ldots, x_k \text{ in } X \text{ and } h \in F(P)$$
$$\text{such that, for all } j = 1, 2, \ldots, k,\ \widetilde{h}(x_j) = y_j \text{ and } X \models Rx_1x_2\ldots x_k$$

We remark that if a sequence of subgroups $(H_1, H_2, \ldots, H_n)$ satisfies conditions (2) and (3) and if for all $i = 1, 2, \ldots, n$, K_i is a subgroup of H_i, then the sequence $(K_1, K_2, \ldots, K_n)$ satisfies also (2) and (3). We will express this fact by saying that (2) and (3) are *negative* conditions.

The correspondence that we have been speaking about is certainly not one to one in general. There may be several solutions corresponding to a given sequence $(H_1, H_2, \ldots, H_n)$. The solution that we have constructed enjoys the following property of "slimness":

Definition 2.1 *Let X be a finite $\mathcal{L}$-structure, P a set of partial automorphism of X. A solution $(Y, (\widetilde{p}\,;\, p \in P))$ of the problem $\mathcal{P}$ is* slim *if: 1) for all $y \in Y$, there exist $x \in X$ and h in $F(P)$ such that $y = \widetilde{h}(x)$; 2) for all R, symbol of the language $\mathcal{L}$ of arity k, and $y_1, y_2, \dots, y_k$ elements in Y, $Y \models Ry_1y_2\dots y_k$ if and only if there exist $x_1, x_2, \dots, x_k$ in X and $h \in F(P)$ such that $X \models Rx_1x_2\dots x_k$ and $y_i = \widetilde{h}(x_i)$ for all $i = 1, 2, \dots k$.*

It is easy to get a slim solution from any solution: if $(Y, (\widetilde{p}\,;\, p \in P))$ is a solution, throw away from Y the elements which are not image by an element of the group generated by $\{\widetilde{p}\,;\, p \in P\}$ of an element of X, and do the same for links. There is one further condition our solutions satisfy. Namely for $x, y \in X$, if there exists $h \in F(P)$ such that $\widetilde{h}(x) = y$ then x and y are in the same orbit relative to P. If we restrict ourself to slim solutions satisfying this further condition, we do get a one to one correspondence.

We will need solutions which satisfy a stronger condition. Consider again the free monoid Σ over $P \cup P^{-1}$ and the homomorphism ς from Σ to $\mathrm{Part}(X)$. We may consider every $w \in \Sigma$ as an element of $F(P)$ and we write again $\widetilde{w}$ for $\varphi(w)$, where φ is the group homomorphism from $F(P)$ into $\mathrm{Aut}(Y)$ defined by: for all $p \in P$, $\varphi(p) = \widetilde{p}$. Of course $\widetilde{w}$ extends $\varsigma(w)$.

Definition 2.2 *The solution $(Y, (\widetilde{p}\,;\, p \in P))$ is* special *if it is slim and, for all t_1, t_2 in X and $h \in F(P)$, if $\widetilde{h}(t_1) = t_2$, then there exists a word $w \in \Sigma$ such that $\varsigma(w)(t_1) = t_2$ and $\widetilde{w} = \widetilde{h}$.*

We show now how to get a special solution from any solution. Let $(Y, (\widetilde{p}\,;\, p \in P))$ a solution. Set:

- $H_i = \{h \in F(P); \varphi(h)(x_i) = x_i\}$;
- K the kernel of φ;
- L_i the subgroup of $F(P)$ generated by
$$\left\{w_y^{-1} \cdot p \cdot w_x \;;\; x, y \in X_i \text{ and } p(x) = y\right\};$$
- $K_i = K \cdot L_i$ (it is also the subgroup of $F(P)$ generated by $K \cup L_i$).

First of all, we see that the sequence $(K_1, K_2, \dots, K_n)$ satisfies the conditions (1) to (3). Condition (1) is insured by the fact that, for all $i = 1, 2, \dots n$, $L_i \subset K_i$. Conditions (2) and (3) are negative conditions and are satisfied by $(H_1, H_2, \dots, H_n)$. Moreover, for all $i = 1, 2, \dots n$, $K_i \subset H_i$. So, $(K_1, K_2, \dots, K_n)$ satisfies the conditions (2) to (3).

It remains to see that the solution corresponding to $(K_1, K_2, \dots, K_n)$ is special. So, let t_1, t_2 in X and $h \in F(P)$ such that $\widetilde{h}(t_1) = t_2$. This implies that t_1 and t_2 belong to the same set $F(P)/K_i$, say $F(P)/K_1$, and we have $t_1 = w_{t_1}x_1 = w_{t_1}K_1$ and $t_2 = w_{t_2}x_1 = w_{t_2}K_1$. Thus, we have $w_{t_2}^{-1}hw_{t_1} \in K_1$, and there exist $k \in K$ and $\ell \in L_1$ such that $w_{t_2}^{-1}hw_{t_1} = k\ell$. So, $w_{t_2}^{-1}hw_{t_1}\ell^{-1} \in K$, and, since K is normal $w_{t_1}\ell^{-1}w_{t_2}^{-1}h \in K$. Since $\ell \in L_1$, it is the product of elements and inverses of the set $\left\{w_y^{-1}pw_x \;;\; x, y \in X_i \text{ and } p(x) = y\right\}$. So ℓ is equal to u for some word $u \in \Sigma$ such that $\varsigma(u)(x_1) = x_1$. Set $w = w_{t_2}uw_{t_1}^{-1}$. Then $\varsigma(w)(t_1) = t_2$ and $w^{-1}h \in K$ and $\widetilde{w} = \widetilde{h}$.

It is clear that, if we start from a finite solution Y, then the special solution constructed above is also finite (since, in this case K has finite index). So we have proved:

Proposition 2.3 *If the problem $\mathcal{P}$ has a finite solution, then it has a finite special solution.*

In fact, we have proved (and will use) more than that. We will want to solve the extension problem, not in the class $\mathcal{C}$ of all $\mathcal{L}$-structures, but in a narrower class $\mathcal{C}_1$. Everything will go through, provided that there we can find a condition, denote it by (*), which is such that:

- if the solution is in $\mathcal{C}_1$, then the corresponding sequence $(H_1, H_2, \ldots, H_n)$ satisfies (*);
- if $(H_1, H_2, \ldots, H_n)$ satisfies (1) to (3) and (*), then the corresponding solution is in $\mathcal{C}_1$;
- (*) is a negative condition.

The classes of triangle free graphs and K_4-free graphs are typical examples of such classes.

It is easy to see that our special solution has the following further property: If the problem $\mathcal{P}$ has a finite solution $(Y, (\widetilde{p}\,; p \in P))$, then there is a finite special solution $(Z, (p^*\,; p \in P))$ and a weak homomorphism $\rho : Z \to Y$ such that for every $a \in Z$ and $p \in P$ $\rho(p^*(a)) = \widetilde{p}(\rho(a))$. For the definition of the notion of weak homomorphism see section 3.2; to define ρ use the equality $\rho(hK_i) = \varphi(h)(x_i)$.

3 From the EPPA to the profinite topology

In this section, we will show how to use the theorem concerning the extension property for automorphisms (to be proved in the next sections) to prove theorems about the profinite topology on free groups.

3.1 A proof of the theorem of Ribes and Zalesskiì

Let $\mathcal{L}$ be the language containing n unary predicate symbols, U_1, U_2,..., U_n and one binary predicate symbol, R. Let $\mathcal{C}$ the class of $\mathcal{L}$-structures M where:

1. the universe is the disjoint union of the sets U_i^M, for $i = 1, 2, \ldots, n$;
2. Rxy implies that, for some $i = 1, 2, \ldots, n-1$, $U_i x \wedge U_{i+1} y$ or $U_n x \wedge U_1 y$;
3. there are no $x_1, x_2, \ldots, x_n$ in M such that

$$M \models Rx_1x_2 \wedge Rx_2x_3 \wedge \ldots \wedge Rx_{n-1}x_n \wedge Rx_nx_1$$

The class $\mathcal{C}$ is called the class of cycle-free n-partitioned graphs.

Theorem 3.1 *The class $\mathcal{C}$ has the extension property for partial automorphisms.*

This theorem will be proved in the next section. We give a proof of the Ribes-Zalesskiì theorem from theorem 3.1 (using the techniques of the preceding section, one could also prove Theorem 3.1 from the theorem of Ribes and Zalesskiì).

Let $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of $F(P)$, and g an element of $F(P)$ not belonging to $H_1 \cdot H_2 \cdot \ldots \cdot H_n$. Let M be the following structure, in the above described language: the universe of M is the disjoint union of the sets $F(P)/H_i$, for $i = 1, 2, \ldots, n$; the interpretation of U_i is just $F(P)/H_i$; finally, for x and y in M, $M \models Rxy$ if and only if: either for some $i = 1, 2, \ldots, n-1$, $x \in F(P)/H_i$, $y \in F(P)/H_{i+1}$, and $x \cap y \neq \emptyset$ or $x \in F(P)/H_n$, $y \in F(P)/H_1$, and $xg^{-1} \cap y \neq \emptyset$.

The fact that $g \notin H_1 H_2 \ldots H_n$, implies that M is in $\mathcal{C}$. Indeed, assume, toward a contradiction, that we can find h_1, h_2, ..., h_n in $F(P)$ such that, for all $i = 1, 2, \ldots, n-1$ $M \models R\, h_i H_i\, h_{i+1} H_{i+1}$ and $M \models R\, h_n H_n\, h_1 H_1$. This implies that $h_1 H_1 \cap h_2 H_2 \neq \emptyset$, which means that $h_1^{-1} h_2 \in H_1 H_2$. Similarly, we see that $h_2^{-1} h_3 \in H_2 H_3$, ..., $h_{n-1}^{-1} h_n \in H_{n-1} H_n$. At last, $M \models R\, h_n H_n\, h_1 H_1$ implies $g \in H_1 h_1^{-1} h_n H_n$. We deduce that $g \in H_1 H_2 \ldots H_n$, a contradiction.

Let X_0 be a finite subset of $F(P)$ which contains g, a set of generators of H_i, for $i = 1, 2, \ldots, n$, and – assuming that these elements have been written as words of $P \cup P^{-1}$ – all final segments of these words. Let M_0 be the finite substructure of M whose universe is

$$\{xH_i \;; x \in X_0, i = 1, 2, \ldots, n\}.$$

For each $p \in P$, let $\bar{p}$ be the partial automorphism of M_0 defined by: for all $x \in M_0$, if $px \in M_0$, then $\bar{p}(x) = px$. If $px \notin M_0$, then $\bar{p}(x)$ is not defined. These partial automorphisms can obviously be extended to automorphisms of M, so by Theorem 3.1, we know that there exist a finite extension M_1 of M_0 in $\mathcal{C}_0$ and automorphisms $\widetilde{p}$ of M_1 extending $\bar{p}$. Let φ be the homomorphism from $F(P)$ into $\mathrm{Aut}(M_1)$ such that $\varphi(p) = \widetilde{p}$. We remark that, if h is one of the generators of one of the H_i, then $\varphi(h)H_i = hH_i$ (thanks to our precaution to have included in M_1 all the final segments of h). Similarly, $\varphi(g)H_i = gH_i$. Set, for $i = 1, 2, \ldots, n$, $K_i = \{h \in F(P)\,;\, \varphi(h)(H_i) = H_i\}$. Obviously, for $i = 1, 2, \ldots, n$, the subgroup K_i has a finite index in $F(P)$, and we have already pointed out that it contains H_i.

We conclude by showing that $g \notin K_1K_2 \ldots K_n$. Assume, toward a contradiction, that $g = k_1k_2 \ldots k_n$. Set $x_1 = H_1$, $x_2 = \varphi(k_1)(H_2)$, …, $x_n = \varphi(k_1k_2 \ldots k_{n-1})(H_n)$. Obviously, $M_1 \models R\, H_1\, H_2$, thus, since $\varphi(k_1)$ is an automorphism of M_1, $M_1 \models Rx_1x_2$. We see in a similar way that, for $i = 1, 2, \ldots, n-1$, $M_1 \models Rx_ix_{i+1}$. Finally, from the fact that $M_1 \models R\; gH_n\; H_1$, we deduce that $M_1 \models R\; \varphi(g)(H_n)\; H_1$, that is $M_1 \models Rx_nx_1$. Thus, M_1 is not cycle-free, a contradiction.

♡

3.2 Statement of the main combinatorial theorem

Before going further, we will need to set up some more notation. In this subsection we will consider a finite relational language $\mathcal{L}$.

If M and M' are $\mathcal{L}$-structures, a weak homomorphism from M to M' is a map h from M to M' which is such that: if n is an integer, R an n-ary predicate symbol of $\mathcal{L}$ and $a_1, a_2, \ldots, a_n$ are elements of M such that $M \models Ra_1a_2 \ldots a_n$, then $M' \models Rh(a_1)h(a_2) \ldots h(a_n)$. If A is a substructure of both M and M', a weak A-homomorphism is a weak homomorphism which leaves the elements of A fixed.

To denote that h is a weak homomorphism from M into M', we write: $h : M \underset{w}{\longrightarrow} M'$. To denote that it is a weak A-homomorphism, we write: $h : M \underset{w,A}{\longrightarrow} M'$.

If M is a structure, a link of M is a tuple $(R, a_1, a_2, \ldots, a_n)$ where R is a n-ary predicate symbol of the language $\mathcal{L}$, $a_1, a_2, \ldots, a_n$ are elements of M and $M \models Ra_1a_2 \ldots a_n$. We say that an element a belongs to or is contained in a link $(R, a_1, a_2, \ldots, a_n)$ if a is one of the a_i.

If M is a $\mathcal{L}$-structure and $\mathcal{T}$ a set of $\mathcal{L}$-structures, we say that M is $\mathcal{T}$-free if there is no structure $T \in \mathcal{T}$ and weak homomorphism $h : T \underset{w}{\longrightarrow} M$.

We can now state a general combinatorial theorem, that will be proved in section 5.

Theorem 3.2 *Let $\mathcal{L}$ be a finite relational language and $\mathcal{T}$ a finite set of finite $\mathcal{L}$-structures. Then the class of $\mathcal{T}$-free $\mathcal{L}$-structures has the EPPA.*

3.3 Back to the free groups

A natural question is: is there a generalisation of the theorem of Ribes-Zalesskiĭ that can be proved using Theorem 3.2 or even which is "equivalent" to it? The answer is yes for both questions, and that is what we are going to expose now.

If H is a subgroup of $F(P)$ and x and y are two elements of $F(P)$, we write $x \equiv y \mod H$ for $xH = yH$.

Let $n \in \omega$ and X be a finite set (the set of unknowns). A left-system is a finite set of equations of the form

$$x \equiv_i y \cdot g \quad \text{where } i \in \{1, 2, \dots n\},\ x,\ y \in X \text{ and } g \in F(P)$$

or of the form

$$x \equiv_i g \quad \text{where } i \in \{1, 2, \dots n\},\ x \in X \text{ and } g \in F(P)$$

Let $\mathcal{H} = (H_i, H_2, \dots, H_n)$ be a sequence of subgroups of $F(P)$. A solution of a left-system (E) in $F(P)$ modulo $\mathcal{H}$ is a family $(g_x\ ;\ x \in X)$ of elements of $F(A)$ such that, for each equation $x \equiv_i y \cdot g$ in (E), $g_x \equiv g_y \cdot g \mod H_i$, and for every $x \equiv_i g$ in (E), $g_x \equiv g \mod H_i$.

Theorem 3.3 *Let $n \in \omega$, $\mathcal{H} = (H_1, H_2, \dots, H_n)$ be a sequence of finitely generated subgroups of $F(P)$ and (E) be a left-system. Assume that (E) has no solution in $F(P)$ modulo $\mathcal{H}$. Then there exist subgroups $K_1, K_2, \dots, K_n$ of finite index in $F(P)$ such that $H_i \subseteq K_i$ for all i, $1 \leq i \leq n$ and such that (E) has no solution in $F(P)$ modulo $\mathcal{K} = (K_1, K_2, \dots, K_n)$.*

We remark that this theorem immediately implies the theorem of Ribes and Zalesskiì: the fact that an element g of $F(P)$ does not belong to $H_1 \cdot H_2 \cdot \ldots \cdot H_n$ means exactly that the left-system

$$\begin{cases} x_n & \equiv_n & g \\ x_{n-1} & \equiv_{n-1} & x_n \\ \dots & & \\ x_2 & \equiv_2 & x_3 \\ x_2 & \equiv_1 & e \end{cases}$$

has no solution in $F(P)$ modulo $(H_1, H_2, \dots, H_n)$.

If H is a subgroup of $F(P)$ and x and y are elements of $F(P)$, we will write $x \sim_H y$ for $HxH = HyH$. We notice that the relation $\sim_H$ is an equivalence relation. We first want to replace the left-system by another kind of system, easier to manage for our purpose. A double-system is a finite set of equations of the form

$$x^{-1} \cdot y \sim_i g \ \text{ where } i = 1, 2, \dots n,\ x,\ y \in X \text{ and } g \in F(P)$$

or of the form

$$x \sim_i g \text{ where } i = 1, 2, \dots n,\ x \in X \text{ and } g \in F(P)$$

Let $\mathcal{H} = (H_1, H_2, \dots, H_n)$ be a sequence of subgroups of $F(P)$. A solution of a double-system (E) in $F(P)$ modulo $\mathcal{H}$ is a family $(g_x\ ;\ x \in X)$ of elements of $F(P)$ such that, for every $x^{-1}y \sim_i g$ in (E), $g_x^{-1} \cdot g_y \sim_{H_i} g$, and for every $x \sim_i g$ in (E), $g_x \sim_{H_i} g$.

We will prove

Proposition 3.4 *Let (F) be a double-system, H_1, H_2, $\dots, H_n$ be finitely generated subgroups of $F(P)$. If the double-system (F) has no solution in $F(P)$ modulo $(H_1, H_2, \dots, H_n)$, then there exist subgroups $K_1, K_2, \dots, K_n$ of finite index in $F(P)$ such that $H_i \subseteq K_i$ for all i, $1 \leq i \leq n$ and such that (F) has no solution in $F(P)$ modulo $(K_1, K_2, \dots, K_n)$.*

We show how to prove Theorem 3.3 from Proposition 3.4. We see that the left-system (E) can be translated into a double-system. Let (F) be the double-system obtained by replacing each equation $x \equiv_i y \cdot g$ of (E) by the two equations:

$$\begin{cases} z^{-1} \cdot x \sim_i e \\ y^{-1} \cdot z \sim_0 g \end{cases}$$

where z is a new unknown (of course, different z should be taken for different equations). In the same way, $x \equiv_i g$ should be replaced by

$$\begin{cases} z^{-1}x \sim_i e \\ z \sim_0 g \end{cases}$$

In this translation, we have to introduce a new subgroup H_0 which will be the trivial subgroup.

It is clear that the double-system (F) has a solution modulo $(H_0, H_1, \ldots, H_n)$ if and only if the original left-system (E) has a solution modulo $(H_1, H_2, \ldots, H_n)$; thus (F) has no solution. By Proposition 3.4, there exist subgroups $K_0, K_1, \ldots, K_n$ of finite index such that $H_i \subseteq K_i$ for all i, $0 \le i \le n$ and such that (F) has no solution modulo $(K_0, K_1, \ldots K_n)$. Again, this implies that the system (E) has no solution modulo $(K_1, K_2, \ldots K_n)$.

Incidently, we notice that a double-system can also easily be translated into a left-system, so that Theorem 3.3 and Proposition 3.4 have exactly the same content.

It remains to prove Proposition 3.4.

Let (F) be a double-system. Write X for its set of unknowns, and let $\mathcal{H} = (H_1, H_2, \ldots, H_n)$ be a sequence of finitely generated subgroups such that (F) has no solution modulo $(H_1, H_2, \ldots, H_n)$. We first remark that we can assume that there is no equation of the form $x \sim_i g$ (so that there will remain only "homogeneous" equations, of the form $y^{-1} \cdot x \sim_i g$). Indeed, add one new unknown z (only one all together), and replace each equation of the form $x \sim_i g$ by $z^{-1} \cdot x \sim_i g$. If the new system had a solution $(g_z, g_x\,;\, x \in X)$, then $(g_z^{-1} \cdot g_x\,;\, x \in X)$ would be a solution of the original system.

We consider the following relational language $\mathcal{L}$: it contains:

- $n+1$ unary predicate symbols U_0, U_1, …, U_n;
- n binary predicate symbols T_i;
- for each equation $E : x^{-1}y \sim_i g$ in (F), a binary predicate symbol R_E.

We now define a structure M:

- its base set is the disjoint union of the sets U_i^M, the interpretations of U_i in M, and U_i^M is the set $F(P)/H_i$ for $i = 1, 2, \ldots, n$, and $U_0^M = F(P)$.
- for $i = 1, 2, \ldots, n$, and $x, y \in M$, $M \models T_i xy$ if and only if $x \in U_0^M$, $y \in U_i^M$ and $x \in y$.
- For all α and β in M and equation $E : x^{-1}y \sim_i g$, $M \models R_E\alpha\beta$ if and only if $\alpha, \beta \in F(P)/H_i$ and $g \in \alpha^{-1} \cdot \beta$ (in other words, if and only if there exist $a \in \alpha$ and $b \in \beta$ such that (equivalently, for all $a \in \alpha$ and $b \in \beta$) $H_i a^{-1} b H_i = H_i g H_i$).

We first notice that for all $h \in F(P)$ the left multiplication by h is an automorphism of M. Call it $\widehat{h}$.

Second, we exploit the fact that (F) has no solution modulo $(\mathcal{H})$. We cannot find elements $c(x,i)$ in M, for $x \in X$ and $0 \le i \le n$ such that the following set of conditions is satisfied:

$$(*) \quad \begin{cases} 1.\ \text{for all } x \in X \text{ and } i,\ 0 \le i \le n,\ M \models U_i c(x,i); \\ 2.\ \text{for all } x \in X, \text{ and } i = 1, 2, \ldots, n,\ M \models T_i\ c(x,0)\ c(x,i); \\ 3.\ \text{If } E : x^{-1} \cdot y \sim_i g \text{ belongs to } (F) \text{ then } M \models R_E\ c(x,i)\ c(y,i). \end{cases}$$

Otherwise, $(c(x,0)\,;\, x \in X)$ would be a solution of (F) mod $\mathcal{H}$.

Write N for the $\mathcal{L}$-structure whose base-set is the set $\{c(x,i)\ ;\ x \in X, 1 \leq i \leq n\}$ and where the only relations are those necessary to make the conditions (*) true. Another way to say that (F) has no solution modulo $(\mathcal{H})$ is to say that N cannot be weakly embedded in M.

Let now C_0 be a finite subset of $F(P)$ containing the parameters occurring in the equations of (F) and for each i, $1 \leq i \leq n$ a set generating H_i. For each element of $c \in C_0$, write it in the form $p_0 \cdot p_1 \cdots \cdot p_m$ with the p_i in $P \cup P^{-1}$, and let P_c be the set $\{p_1 p_2 \cdots p_m, p_2 p_3 \cdots p_m, \ldots, p_{m-1} p_m, p_m\}$. Set $C = \bigcup_{c \in C_0} P_c$ and let M_0 be the substructure of M whose base set is $C \cup \{cH_i\ ;\ c \in C,\ 1 \leq i \leq n\}$. For all $p \in P$, we can define a partial automorphism $\widetilde{p}$ on M_0 as the restriction of $\widehat{p}$, the left multiplication by p, to M_0.

Applying Theorem 3.2, we deduce that there exists a finite $\mathcal{L}$-structure M_1, extending M_0, inside which N cannot be weakly embedded and for each $p \in P$, an automorphism $\widetilde{p}$ of M_1 extending p. Then, there is a natural group homomorphism $h \longmapsto \widetilde{h}$ from $F(P)$ onto $\mathrm{Aut}(M_1)$. The point is that for $i = 1, 2, \ldots, n$, and $c \in C$, $\tilde{c}(H_i) = cH_i$. In particular, this is true for a set generating H_i. In other words, if we set

$$K_i = \left\{ h \in F(P)\ ;\ \widetilde{h}(H_i) = H_i \right\}$$

then K_i contains H_i. Moreover, K_i is a subgroup of finite index of $F(P)$.

Set $\mathcal{K} = (K_1, K_2, \ldots, K_n)$. We shall conclude by proving that (F) has no solution modulo $\mathcal{K}$.

Assume, for a contradiction, that $(g_x\ ;\ x \in X)$ is a solution of (F) modulo $\mathcal{K}$. For $x \in X$ and i, $1 \leq i \leq n$, set $x_i = \widetilde{g_x}(H_i)$ and $x_0 = \widetilde{g_x}(e)$. We prove that the x_i satisfy the conditions (*), and thus that N is weakly embedded in M_1, which is contradictory.

1. For all $x \in X$ and i, $1 \leq i \leq n$, we have $M_1 \models U_i(\widetilde{g_x}(H_i))$ since $M_1 \models U_i H_i$ and $\widetilde{g_x}$ is an automorphism. Similarly, $M_1 \models U_0 \widetilde{g_x}(e)$

2. For $i = 1, 2, \ldots n$ and $x \in X$, we have $M_1 \models T_i\ \widetilde{g_x}(e)\ \widetilde{g_x}(H_i)$ since $M_1 \models T_i e H_i$ and $\widetilde{g_x}$ is an automorphism.

3. If $E : x^{-1}y \sim_i g$ is in (F), then there exist k, k' in K_i such that $(g_x)^{-1} \cdot g_y = k \cdot g \cdot k'$, so $\widetilde{g_x}^{-1} \cdot \widetilde{g_y} \cdot H_i = \widetilde{k \cdot g \cdot k'} \cdot H_i = \widetilde{k \cdot g} \cdot H_i = \widetilde{k} \cdot g \cdot H_i$. It is clear that $M_1 \models R_E\ H_i\ gH_i$, and since $\widetilde{k}$ is an automorphism of M_1, $M_1 \models R_E\ \ \widetilde{k}H_i\ \ \widetilde{k}gH_i$, thus $M_1 \models R_E\ \ H_i\ \ \widetilde{g_x}^{-1}\widetilde{g_y}H_i$. Now, $\widetilde{g_x}$ is an automorphism, so $M_1 \models R_E\ \ \widetilde{g_x}(H_i)\ \ \widetilde{g_y}(H_i)$.

♡

We just proved Theorem 3.3 using Theorem 3.2 and we will prove Theorem 3.2 directly in the following section. But let us point out that the method of section 2 provides a short proof of Theorem 3.2 using Theorem 3.3. One has to translate the condition of being $\mathcal{T}$-free into a finite system of equations.

We can give an alternative formulation to Theorem 3.3: fix the finite alphabet P and consider equations of the form:

$$x \equiv y \cdot v \qquad \mathrm{mod}\, \langle w_1, w_2, \ldots, w_n \rangle$$

where the x and y are unknowns and the v and the w are words in the alphabet $P \cup P^{-1}$. Given a group G and, for each $p \in P$, a value $\bar{p}$ of p, this equation has an obvious meaning: each of the words v or w_i occurring in these equations is interpreted by the element of G obtained by replacing the p by the $\bar{p}$, and we must find values in G for the unknowns satisfying all the equations. Then Theorem 3.3 can be rephrased as follows:

Theorem 3.5 *Let S be a finite set of equations of the above form. If for all finite groups G and interpretations of the p in G, the system S has a solution, then for every group G' and for all interpretations of the p in G' the system S has a solution.*

4 Extension Lemmata

4.1 A simple combinatorial proof of the theorem of Hrushovski

In this subsection, we will give a simple combinatorial proof of Theorem 1.4. We begin with two definitions.

Definition 4.1 *Let X be a finite set and n a positive integer; then $\Gamma(X,n)$ denotes the graph whose base is $[X]^n$, the set of subsets of X of cardinality exactly n, and where the binary relation R is defined by: for $a,b \in [X]^n$, Rab if and only if $a \cap b \neq \emptyset$.*

Definition 4.2 *A subgraph Γ_0 of $\Gamma(X,n)$ is poor if: 1) for all $x \in X$, $\mathrm{card}\{a \in \Gamma_0; x \in a\} \leq 2$; 2) for all $a,b \in \Gamma_0$, if $a \neq b$ then $\mathrm{card}(a \cap b) \leq 1$.*

If α is a permutation of X, we will denote by α^* the induced automorphism of $\Gamma(X,n)$.

The theorem is a immediate consequence of the two following lemmata.

Lemma 4.3 *Every finite graph is poorly represented: if Γ is a finite graph, then there exist a finite set X, a positive integer n and a poor subgraph Γ_0 of $\Gamma(X,n)$ isomorphic to Γ.*

Proof. Let X_0 be the set of edges of Γ. For each point a in Γ, let $f(a) = \{x \in X_0 \; ; \; x \text{ is adjacent to } a\}$. If the cardinality of $f(a)$, for $a \in \Gamma$, is constant equal to n bigger than one, then we are done, because f is an isomorphism from Γ to a poor subgraph of $\Gamma(X_0,n)$. In the general case, let $n = \sup(\sup(f(a) \; ; \; a \in \Gamma), 2)$ and let X be a finite set containing X_0 and sufficiently large so that it is possible to define a map h from Γ to $[X]^n$ such that for all distinct a and b in Γ: $f(a) \subseteq h(a)$, $h(a) - f(a) \subset X - X_0$, $(h(a) - f(a)) \cap (h(b) - f(b)) = \emptyset$.

♡

Lemma 4.4 *Let Γ_0 and Γ_1 be two poor subgraphs of $\Gamma(X,n)$, and f an isomorphism from Γ_0 to Γ_1. Then there exist a permutation α of X such that α^* extends f.*

Proof. First define $\alpha(x)$ for $x \in X$ belonging to 2 elements a and b of Γ_0: there is no choice, it has to be the unique element of $f(a) \cap f(b)$; then, for all $a \in \Gamma_0$, extend α to a by defining a bijection between

$$\{x \in a \; ; \; \text{for all } b \in \Gamma_0 - \{a\}, \, x \notin b\}$$

and

$$\{x \in f(a) \; ; \; \text{for all } b \in \Gamma_1 - \{f(a)\}, \, x \notin b\}$$

This is possible because these two sets have the same cardinality. Then extend to a permutation of X.

♡

Remark: in his paper, Hrushovski remarks that the cardinality of the resulting homogeneous graph Z is bounded by something like 2^{2^k}, if k is the cardinality of Γ. He asks whether it is possible to find a graph Z of cardinality bounded by 2^{ck^2}, for some constant c.

The above proof show that, in fact the graph Z can be found of cardinality less than k^{2k}. We will make the precise computation.

Let k be the cardinality of Γ and n the valency of Γ, that is the maximal number of edges adjacent to a given vertex. The "homogeneous" graph Z is a graph $\Gamma(X,n)$; let us compute precisely the cardinality of X. Let m be the number of edges of Γ, and for every $a \in \Gamma$, $c(a)$ the number of edges adjacent to a. We have

$$\sum_{a\in\Gamma} c(a) = 2m$$

On the other hand, the set X is the disjoint union of the set of edges and, for all $a \in \Gamma$, of a set of cardinality $n - c(a)$. So the cardinality of X is

$$m + \sum_{a\in\Gamma}(n - c(a)) = nk - m$$

and the cardinality of Z is bounded by $(nk)^n$.

So, for graphs of bounded valency, the cardinality of Z is bounded polynomially in the cardinality of Γ (but we should be careful that the graph Z has a much bigger valency).

If we do not want to take the valency into account, a first estimation gives k^{2k} for the bound of the cardinality of Z. But we can get a slightly better bound: we may assume that m the number of edges in Γ is bigger than or equal to the number of non-edges, so the cardinality of X can be bounded by $3k^2/4$ and the cardinality of Z can be bounded by

$$\left(\frac{3k^2}{4}\right)^k \times \frac{1}{k!}$$

Remarking that $k! \geq (k/e)^k$, we see that the cardinality of Z can be bounded by $(3ek/4)^k$.

4.2 Generalisation to arbitrary relational languages

Theorem 4.5 *Let $r > 1$. Suppose the language $\mathcal{L}$ consists of just one r-ary predicate R. Let A be an $\mathcal{L}$-structure of cardinality c. There exists an $\mathcal{L}$-structure B with $A \subseteq B$ and* $\operatorname{card}(B) \leq 2^{r!rc^r}$ *such that every partial automorphism on A extends to an automorphism of B.*

Before we prove the theorem we give some helpful definitions.

Definition 4.6 *Let X be a finite set. We define the $\mathcal{L}$-structure $M(X)$: its domain is $(\wp(X))^r$, so its elements are r-tuples of subsets of X. For an element $a \in M(X)$ we denote by a_j the j-th coordinate of a $(1 \leq j \leq r)$. We define the r-ary relation R on $M(X)$. For $a^1, \ldots, a^r \in M(X)$:*

$$M(X) \models R(a^1, \ldots, a^r) \textit{ if and only if } \bigcap_{1\leq i\leq r} a^i_i \neq \emptyset$$

Note that the group $\operatorname{Sym}(X)$ of permutations of X acts as automorphisms on $M(X)$.

Definition 4.7 *Let k be an integer. We say that a substructure N of $M(X)$ is k-regular, if there exists an integer $p_k > 0$ such that, for every $a^1, \ldots, a^k \in N$ and every $i_1, \ldots, i_k \in \{1, \ldots, r\}$, if $i_1, \ldots i_k$ are pairwise distinct, then* $\operatorname{card}(\bigcap_{j=1}^k a^j_{i_j}) = p_k$. *We say that N is regular, if*

- *it is k-regular for every $k < r$;*
- *for $a, b \in N$ distinct and $1 \leq i \leq r$ we have $a_i \cap b_i = \emptyset$;*
- *for $a^1, \ldots, a^r \in N$, if $Ra^1 \ldots a^r$ then* $\operatorname{card}(\bigcap_{i=1}^r a^i_i) = 1$.

For m an integer we denote by $M(X,m)$ the substructure of M consisting of all r-tuples of sets of size m. Note that if N is a 1-regular substructure of $M(X)$ then there exists an m such that N is a substructure of $M(X,m)$. Furthermore $\mathrm{Sym}(X)$ also acts on $M(X,m)$.

The theorem follows immediately from the following two Lemmata:

Lemma 4.8 *Let A be an $\mathcal{L}$-structure of cardinality c. There exist a set X with* $\mathrm{card}(X) \leq r!c^r$ *and a regular substructure of $M(X)$ which is isomorphic to A.*

Lemma 4.9 *If N is a regular substructure of $M(X)$, then every partial automorphism of N extends to an automorphism of $M(X)$, which is induced by the action of* $\mathrm{Sym}(X)$ *on $M(X)$.*

Proof of the first Lemma:

Let X_0 be the set of links of A (here, the r-tuple $(a^1, a^2, \ldots, a^r)$ of A such that $A \models Ra^1a^2\ldots a^r$). We first embed A into $M(X_0)$: for each $a \in A$ we let $\alpha(a) = (t_1, \ldots, t_r)$, where $t_i = \{q \in X_0; q_i = a\}$. We first remark that the last two conditions for regularity are satisfied. We will increase the set X_0 step by step and change the embedding α such that $\alpha[A]$ becomes k-regular for every k with $r-1 \geq k \geq 1$. Do not worry that α is not necessarily an embedding to begin with. All the isolated points get mapped to the same r-tuple. In our construction we will maintain the condition, that for $a, b \in A$ distinct and $1 \leq j \leq r$ we have $\alpha(a)_j \cap \alpha(b)_j = \emptyset$. So the final mapping α will be injective.

We are first aiming for $(r-1)$-regularity. Consider all sets of the form $\bigcap_{1\leq j\leq r-1} \alpha(a^j)_{i_j}$ for $a^1, a^2, \ldots, a^{r-1} \in A$ and $i_1, \ldots, i_{r-1} \in \{1, \ldots, r\}$ pairwise distinct. Let v be the maximal cardinality of these sets. You may think of v as the (maximal) valency of the $\mathcal{L}$-structure A. We let $p_{r-1} = v$. Easily $p_{r-1} \leq c$. For every $a^1, \ldots, a^{r-1} \in A$ and $i_1, \ldots, i_{r-1} \in \{1, \ldots, r\}$ pairwise distinct, if $\mathrm{card}(\bigcap_{j=1}^{r-1} \alpha(a^j)_{i_j}) = p' < p_{r-1}$ we choose $(p_{r-1} - p')$ new elements which we add to each set $\alpha(a^j)_{i_j}$ (for $1 \leq j \leq (r-1)$). As every new element will belong to exactly $(r-1)$ sets of the form $\alpha(a)_j$ with different j each, the second two conditions for regularity will remain true. If we let X_1 be the set X_0 together with the new points, and if we change α as indicated we will have $\alpha[A]$ is $(r-1)$-regular in $M(X_1)$. Note that $X_1 = \bigcup\{\bigcap_{j=1}^{r-1} \alpha(a^j)_{i_j}; a^1, \ldots, a^{r-1} \in A, i_1, \ldots, i_{r-1} \in \{1, \ldots, r\}$ pairwise distinct$\}$.

Now we suppose we already have constructed a set X_t $(1 \leq t < (r-1))$ and an embedding $\alpha : A \hookrightarrow M(X_t)$ which is j-regular for $(r-1) \geq j \geq (r-t)$, the constant for $(r-t)$-regularity being $p_{r-t} = t! \cdot c^{t-1} \cdot v$. Furthermore we suppose $X_t = \bigcup\{\bigcap_{j=1}^{r-t} \alpha(a^j)_{i_j}; a^1, \ldots, a^{r-t} \in A, i_1, \ldots, i_{r-t} \in \{1, \ldots, r\}$ pairwise distinct$\}$. Now we consider all sets of the form $\bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j}$ for $a^1, \ldots, a^{r-(t+1)} \in A$ and $i_1, \ldots, i_{r-(t+1)} \in \{1, \ldots, r\}$ pairwise distinct. As we have

$$\bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j} = \bigcup\{\bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j} \cap \alpha(b)_k; b \in A, k \in \{1, \ldots, r\} - \{i_1, \ldots, i_{r-(t+1)}\}\},$$

we have $\mathrm{card}(\bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j}) \leq (t+1) \cdot c \cdot p_{r-t}$. This means we can set $p_{r-(t+1)} = (t+1)! \cdot c^t \cdot v$. Now we add $(p_{r-(t+1)} - q)$ many new points to a set of the form $\bigcap_{j=1}^{r-(t+1)} \alpha(a^j)_{i_j}$ of cardinality q to define the set X_{t+1}.

At the end we constructed a set $X = X_{r-1}$ and an embedding $\alpha : A \hookrightarrow M(X)$ such that $\alpha[A]$ is regular, the constant for 1-regularity being $p_1 = (r-1)!c^{r-2}v$. Every point of X is of the form $\alpha(a)_i$ with $a \in A$ and $i \in \{1, \ldots, r\}$. That means $\mathrm{card}(X) \leq r \cdot c \cdot p_1 = r! \cdot c^{r-1} \cdot v \leq r! \cdot c^r$. This proves the Lemma. Note that we could get a slightly better bound in the theorem by letting $B = M(X, p_1)$: $\mathrm{card}(B) \leq (e \cdot r \cdot c)^{(r! \cdot c^{r-2} \cdot v)} \leq (erc)^{r!c^{r-1}}$.

♡

Before we prove Lemma 4.9, we do a little preparation:

Definition 4.10 *Let X, Y be sets of the same cardinality and let q be a partial function from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. We say q is induced by a bijection $\pi : X \to Y$, if for every a in the domain of q $q(a) = \pi[a]$.*

Lemma 4.11 *Let X, Y be finite sets of the same cardinality and let q be a partial function from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. Then q is induced by a bijection $\pi : X \to Y$ if and only if for every subset s of $\mathrm{dom}(q)$, the domain of q, we have: $\mathrm{card}(\bigcap_{a \in s} a) = \mathrm{card}(\bigcap_{a \in s} q(a))$.*

Suppose for every $s \subseteq \mathrm{dom}(q)$, $\mathrm{card}(\bigcap_{a \in s} a) = \mathrm{card}(\bigcap_{a \in s} q(a))$. We can suppose $\mathrm{dom}(q)$ is closed under intersection, as for $a, b \in \mathrm{dom}(q)$ we can define $q(a \cap b)$ to be $q(a) \cap q(b)$ and still maintain the condition on q. We can also suppose $\mathrm{dom}(q)$ is closed under complements, as for every $a \in \mathrm{dom}(q)$ we can define $q(a^c) = q(a)^c$. Here we are using the finiteness of X and Y. Finally we can suppose that for every $x \in X$, $\{x\} \in \mathrm{dom}(q)$: for $x \in X$ we define $s_x = \{b \in \mathrm{dom}(q) \mid x \in b\}$ and by the condition on q, we have $\bigcap_{b \in s_x} q(b) \neq \emptyset$. We choose $y \in \bigcap_{b \in s_x} q(b)$ and let $q(\{x\}) = \{y\}$ and check that we still have the condition on q. This means we can assume $\mathrm{dom}(q) = \mathcal{P}(X)$ and in this case we can define π by letting $\pi(x)$ be the unique element of $q(\{x\})$. Easily we have for every $a \in \mathcal{P}(X)$ $q(a) = \pi[a]$.

♡

Now we prove the Lemma 4.9: Let p be a partial automorphism of N with domain D. Define a partial map q from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ with domain $\{a_i;\ a \in D,\ 1 \leq i \leq r\}$ by defining for $a \in D$ and $1 \leq i \leq r$: $q(a_i) = p(a)_i$. Now let $k \in \omega$ and let $a^1, \ldots a^k \in D$ and $j_1, \ldots, j_k \in \{1, \ldots, r\}$. We want to check that $\mathrm{card}(a^1_{j_1} \cap \ldots \cap a^k_{j_k}) = \mathrm{card}(p(a^1)_{j_1} \cap p(a^2)_{j_2} \cap \ldots \cap p(a^k)_{j_k})$. We can suppose that $(a^1, j_1), \ldots, (a^k, j_k)$ are pairwise distinct. Also we can suppose that $j_1, \ldots j_k$ are pairwise distinct as otherwise both cardinalities are 0 by the second condition for regularity. If $k < r$ then by k-regularity both cardinalities are equal to p_k. If $k = r$ by changing the enumeration we can suppose that $j_1 = 1, \ldots, j_r = r$. In that case only the cardinalities 0 and 1 appear. We have $\mathrm{card}(a^1_1 \cap \ldots \cap a^r_r) = 1$ if and only if $Ra^1 a^2 \ldots a^r$ if and only if $R\ p(a^1)\ p(a^2)\ \ldots\ p(a^r)$ if and only if $\mathrm{card}(p(a^1)_1 \cap \ldots \cap p(a^r)_r) = 1$.

Thus q and therefore also p is induced by a permutation of X.

♡

The following lemma shows how to compute bounds for bigger languages:

Lemma 4.12 *Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be a finite relational language. Suppose $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$. Let A be an $\mathcal{L}$-structure. Let B_1 be an $\mathcal{L}_1$-structure with $A \subset B_1$ such that every partial automorphism of $A_{|\mathcal{L}_1}$ can be extended to an automorphism of B_1 and let B_2 be an $\mathcal{L}_2$-structure with $A \subset B_2$ such that every partial automorphism of $A_{|\mathcal{L}_2}$ can be extended to an automorphism of B_2. Suppose $\mathrm{card}(B_1) = n_1$ and $\mathrm{card}(B_2) = n_2$.*

There exists an $\mathcal{L}$-structure B with $\mathrm{card}(B) = n_1 \cdot n_2$ and $A \subset B$, such that every partial automorphism of A extends to an automorphism of B.

Proof. The domain of B is $B_1 \times B_2$.

For $R \in \mathcal{L}_1$, r-ary and $(a^1_1, a^1_2), \ldots, (a^r_1, a^r_2) \in B$ we define:

$$R^B\ (a^1_1, a^1_2)\ \ldots\ (a^r_1, a^r_2) \Longleftrightarrow R^{B_1} a^1_1 \ldots a^r_1$$

and for $R \in \mathcal{L}_2$, r-ary and $(a^1_1, a^1_2), \ldots, (a^r_1, a^r_2) \in B$ we define:

$$R^B\ (a^1_1, a^1_2) \ldots\ (a^r_1, a^r_2) \Longleftrightarrow R^{B_2} a^1_2 \ldots a^r_2.$$

Note that for $\alpha_1 \in \mathrm{Aut}(B_1)$ and $\alpha_2 \in \mathrm{Aut}(B_2)$ we have that $(\alpha_1, \alpha_2) \in \mathrm{Aut}(B)$.

Let $i_1 : A_{|\mathcal{L}_1} \hookrightarrow B_1$ and $i_2 : A_{|\mathcal{L}_2} \hookrightarrow B_2$ be the inclusion mappings. Then $(i_1, i_2) : A \hookrightarrow B$ is an embedding of $\mathcal{L}$-structures.

Via this embedding we can suppose that $A \subset B$. Let p be a partial automorphism of A. As p is a partial automorphism of $A_{|\mathcal{L}_1}$ it can be extended to an automorphism α_1 of B_1 and for the analogous reason to an automorphism α_2 of B_2. Now $(\alpha_1, \alpha_2) \in \mathrm{Aut}(B)$ extends p.

♡

Corollary 4.13 *Let $\mathcal{L}$ be a finite relational language. Let r be the maximal arity of symbols in $\mathcal{L}$. For $1 \le j \le r$ let l_j be the number of j-ary symbols in $\mathcal{L}$. Let A be a finite $\mathcal{L}$-structure of cardinality c. There exists a structure B such that $A \subset B$ and every partial automorphism of A extends to an automorphism of B and* $\mathrm{card}(B) \le c \cdot 2^{p(c)}$ *where* $p(c) = \sum_{j=2}^{r} l_j j! j c^j$.

The proof consists of putting together Theorem 4.5 and Lemma 4.12 and observing that if R only consists of unary predicates then $B = A$ will do.

♡

4.3 The case of the cycle-free n-partitioned graph

The language L consists of n unary predicates $U_1, \ldots, U_n$ and one binary predicate R. We write $a \to b$ for Rab. Denote by $\mathcal{K}$ the class of cycle-free, n-partitioned directed graphs.

If M is in $\mathcal{K}$ and $a, b \in M$ then we define $a \to_M^* b$ if there exist $j \le k$ and $a_j \in U_j^M, \ldots, a_k \in U_k^M$ such that $a = a_j \to a_{j+1} \to \ldots \to a_k = b$.

Lemma 4.14 *$\mathcal{K}$ has the EPPA.*

Proof. Let $A \in \mathcal{K}$ be finite, and let $p_1, \ldots, p_t$ be partial automorphisms on A. We assume there exist $M \in \mathcal{K}$ and $g_1, \ldots, g_t \in \mathrm{Aut}(M)$ such that g_i extends p_i. We let D_i be the domain of p_i and D_i' be its range.

First step.

Without loss of generality we can suppose that for $a, b \in A$, if there exists an i such that $a, b \in D_i$ or $a, b \in D_i'$, then $a \to_M^* b$ if and only if $a \to_A^* b$. (Just put enough points as witnesses into the structure A without extending p_i.) Now for every $a, b \in D_i$: $a \to_A^* b$ if and only if $p_i(a) \to_A^* p_i(b)$ (because g_i is an automorphism it is clear that $a \to_M^* b$ if and only if $g_i(a) \to_M^* g_i(b)$.) This will be the only place where we use the existence of the structure M; later we will only use the fact that now p_i also respects $\to_A^*$. Now we can forget the structure M and use $\to^*$ for $\to_A^*$

Second step.

For $U \subseteq A$ define $\mathrm{cl}(U) := \{b \in A \ ; \ \exists a \in U : \ a \to^* b\}$. We say U is closed, if $\mathrm{cl}(U) = U$. The first level U_1^A of our structure will play a special role in the argument, we define $A_u := A - U_1^A$ (the upper part of A). We order $\mathbb{N}^{n-1}$ lexicographically and define a dimension function $\dim : \wp(A_u) \to \mathbb{N}^{n-1}$ by:

If U is closed then $\dim_2(U) = \mathrm{card}(U \cap U_2^A)$ and for $k \ge 2$ $\dim_{k+1}(U) = \mathrm{card}((U - \mathrm{cl}(U \cap U_k^A)) \cap U_{k+1}^A)$ and $\dim(U) := (\dim_2(U), \ldots, \dim_n(U))$; finally if U is arbitrary: $\dim(U) := \dim(\mathrm{cl}(U))$. Informally $\dim_k(U)$ counts the points in the k-th level of U, which are not already in the closure of a point of lower level of U. If $U \subseteq A_u$ is arbitrary one can use the following method, to determine $\dim(U)$: call $b \in U$ a root of U if there does not exist an element $a \in U$ such that $a \ne b$, $a \to^* b$. Then $\dim_k(U) = \mathrm{card}(\{b \in U \cap U_k^A \ ; \ b \text{ a root of } U\})$.

It is not hard to check, that

- $U \subseteq V$ implies $\mathrm{cl}(U) \subseteq \mathrm{cl}(V)$;
- $U \subseteq V$ implies $\dim(U) \leq \dim(V)$;
- $\mathrm{cl}(U) \subsetneq \mathrm{cl}(V)$ implies $\dim(U) < \dim(V)$;
- if $U \subseteq D_i$ then $\dim(U) = \dim p_i(U)$.

For the last statement use the fact that p_i respects the relation $\to^*$ and use the above mentioned method to compute the dimension.

Lemma 4.15 *There exist a set C of unary predicates (called colors), containing for every $a \in U_1^A$ a color Q_a and an expansion $(A, (Q^A)_{Q \in C})$ of the structure A (a coloring of A), such that:*

- $Q_a^A = \mathrm{cl}(a)$;
- *For $Q \in C - \{Q_a \ ; \ a \in U_1^A\}$, Q^A is included in A_u and is closed;*
- *"For a closed subset V the number of colors of V (i.e. the colors Q such that $V \subseteq Q^A$) only depends on the dimension of V". More formally:*
  There exists a function $f : \mathbb{N}^{n-1} \to \mathbb{N}$, such that for every $V \subseteq A_u$, V closed,

$$\mathrm{card}\{Q \in C \ ; \ V \subseteq Q^A\} = f(\dim(V))$$

Let $\{d_1, \ldots, d_r\} = \{\dim V \ ; \ V \subseteq A_u\}$, $d_1 < \ldots < d_r$ lexicographically. For every single closed $V \subseteq A_u$ we will decide how many colors Q we want to put into C with $Q^A = V$. We will do this and define the function f by downward induction on $\dim(V)$. Note that the value of f only has to be defined for $d_r, \ldots, d_1$.

Let $C_r = \{Q_a \ ; \ a \in U_1^A\}$; we will define $C_r \subseteq \ldots \subseteq C_0 = C$. Suppose (by induction) that for a given $i < r$, C_{i+1} is already defined and for $d' > d_{i+1} = d$ $f(d')$ is already defined such that for every $V \subseteq A_u$: If $\dim V > d$ then $\mathrm{card}\{Q \in C_{i+1} \ ; \ V \subseteq Q^A\} = f(\dim(V))$.

Define $f(d) = \max\{\mathrm{card}\{Q \in C_{i+1} \ ; \ V \subseteq Q^A\}; V \subseteq A_u, V \text{ closed}, \dim V = d\}$ and add for every closed $V \subseteq A_u$ with $\dim V = d$ enough colors Q with $Q^A = V$ (i.e. $f(d) - \mathrm{card}\{Q \in C_{i+1} \ ; \ V \subseteq Q^A\}$ many) to C_{i+1} to define C_i such that $\mathrm{card}\{Q \in C_i \ ; \ V \subseteq Q^A\} = f(d)$.

The strict monotonicity of the dimension ensures that this works (no different closed subsets of dimension d are contained in each other and if $\dim V > d$ the equation $\mathrm{card}\{Q \in C_i \ ; \ V \subseteq Q^A\} = f(\dim V)$ will be maintained). $C = C_0$ fulfils the requirement of the claim.

♡

Note that also for arbitrary $V \subseteq A_u$, we have

$$\mathrm{card}\{Q \in C \ ; \ V \subseteq Q^A\} = f(\dim V)$$

because $\{Q \in C \ ; \ V \subseteq Q^A\} = \{Q \in C_i \ ; \ \mathrm{cl}(V) \subseteq Q^A\} = f(\dim(\mathrm{cl}(V)))$ and $\dim(\mathrm{cl}(V)) = \dim V$.

Definition 4.16 *Let $D \subseteq A$ (think of $D = D_i \cap A_u$) and let $V \subseteq D$. V is called relatively closed in D, if $V = \mathrm{cl}(V) \cap D$ or, equivalently, if*

$$\forall a \in V, \forall b \in D : (a \to^* b \Rightarrow b \in V).$$

Note that for every $Q \in C$ and $D \subseteq A_u$, $Q^A \cap D$ is relatively closed in D.

Lemma 4.17 *Let $1 \leq i \leq t$ and let $V \subseteq D_i \cap A_u$ be relatively closed in $D_i \cap A_u$. Then $V' := p_i(V)$ is relatively closed in $D_i' \cap A_u$ and*

$$\mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i \cap A_u = V\} = \mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i' \cap A_u = V'\}.$$

We will prove the equality by downward induction on $\dim V$. First note that the lattice of relatively closed subsets of $D_i \cap A_u$ is isomorphic via p_i to the corresponding lattice for $D_i' \cap A_u$ and this isomorphism respects dimensions.

Suppose $V \subseteq D_i \cap A_u$ is relatively closed and suppose that for $W \subseteq D_i \cap A_u$ relatively closed with $V \subsetneq W$. We already know that:

$$\mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i \cap A_u = W\} = \mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i' \cap A_u = p_i(W)\}.$$

(The same equation for W not relatively closed is trivial: then both sides are 0.)

Write $\mathcal{S} := \{W \ ; \ V \subsetneq W \subseteq D_i \cap A_u\}$ and $\mathcal{S}' := \{W \ ; \ V' \subsetneq W \subseteq D_i' \cap A_u\}$. Then:

$$\mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i \cap A_u = V\} =$$

$$\mathrm{card}\{Q \in C \ ; \ V \subseteq Q^A\} - \sum_{W \in \mathcal{S}} \mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i \cap A_u = W\} =$$

$$f(\dim V) - \sum_{W \in \mathcal{S}} \mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i' \cap A_u = p_i(W)\} =$$

$$f(\dim V') - \sum_{W' \in \mathcal{S}'} \mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i' \cap A_u = W'\} =$$

$$\mathrm{card}\{Q \in C \ ; \ Q^A \cap D_i' \cap A_u = V'\}.$$

♡

Now we want to get the colors into the language by introducing a new binary predicate relating the points to their colors.

Definition 4.18 *We let $L' = \{R, U_1, \ldots, U_n\} \cup \{C, D\}$. We define an L'-structure B with domain $A \cup C$. We let $R^B = R^A$, $U_1^B = U_1^A, \ldots, U_n^B = U_n^A$ and $C^B = C$ and we put $D^B qa$ if and only if $q \in C$ and $a \in A$ and a is of color q.*

Lemma 4.19 *For every i there is a permutation χ_i of the set C of colors such that $p_i \cup \chi_i$ is a partial automorphism of B.*

Certainly every color $Q \in C$ belongs to exactly one set of the form

$$\{Q \in C \ ; \ Q^A \cap D_i \cap A_u = V\}$$

(for some $V \subset D_i$), so we can define a permutation χ_i mapping $\{Q \in C \ ; \ Q^A \cap D_i \cap A_u = V\}$ bijectively to $\{Q \in C \ ; \ Q^A \cap D_i' \cap A_u = p_i(V)\}$. Furthermore, we can find the χ_i such that for $a \in D_i \cap U_1^A$, $Q_a^{\chi_i} = Q_{p_i(a)}$ (because: if $Q_a \cap D_i \cap A_u = V$ then $Q_{p_i(a)} \cap D_i' \cap A_u = p_i(V)$ and that is because p_i respects $\rightarrow^*$).

Now it remains to show, that $r_i := p_i \cup \chi_i$ is a partial automorphism of B; so let $a \in D_i$, $Q \in C$. We have to show that $a \in Q^A$ if and only if $a' \in Q'^A$ (where a' is $p_i(a)$ and Q' is $\chi_i(Q)$).

If $a \in U_1^A$, then $a \in Q^A$ if and only if $Q = Q_a$, if and only if $Q' = Q_{a'}$, if and only if $a' \in Q'^A$.

If $a \in A_u$, let $V := Q^A \cap D_i \cap A_u$ so by definition of χ_i, $Q'^A \cap D_i' \cap A_u = p_i(V)$. So $a \in Q^A$ if and only if $a \in V$, if and only if $a' \in p_i(V)$, if and only if $a' \in Q'^A$.

Third step

The structure B has the following properties:

- for every element a of U_1 there exists exactly one $q \in C$ (namely $q = Q_a$) such that Dqa;
- if $a \in U_i$ $(1 \leq i < n)$ and $a \to b$ and Dqa then Dqb (just because the interpretation of the color q in A is closed);
- if $a \in U_s^A$ and $a \to b$ and Dqa then not Dqb.

For the last point suppose Dqb; because $b \in U_1^A$, it follows that $q = Q_b$, but $a \in Q_b$ means that $b \to^* a$, and this would mean that there is an s-cycle in A.

Now we are using the Corollary 4.13. So we know, there exists a finite $\mathcal{L}'$–structure E, $B \subseteq E$, and $g_1, \ldots, g_n \in \mathrm{Aut}(E)$, g_i extending p_i for every i. By the remark after Definition 2.1 we can choose B to be slim. Note that this automatically ensures that $C^E = C^B = C$.

Claim: The structure E has the following properties:

- for every point $a \in U_1$ there exists exactly one $q \in C$ such that Dqa;
- if $a \in U_i$ $(1 \leq i < n)$ and $a \to b$ and Dqa then Dqb;
- if $a \in U_s^A$ and $a \to b$ and Dqa then not Dqb.

This follows directly from the fact that E is slim, and the corresponding property of A. Suppose e.g. that $a \in U_i^E$ $(1 \leq i < n)$ and $a \to b$ and Dqa. Then we know that there exists $h \in F(P)$ such that $\widetilde{h}(a), \widetilde{h}(b) \in A$ and of course $\widetilde{h}(a) \to \widetilde{h}(b)$. So we have $D\widetilde{h}(q)\widetilde{h}(a)$, which implies $D\widetilde{h}(q)\widetilde{h}(b)$ which implies Dqb.

Now to complete the proof, we have to check that E is cycle-free. Suppose there are elements $a_1 \in U_1^E, \ldots, a_n \in U_n^E$, such that $a_1 \to a_2 \to \ldots \to a_n \to a_1$. Choose $q \in C$ such that Dqa_1, inductively follows Dqa_n and not Dqa_1, a contradiction.

5 Proof of Theorem 3.2

This section is entirely devoted to the proof of Theorem 3.2. The strategy is to reduce the problem, by successive steps, to an easy one.

5.1 Reduction to stretched structures

To begin with, we will assume that all structures we are considering are irreflexive: a structure M in a language $\mathcal{L}$ is irreflexive if, for every n-ary predicate R in $\mathcal{L}$ and $a_1, a_2, \ldots, a_n \in M$, if $M \models Ra_1a_2\ldots a_n$, then the a_i are pairwise distinct. We can do that without loss of generality (see the last section of [6]).

The first real reduction states that it is enough to prove it for a certain kind of structures, which we will call the *stretched structures* and which we define now.

The language $\mathcal{L}$ of a stretched structure M should contain unary predicates U_0, U_1, …, U_k. The universe of M is the disjoint union of the sets U_i^M. Moreover, for each n-ary relation symbol R of $\mathcal{L}$, and for all $a_1, a_2, \ldots a_n$ in M, if $M \models Ra_1a_2\ldots a_n$, then the set $\{a_1, a_2, \ldots, a_n\}$ intersects each set U_i, for $1 \leq i \leq k$ in at most one element (notice that U_0 has a special status).

A *small structure* M is a stretched structure such that for all i, $0 \leq i \leq k$, U_i^M has at most one element.

The first reduction is a very important one (we will consistently call each of these reductions 'proposition').

Proposition 5.1 *Let $\mathcal{T}$ be a finite set of small structures. Then the class of stretched $\mathcal{T}$-free structures has the EPPA.*

Proof. We deduce Theorem 3.2 from Proposition 5.1: suppose we are given a language $\mathcal{L}$, a finite set $\mathcal{T}$ of finite $\mathcal{L}$-structures, A a finite $\mathcal{T}$-free $\mathcal{L}$-structure, p_1, p_2, ..., $p_n \in \mathrm{Part}(A, A)$, a $\mathcal{T}$-free structure M extending A and automorphisms α_1, α_2, ..., $\alpha_n \in \mathrm{Aut}(M)$ extending respectively p_1, p_2, ..., p_n. Let $k = \max(\mathrm{card}(T); T \in \mathcal{T})$. From A we want to define a stretched structure $\widehat{A}$. We first add to the language k new unary predicates U_0, U_1, ..., U_{k-1}. We will write $\mathcal{L}^+$ for the language that we obtain this way. The universe of $\widehat{A}$ is $A \times \{0, 1, \ldots, k-1\}$. The interpretation of U_i in $\widehat{A}$ (for $0 \leq i \leq k-1$) is $A \times \{i\}$. If R is an s-ary relation symbol in the language $\mathcal{L}$ of A, the interpretation of R in $\widehat{A}$ is defined by:

$\widehat{A} \models R\ (a_1, i_1)\ (a_2, i_2) \ldots (a_s, i_s)$ if and only if $A \models Ra_1a_2 \ldots a_s$ and for all i, $1 \leq i \leq k-1$ there is at most one m, $1 \leq m \leq s$, such that $i_m = i$.

It is clear that $\widehat{A}$ is a stretched structure. Moreover, the map π from $\widehat{A}_{|\mathcal{L}}$ onto A defined by $\pi((a, i)) = a$ is a weak homomorphism.

For every element T of $\mathcal{T}$ choose a small $\mathcal{L}^+$-structure T^+ which expands T (this is possible because k has been chosen sufficiently large). Let $\mathcal{T}^+ = \{T^+; T \in \mathcal{T}\}$. We can see that $\widehat{A}$ is $\mathcal{T}^+$-free: if h were a weak homomorphism from some T^+ into $\widehat{A}$, then $\pi \circ h$ would be a weak homomorphism from $T^+_{|\mathcal{L}}$ (which is equal to T) to A.

Now, for each partial automorphism p_i of A, we may define a partial map $\widehat{p_i}$ of $\widehat{A}$ by: if $a \in Dom(p)$ and $0 \leq i \leq k-1$, $\widehat{p_i}((a, j)) = (p_i(a), j)$. It is straightforward to check that these maps are partial automorphisms. We may also define analogously a stretched $\mathcal{L}^+$-structure $\widehat{M}$ from M, and automorphisms $\widehat{\alpha_1}$, $\widehat{\alpha_2}$, $\ldots \widehat{\alpha_n}$ of $\widehat{M}$. As above, $\widehat{M}$ is $\mathcal{T}^+$-free, and it is clear that $\widehat{M}$ extends $\widehat{A}$ and that the $\widehat{\alpha_i}$ extend the $\widehat{p_i}$.

Thus we may apply Proposition 5.1: we get a finite $\mathcal{T}^+$-free $\mathcal{L}^+$-structure C extending $\widehat{A}$ and automorphisms γ_1, γ_2, ..., γ_n of C extending the $\widehat{p_i}$. Moreover, if we translate the problem into a problem in the free group, as has been done in subsection 2.2, we see that to be $\mathcal{T}^+$-free can be forced by negative conditions, so, we may apply Proposition 2.3 and assume that C is a special extension of $\widehat{A}$. (Another way to see this is as follows: By a remark after Proposition 2.3 we find a special solution D and $\rho : D \underset{w}{\longrightarrow} C$. As C is $\mathcal{T}^+$-free also D is $\mathcal{T}^+$-free and therefore we can suppose C to be special.)

We restrict our attention to the $\mathcal{L}$-structure B that we get in the following way: first we take the substructure of C whose universe is the set of elements of C which belong to U_0. Then we take the $\mathcal{L}$-reduct of this structure to obtain B. Some fact are immediately clear: Since C is special, the U_i partition the universe of C. The γ_i leave the set B fixed, so they induce permutations β_i of B, and these permutations are in fact automorphisms of B. We may identify the U_0 part of $\widehat{A}$ with A (identifying $(a, 0)$ with a). Doing this, B will be viewed as an extension of A, and the β_i as extensions of the p_i.

Thus, it will suffice to prove that B is $\mathcal{T}$-free.

Let $T \in \mathcal{T}$ and let $\{t_1, t_2, \ldots, t_s\}$ be the universe of T. Let T' be the expansion of T to $\mathcal{L}^+$, where all the t_i for $1 \leq i \leq s$ belong to U_0. We may construct a sequence $T_0, T_1, \ldots, T_s$ of stretched $\mathcal{L}^+$-structures which all are expansions of T, beginning with $T_0 = T^+$ and ending with $T_s = T'$ such that the only possible difference between T_j and T_{j+1} is that t_{j+1}, which satisfies some U_r in T_j, satisfies U_0 instead in T_{j+1}. We prove by induction on j that C is T_j-free. We already know that it is T_0-free, and once we will know it is T_s-free, we will know that B is T-free.

By way of contradiction, assume that C is T_j-free, and that h is a weak homomorphism from T_{j+1} into C. Let G be the group of automorphisms of C generated by $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Since every element of C is the image by an element of G of an element of $\widehat{A}$, we may assume that $h(t_{j+1}) \in \widehat{A}$.

Since $T_{j+1} \models U_0\ t_{j+1}$ and h is a weak homomorphism, $h(t_{j+1}) = (a, 0)$ for some $a \in A$. Let m be the positive integer such that $T_j \models U_m(t_{j+1})$ and h' the map from T_j into C equal to h except in t_{j+1} where $h'(t_{j+1}) = (a, m)$. We show that h' is a weak homomorphism from T_j into C, and get a contradiction.

By construction, h' preserves the predicates U_i. So let R be a predicate symbol of L and assume that $T_j \models Rt_{j+1}\bar{t}$ where $\bar{t}$ is a sequence of t_i . Thus, we also have $T_{j+1} \models Rt_{j+1}\bar{t}$, and since h is a weak homomorphism, $C \models R\ h(t_{j+1})\ h(\bar{t})$. Because C is slim, there exist $b \in \widehat{A}$, a sequence $\bar{c}$ of elements of $\widehat{A}$ such that $\widehat{A} \models Rb\bar{c}$ and such that $h(t_{j+1})h(\bar{t}) = \gamma(b\bar{c})$ for some $\gamma \in G$. In particular $\gamma(b) = h(t_{j+1}) = (a, 0)$. Let $b' = \pi(b)$ (that is, $b = (b', 0)$), $\bar{c}' = \pi(\bar{c})$ and $b_1 = (b', m)$. By construction of $\widehat{A}$ $A \models Rb'\bar{c}'$ and $\widehat{A} \models Rb_1\bar{c}$.

We now use the real strength of the hypothesis that C is special: there exist a word w of the free group $F(x_1, x_2, \ldots, x_n)$ such that $\gamma = w(\beta_1, \beta_2, \ldots, \beta_n)$ and $w(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_n)(b) = (a, 0)$. Since $w(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_n)(b_1) = (a, m)$, we have $\gamma(b_1) = w(\beta_1, \beta_2, \ldots, \beta_n)(b_1) = (a, m) = h'(t_{j+1})$. From $\widehat{A} \models Rb_1\bar{c}$ we deduce $C \models R\gamma(b_1)\gamma(\bar{c})$. Because T is irreflexive $h'(\bar{t}) = h(\bar{t}) = \gamma(\bar{c})$. Thus $C \models Rh'(t_{j+1})h'(\bar{t})$.

♡

5.2 Short extensions

From now on and until the end of the proof of Theorem 3.2, we will only deal with stretched structures in a fixed language $\mathcal{L}$. To fix the notations, we will suppose that U_0, U_1, ..., U_k are the unary predicates necessary to make our structures stretched. Before going any further, we need some definitions.

Definition 5.2 *Let M_0 be a stretched structure, M_1 and M_2 be two extensions of M_0. Assume that $M_1 \cap M_2 = M_0$. The* free amalgam *of M_1 and M_2 over M_0, denoted $M_1 *_{M_0} M_2$, is the structure whose universe is $M_1 \cup M_2$ and whose set of links is the union of the set of links of M_1 and of the set of links of M_2.*

If $M_1 \cap M_2 \neq M_0$, then $M_1 *_{M_0} M_2$ is defined up to M_0-isomorphism, and is equal to $M_1' *_{M_0} M_2'$ where M_1' and M_2' are extensions of M_0 which are respectively M_0-isomorphic to M_1 and M_2 and such that $M_1' \cap M_2' = M_0$.

Now come the main devices of the proof.

Definition 5.3 *1) Let A be a structure. A short extension of A is a structure which can be written as $A *_{A_0} C$ where A_0 is a small substructure of A or the empty set, and C is an extension of A_0 which is also small.*

2) Let $D \subseteq A$. A short extension is based on D if it can be written as $A *_{A_0} C$ as above, with the requirement $A_0 \subseteq D$.

3) Let $A \subseteq B$ be two $\mathcal{L}$-structures. We say that B is a strong extension of A if, for all short extension C of A, if there exists $h : C \underset{w,A}{\longrightarrow} B$, then there exists $h' : C \underset{w,A}{\longrightarrow} A$. To denote that B is a strong extension of A, we will write $A \subseteq_s B$.

4) Let A and B be two $\mathcal{L}$-structures and p a partial isomorphism from A to B with domain $D \subseteq A$ and image $D' \subseteq B$. We say that p is strong in (A, B) if for every short extension D_1 of D and $k : D_1 \underset{w,D}{\longrightarrow} A$, there exists $p' : D_1 \underset{w}{\longrightarrow} B$ such that p' extends p, and conversely, for every short extension D_1' of D' and $k' : D_1' \underset{w,D'}{\longrightarrow} B$, there exists $p' : D_1' \underset{w}{\longrightarrow} A$ such that p' extends p^{-1}.

Remark for 1): *A priori*, the structures A_0 and C are not uniquely determined. But there is a kind of canonical decomposition of a short extension B: set $C_1 = B - A$ and

$$A_1 = \{a \in A;\ a \text{ is linked with an element of } C_1\}.$$

From the definition, it should be clear that, if $B = A *_{A_0} C$, then $A_1 \subseteq A_0$, $A_1 \cup C_1 \subseteq C$ and $B = A *_{A_1} (A_1 \cup C_1)$. It is this decomposition which will be used implicitly when a decomposition is needed.

It should be remarked that, in the above definition 2), the fact that p is strong does not depend only on p itself, but also on the way that D and D' sit in A and B respectively. That is why we add "in (A, B)". If $p \in \mathrm{Part}(A, A)$, we will say "$p$ is strong in A" instead of "p is strong in (A, A)".

The following facts are easy to prove:

Lemma 5.4 *1. If A is $\mathcal{T}$-free and $A \subseteq_s B$ is a strong extension of A, then B is $\mathcal{T}$-free.*

2. *If $B \subseteq B_1$, then $B \subseteq_s B_1$ if and only if for every small structure $C \subseteq B_1$, there exists $h : C \underset{w, B\cap C}{\longrightarrow} B$.*
3. *If $A \subseteq_s B$ and $B \subseteq_s C$, then $A \subseteq_s C$.*
4. *If $A \subseteq_s B$ and $A \subseteq C \subseteq B$, then $A \subseteq_s C$.*
5. *If $p \in \mathrm{Part}(A, A)$ is strong in A and $A \subseteq_s B$, then p is strong in B*
6. *If B is a short extension of A, then $A \subseteq_s B$ if and only if there exists $h : B \underset{w,A}{\longrightarrow} A$.*
7. *If B is a short extension of A and $B = A *_{A_0} C$ (where A_0 and C are small structures), then $A \subseteq_s B$ if and only if there exists $h : C \underset{w,A_0}{\longrightarrow} A$.*
8. *Assume that $A \subseteq_s B$, $A \subseteq B'$ $h : B' \underset{w,A}{\longrightarrow} B$. Then $A \subseteq_s B'$.*
9. *If $A \subseteq_s B$ and $A \subseteq C$, then $C \subseteq_s B *_A C$.*
10. *If $A \subseteq_s B$ and $A \subseteq_s C$, then $A \subseteq_s B *_A C$.*

We now get to our second reduction step:

Proposition 5.5 *Let $\mathcal{T}$ be a finite set of small structures, A be a finite $\mathcal{T}$-free structure and p_1, p_2, …, p_n be partial automorphisms of A. Suppose that the p_i are strong in A. Then there exists a finite $\mathcal{T}$-free structure B extending A and automorphisms β_1, β_2, …, $\beta_n \in \mathrm{Aut}(B)$ extending respectively p_1, p_2, …, p_n.*

Proof. We show how to deduce proposition 5.1 from proposition 5.5. Let M and α_i as in the hypothesis of proposition 5.1. The point is that, even if the p_i are not strong in A, they are obviously strong in M.

For all i between 1 and n, let D_i be the domain of p_i and D'_i its image. Obviously, there is only a finite number of short extensions of D_i, up to D_i-isomorphism. Thus we may find a finite structure B_i, $A \subseteq B_i \subseteq M$ such that: for every short extension E of D_i and $k : E \underset{w,D_i}{\longrightarrow} M$, there exists $k' : E \underset{w,D_i}{\longrightarrow} B_i$. Thus if C_i is any substructure of M containing B_i and $\alpha_i[B_i]$, we see that the following is true: for every short extension E of D_i and $k : E \underset{w,D_i}{\longrightarrow} M$, p_i can be extended to a weak homomorphism from E to C_i. Repeating the operation for D'_i in place of D_i, and then for all i, $1 \leq i \leq n$, we get a finite substructure C in which all the p_i are strong. Then we may apply proposition 5.5 to get the structure B.

♡

The next reduction tells us that we can work sort by sort.

Proposition 5.6 *Let A be a finite structure and p_1, p_2, ..., p_n be elements of* $\mathrm{Part}(A, A)$*, and suppose that the p_i are strong in A. Fix an integer j, $1 \leq j \leq k$. There exist a finite strong extension B of A and partial automorphisms q_1, q_2, ..., $q_n \in \mathrm{Part}(B, B)$ extending respectively p_1, p_2, ..., p_n such that all the q_i are strong in B and induce a permutation on U_j^B.*

Proof. We prove Proposition 5.5 using Proposition 5.6. So we start with the hypothesis of Proposition 5.5. We define a sequence of finite structures $A = A_0 \subseteq_s A_1 \subseteq_s \ldots \subseteq_s A_k$, and we extend successively the partial automorphisms $p_i^0 = p_i$: $p_i^0 \subseteq p_i^1 \subseteq \ldots \subseteq p_i^k$, $p_i^j \in \mathrm{Part}(A_j, A_j)$ such that p_i^j is strong in A_j. Furthermore for $j \geq 1$, p_i^j induces a permutation on $U_j^{A_j}$. Clearly, Proposition 5.6 allows us to do that. Next we take C to be $A \cup U_1^{A_1} \cup U_2^{A_2} \cup \ldots \cup U_k^{A_k} \subseteq A_k$ and take r_i to be the restriction of p_i^k to C. It is a partial automorphism on C and for $1 \leq j \leq k$, r_i induces a permutation on $U_j^C = U_j^{A_j}$. Since A_k is a strong extension of A, it is $\mathcal{T}$-free, and so is C.

It remains to take care of U_0. By Corollary 4.13, we can find a finite L–structure B extending C, and automorphisms α_i of B extending the r_i. By proposition 2.3 we can take $(B, \alpha_1, \alpha_2, \ldots, \alpha_n)$ to be a slim extension of $(C, r_1, \ldots, r_n)$, which is automatically stretched.

Write G for the group of automorphisms of B generated by $\{\alpha_1, \ldots, \alpha_n\}$. For $1 \leq j \leq k$, if $b \in U_j^B$, there exist $c \in U_j^C$ and $\beta \in G$ such that $b = \beta(c)$. Since each $\beta \in G$ induces a permutation on U_j^C, we see that $b \in U_j^C$, so $U_j^C = U_j^B$.

We have to prove that B is $\mathcal{T}$–free. The fact that B is a slim extension implies: for $b \in U_0^B$ there exists $\beta \in G$ such that $\beta(b) \in U_0^C$. Suppose now that there exist $T \in \mathcal{T}$ and r a weak homomorphism from T to B. T is a small stretched structure. If $U_0^T = \emptyset$ then r is a weak homomorphism from T to $\bigcup\{U_j^B; 1 \leq j \leq s\}$, and this is impossible since C is $\mathcal{T}$-free. If U_0^T has one element, say t then we choose $\gamma \in G$ such that $\gamma \circ r(t) \in U_0^C$ and then $\gamma \circ r$ is a weak homomorphism from T to C, which is again impossible.

♡

5.3 Types

To simplify the notation slightly, we prove proposition 5.6 for $j = 1$. Given $a \in U_1^A$ and concentrating on a partial automorphism, say p_1, the first step is to find an image for a, that is an element b in some extension B of A such that, if we extend p_1 to q_1 by setting: $q_1(a) = b$, then q_1 is still a strong partial automorphism of B. Intuitively, the conditions to be satisfied by this element b are: 1) (for q_1 to be a partial automorphism) the relations between b and D_1' (the image of p_1) should be exactly the image of the relations existing between a and D_1 (the range of p_1); 2) (for q_1 to be a *strong* partial automorphism) for every small substructure C of A containing a, there should be in B a small substructure $C'$of B containing b, and a weak homomorphism sending C to C' extending q_1; 3) same as 2), permuting the roles of a and b and replacing q_1 by q_1^{-1}. That is where the notion of type comes in: the type of a over D is meant to collect the necessary data.

For the rest of the proof, when considering a short extension B of A, we will implicitly assume that there is a (unique) element in U_1^B not in A. We will denote this element $u(B)$.

We can preorder the set of short extensions of A in the following way: given B and C two such extensions, we write $B \leq C$ if there exists a weak A-homomorphism h from B into C such that $h(u(B)) = u(C)$. It is obviously a reflexive and transitive relation (but not antisymmetric in general, so it is just a preordering).

To define the type of an element b of B, we need to consider the short extensions C of A such that there exists $h : C \underset{w,A}{\longrightarrow} B$ such that $h(u(C)) = b$. To deal only with a finite number of them, we will consider a finite set $\mathcal{S}$ of short extensions of A, such that, for all short extensions C of A, C is A-isomorphic to one and only one element of $\mathcal{S}$. Moreover, we will choose an element x which does not belong to A and assume that, for all $C \in \mathcal{S}$, $u(C) = x$, and that for all $C, C' \in \mathcal{S}$, if $C \neq C'$, then $C \cap C' = A \cup \{x\}$.

Definition 5.7 *Let B is be strong extension of A and $b \in U_1^B$. Then:*

- *$\Gamma_B(b)$ is the extension of A whose universe is $A \cup \{x\}$ and such that, for all $R \in \mathcal{L}$, R different from the equality, and for all sequences $\bar{a}_1, \bar{a}_2$ from A, $\Gamma_B(b) \models R\bar{a}_1 x \bar{a}_2$ if and only if $B \models R\bar{a}_1 b \bar{a}_2$;*
- *$\mathcal{E}_B(b)$ is the set $\{C \in \mathcal{S}$; there exists $h : C \underset{w,A}{\longrightarrow} B$ such that $h(x) = b\}$.*

More generally, if $D \subseteq A$, we define:

- *$\Gamma_B(b/D)$ is the structure whose base set is $A \cup \{x\}$ and such that, for every $R \in \mathcal{L}$ and sequence $\bar{a}$ from $A \cup \{x\}$, $\Gamma_B(b/D) \models R\bar{a}$ if and only if $\Gamma_B(b) \models R\bar{a}$ and either all the elements of $\bar{a}$ belong to $D \cup \{x\}$ or all elements of $\bar{a}$ belong to A.*
- *$\mathcal{E}_B(b/D) = \{C \in \mathcal{E}_B(b)$; C is based on $D\}$.*

Lemma 5.8 *Let $D \subseteq A \subseteq_s B \subseteq_s B'$. Let $b \in U_1^B$. Then $\Gamma_B(b) = \Gamma_{B'}(b)$, $\Gamma_B(b/D) = \Gamma_{B'}(b/D)$, $\mathcal{E}_B(b) = \mathcal{E}_{B'}(b)$, $\mathcal{E}_B(b/D) = \mathcal{E}_{B'}(b/D)$*

Proof. It is clear that $\Gamma_B(b) = \Gamma_{B'}(b)$ and that $\Gamma_B(b/D) = \Gamma_{B'}(b/D)$. We prove $\mathcal{E}_B(b) = \mathcal{E}_{B'}(b)$ (the fourth equality follows immediately). Again, it is clear that $\mathcal{E}_B(b) \subseteq \mathcal{E}_{B'}(b)$. So, let $C \in \mathcal{E}_{B'}(b)$, and let us prove $C \in \mathcal{E}_B(b)$. We may decompose C: $C = A *_{A_0} C_0$, where A_0 and C_0 are small structures, and we know that there exists $h : C_0 \underset{w,A_0}{\longrightarrow} B'$ with $h(x) = b$. Let $C_1 = h[C_0]$ and $C_2 = C_1 \cap B$. Thus $b \in C_2$ and $A_0 \subseteq C_2$. Since $B \subseteq_s B'$, there exists $h_1 : C_1 \underset{w,C_2}{\longrightarrow} B$, and $h_1 \circ h : C_0 \underset{w,A_0}{\longrightarrow} B$, and this proves that $C \in \mathcal{E}_B(b)$.

♡

Assume that B is a short extension of A based on $D \subseteq A$, let p be a strong partial automorphism of A whose domain contains D, and let $D' = p[D]$. We define an extension $p(B)$ of A as follows: say that $B = A *_{A_0} C_0$ according to Definition 5.3 (where $A_0 \subseteq D$). Let $A'_0 = p[A_0]$. It is clear that we can find a small structure C'_0 containing A'_0 and an isomorphism g from C_0 onto C'_0 extending $p_{|A_0}$, and moreover we may assume that $A \cap C'_0 = A'_0$. By definition $p(B) = A *_{A'_0} C'_0$. One should consult the remark following Definition 5.3 to be sure that this definition is legal (that is, does not depend on the particular A_0 chosen). Moreover, we may assume that, if $B \in \mathcal{S}$, then $p(B) \in \mathcal{S}$.

Lemma 5.9 *With the above hypothesis, and supposing moreover that B is a strong extension of A, $p(B)$ is a strong extension of A based on D'.*

Proof. It suffices to show that there is a weak A-homomorphism from $p(B)$ into A. We use the notations of the previous paragraph: $p(B) = A *_{A'_0} C'_0$ and g is the isomorphism from C_0 to C'_0. It is enough to show that there is a weak A'_0-homomorphism from C'_0 into A.

Because B is a strong extension of A, there exists a weak A-homomorphism from B to A, so there exits $h : D *_{A_0} C_0 \underset{w,D}{\longrightarrow} A$. Since p is strong, p can be extended to a weak homomorphism k from $D *_{A_0} C_0$ into A. Then $k_{|C_0} \circ g^{-1}$ is a weak A'_0-homomorphism from C'_0 into A.

♡

Lemma 5.10 *Let B and C be two short extensions of A based on D and p a strong partial automorphism of A of domain D. Then $B \leq C$ if and only if $p(B) \leq p(C)$.*

Proof. Since $p^{-1}(p(B)) = B$ and $p^{-1}(p(C)) = C$, it suffices to prove that, if $B \leq C$, then $p(B) \leq p(C)$.

We need some notation: let D' be the image of p, so that p is an isomorphism from D to D'. We may write $B = A *_{B_2} B_1$, where $B_2 \subseteq B_1$ are small structures and $B_2 \subseteq D$. Similarly $C = A *_{C_2} C_1$. Set $B' = p(B)$ and $C' = p(C)$. Let $B'_2 = p[B_2]$ and $B' = A *_{B'_2} B'_1$ and $g : B_1 \longrightarrow B'_1$ be the isomorphism given by the definition of $p(B)$. Let $F = C_1 \cup D$ so that $C = A *_D F$. We can write $C' = A *_{D'} F'$ and $f : F \longrightarrow F'$ is an isomorphism extending p. We have to show that there exists $k_1 : B' \underset{w,A}{\longrightarrow} C'$; it suffices to find $k : B'_1 \underset{w,B'_2}{\longrightarrow} C'$ with $k(u(B')) = u(C')$.

As $B \leq C$ there exists $h_1 : B \underset{w,A}{\longrightarrow} C$ such that $h(u(B)) = u(C)$. Let $h = (h_1)_{|B_1}$, $h : B_1 \underset{w,B_2}{\longrightarrow} C$. Let $E_1 = h^{-1}(A)$, $E_2 = h^{-1}(F)$, $E_3 = h^{-1}(D)$. As $C = A *_D F$ and h is a weak homomorphism we have $B_1 = E_1 *_{E_3} E_2$. The structure $G = D *_{h[E_3]} h[E_1]$ is a small extension of D. Because p is strong there exists $r : G \underset{w}{\longrightarrow} A$, r extending p. Now we are gluing together $f_{|h[E_2]} \circ h_{|E_2}$ and $r_{|h[E_1]} \circ h_{|E_1}$ which coincides on E_3 with $p_{|h[E_3]} \circ h_{|E_3}$ to get $g' : B_1 \underset{w}{\longrightarrow} C'$. Finally let $k = g' \circ g^{-1}$, $k : B'_1 \underset{w,B'_2}{\longrightarrow} C'$.

♡

Now, suppose that $D \subseteq A \subseteq_s B$, that p is a strong partial automorphism of A of domain D and image D' and that $b \in U_1^B$. We may define $p(\mathcal{E}_B(b/D))$ as $\{p(C); C \in \mathcal{E}_B(b/D)\}$. We may also define $p(\Gamma_B(b/D))$: it is the structure whose base set is $A \cup \{x\}$ and such that, for every $R \in \mathcal{L}$ and sequence $\bar{a}$ from $A \cup \{x\}$, $p(\Gamma_B(b/D)) \models R\bar{a}$ if and only if one of the two following cases holds:
1. all the elements of $\bar{a}$ belong to A and $A \models R\bar{a}$ or
2. all elements of $\bar{a}$ belong to $D' \cup \{x\}$, say to simplify that $\bar{a} = \bar{a}'x$, where $\bar{a}'$ is a sequence from D', and $\Gamma_B(b/D) \models Rp^{-1}(\bar{a}')x$.

The next lemma shows that, as announced, we have collected the relevant information.

Lemma 5.11 *Let p be a strong partial automorphism of A with domain D and image D'. Let B be a strong extension of A, and let a and a' be elements of U_1^B, $a \notin D$ and $a' \notin D'$. Define a map q with domain $D \cup \{a\}$ by: $q_{|D} = p$ and $q(a) = a'$. Then q is a strong partial automorphism if and only if $p(\Gamma_B(a/D)) = \Gamma_B(a'/D')$ and $p(\mathcal{E}_B(a/D)) = \mathcal{E}_B(a'/D')$.*

Proof. It follows immediately from the definitions that q is a partial automorphism if and only if $p(\Gamma_B(a/D)) = \Gamma_B(a'/D)$.

Suppose first that q is strong. Let $C \in \mathcal{E}_B(a/D)$. Let $C = A *_{A_0} C'$, where C' is a small structure including A_0, and $A_0 \subseteq D$. By definition, $p(C) = A *_{A'_0} C''$, where $A'_0 = p[A_0]$ and C' is isomorphic to C'', via an isomorphism p^* extending $p_{|A_0}$ and such that $p^*(x) = x$.

Since $C \in \mathcal{E}_B(a/D)$, there exists $h : C' \underset{w,A_0}{\longrightarrow} B$ such that $h(x) = a$. Since q is strong, there exists a weak homomorphism h' from C' into B, extending $q_{|A_0}$ and such that $h'(x) = a'$. Thus, if we set $k = h' \circ p^{*-1}$, k is a weak A'_0-homomorphism from C'' into B, and $k(x) = a'$.

This proves that $p(C) \in \mathcal{E}_B(a'/D')$. So $p(\mathcal{E}_B(a/D)) \subseteq \mathcal{E}_B(a'/D')$, and, for the same reason, $\mathcal{E}_B(a'/D')) \subseteq p(\mathcal{E}_B(a/D))$.

Conversely, suppose that $p(\mathcal{E}_B(a/D)) = \mathcal{E}_B(a'/D')$. Assume that C is a short extension of $D \cup \{a\}$, and $h : C \underset{w, D\cup\{a\}}{\longrightarrow} B$. As usual C can be written as $(D \cup \{a\}) *_{A_0} C'$. If $a \notin A_0$, we may assume that $a \notin C'$. Because p is strong, there exists a weak homomorphism k from C' into B extending $p_{|A_0}$. Then the map from $(D \cup \{a\}) *_{A_0} C'$ into B extending both k and q is a weak homomorphism, so there exists a weak homomorphism from C to B extending q.

If $a \in A_0$, let $A_1 = A_0 - \{a\}$ and $A'_1 = p[A_1]$. Obviously, h is also a weak D-homomorphism from C to B. Intuitively, this implies that $A *_{A_1} C'$ belongs to $\mathcal{E}_B(a/D)$. More precisely, there exist $C_1 = A *_{A_1} C'_1 \in \mathcal{E}_B(a/D)$ and an A_1-isomorphism k from C' onto C'_1 satisfying $k(a) = x$. Let $p(C_1) = A *_{A'_1} C'_2$ and let p^* denote the isomorphism from C'_1 onto C'_2 extending $p_{|A_1}$ satisfying and $p^*(x) = x$. Since $p(C_1) \in p(\mathcal{E}_B(a/D)) = \mathcal{E}_B(a'/D')$, there exists a weak homomorphism $k' : C'_2 \underset{w, A'_1}{\longrightarrow} B$ such that $k'(x) = a'$. Then $p_1 = k' \circ p^* \circ k : C' \underset{w}{\longrightarrow} B$ extends $p_{|A_1}$ and satisfies $p_1(a) = a'$. Thus, the map from C into B which extends both p_1 and q is a weak homomorphism and it is exactly what we had to find.

♡

Definition 5.12 *Let $A \subseteq B$ be two structures. We say that B is irreducible over A if, whenever B is isomorphic to $B_1 *_A B_2$, one of the structures B_1 or B_2 is equal to B.*

So, B irreducible over A exactly means that $B - A$ is not the disjoint union of two non empty subsets B_1 and B_2 such that there is no link containing an element of B_1 and an element of B_2.

Definition 5.13 *Let A be a finite structure. A* tiny *extension of A is a short and strong extension B of A which is irreducible over A.*

Lemma 5.14 *Suppose B is a tiny extension of A based on D and that p is a strong partial automorphism of A of domain D and image D'. Then $p(B)$ is a tiny extension of A based on D'.*

Proof. To prove that $p(B)$ is irreducible use the remark after Definition 5.12.

♡

Definition 5.15 *Let $D \subseteq A \subseteq_s B$ and $b \in U_1^B$. Then* the type of b in B over D *is the pair $(\Gamma_B(b/D), \mathcal{E}_B^{irr}(b/D))$ where $\mathcal{E}_B^{irr}(b/D)$ is the set of maximal elements (for the ordering $>$) of the set*

$$\{C; C \in \mathcal{E}_B(b/D) \text{ and } C \text{ is irreducible}\}.$$

We will denote $t_B(b/D)$ this type. From Lemma 5.8, it follows that if B' is a strong extension of B, then $t_B(b/D) = t_{B'}(b/D)$.

Lemma 5.16 *$D \subseteq A \subseteq_s B$ and $b \in U_1^B$, and let C be a short extension of A. Then $C \in \mathcal{E}_B(b/D)$ if and only if there exists $C' \in \mathcal{E}_B^{irr}(b/D)$ such that $C \leq C'$.*

Proof. One direction is clear: if $C' \in \mathcal{E}_B^{irr}(b/D)$ and $C \leq C'$, then $C \in \mathcal{E}_B(b/D)$. Conversely, let $C \in \mathcal{E}_B(b/D)$. We can write C as $C_1 *_A C_2$ where C_1 is a tiny extension of A and C_2 is a short extension of A such that $C_2 - A$ contains no point in U_1. Clearly $C_1 \in \mathcal{E}_B(b/D)$, so there exists $C' \in \mathcal{E}_B^{irr}(b/D)$ such that $C_1 \leq C'$. As $A \subseteq_s B$ there exists $h : C_2 \underset{w,A}{\longrightarrow} A$ and thus $C \leq C_1$.

♡

Definition 5.17 *A type is an object of the form $t_B(b/A)$, where $A \subseteq_s B$ and $b \in B$. A type based on D (where D is a subset of A) is an object of the form $t_B(b/D)$.*

These definitions are justified by the following lemma:

Lemma 5.18 *Let $D \subseteq A \subseteq_s B$ and $b \in U_1^B$; then $t_B(b/D)$ is a type.*

Proof. Let B^* be an isomorphic copy of B: its universe is $B^* = \{a^* \ ; \ a \in B\}$, $B \cap B^* = \emptyset$ and the map α from B onto B^* defined by $\alpha(a) = a^*$ is an isomorphism. Let $B_1 = A \cup B^*$ and let h be the map from B_1 on B defined by: for all $a \in A$, $h(a) = a$ and for all $a \in B : h(a^*) = a$. We endow B_1 with an $\mathcal{L}$-structure by setting: for R an n-ary predicate symbol and $a_1, a_2, \ldots, a_n$ elements in B_1, $B_1 \models Ra_1a_2\ldots a_n$ if and only if $B \models R\ h(a_1)\ h(a_2)\ \ldots\ h(a_n)$ and either all the a_i belong to A or all the a_i belong to $B^* \cup D$. With this definition, we see that B_1 is an extension of both A and B^*, and that h is a weak A-homomorphism from B_1 to B, so by Lemma 5.4 B_1 is a strong extension of A.

We want to check that $t_{B_1}(b^*/A) = t_B(b/D)$. It is quite clear that $\Gamma_{B_1}(b^*/A) = \Gamma_B(b/D)$.

Let $t_{B_1}(b^*/A) = (\Gamma, \mathcal{E})$, and $C \in \mathcal{E}$ (so C is a maximal irreducible element of $\mathcal{E}_{B_1}(b^*/A)$). In order to show that C is based on D, we need the two following general lemmata:

Lemma 5.19 *Let C be a tiny extension of A, and assume that $h : C \underset{w,A}{\longrightarrow} C$. Then h is the identity or $h[C] = A$.*

Proof. Write $C = A *_{A_0} C'$ where C' is a small structure, and split the set $C - A$ in two sets: $C_1 = \{a \in C - A \ ; \ h(a) = a\}$ and $C_2 = \{a \in C - A \ ; \ h(a) \in A\}$. We have to prove that one of the sets C_1 or C_2 is empty.

If not, since C is irreducible, there is a link containing an element $a \in C_1$ and an element $b \in C_2$. Thus, there is also a link containing a and $c = h(b)$. We see that b and c are distinct elements, both linked to a, so both belong to C', and belong to the same level U_i: this contradicts the fact that C' is small.

♡

As a corollary, we see that the preordering $<$ when restricted to the set of tiny extensions in $\mathcal{S}$ is an ordering.

Lemma 5.20 *Let $A \subseteq_s B$, $b \in B - A$ and $t_B(b/A) = (\Gamma, \mathcal{E})$. Let $C \in \mathcal{E}$. So C is a short extension of A and there exists $k : C \underset{w,A}{\longrightarrow} B$ such that $k(x) = b$. Then, for every $c \in C - A$, $k(c) \notin A$.*

Proof. : Write $C = A *_{A_0} C'$ where C' is a small structure. Assume for a contradiction that there exists $k : C' \underset{w,A_0}{\longrightarrow} B$ with $k(x) = b$ and $c \in C' - A$ with $k(c) \in A$. Choose k such that the set $X = \{c \in C' - A \ ; \ k(c) \in A\}$ is maximal. Let $C_1' = k[C']$ and $A_1' = C_1' \cap A$. Intuitively this implies that $C_1 = A *_{A_1'} C_1'$ belongs to $\mathcal{E}_B(b/A)$. More precisely there exists a small structure C_2' containing A_1' and x, and an A_1'-isomorphism k' from C_1' to C_2' satisfying $k'(b) = x$ such that $A *_{A_1'} C_2' \in \mathcal{E}_B(b/A)$. Let $C_2 = A *_{A_1'} C_2'$. Let us first check that C_1' is irreducible over A_1': suppose not, say $C_1' = C_3 *_{A_1'} C_4$ with $A_1' \subsetneq C_3$ and $A_1' \subsetneq C_4$. Let us say $b \in C_3$. As $A \subseteq_s B$ there exists $h : C_4 \underset{w,A_1'}{\longrightarrow} A$. Let $h' : C_1' \underset{w,A_1'}{\longrightarrow} B$ be the homomorphism which coincides with h on C_4 and is the identity on C_3. Then $h' \circ k$ contradicts the choice of k as $\{c \in C - A \ ; \ k(c) \in A\} \subsetneq \{c \in C - A \ ; \ h' \circ k(c) \in A\}$. Thus C_1' is irreducible over A_1', i.e. C_1 is irreducible over A. Thus C_2 is irreducible over A and in $\mathcal{E}_B(b/A)$. Let $\ell : C \underset{w,A}{\longrightarrow} C_1$ and

$\ell' : C_1 \underset{w,A}{\longrightarrow} C_2$ be the extension of k and k' respectively by the identity on A. As $\ell' \circ \ell : C \underset{w,A}{\longrightarrow} C_2$ and $\ell' \circ \ell(x) = x$, we have $C \leq C_2$. Now, since C is maximal among the irreducible elements of $\mathcal{E}_B(b/A)$, we see that there exists $k_1 : C_2 \underset{w,A}{\longrightarrow} C$ with $k_1(x) = x$, so, by the previous lemma, $\ell' \circ \ell$ is injective, a contradiction.

♡

Now, we go back to $C \in \mathcal{E}$. We know that there exists $k : C \underset{w,A}{\longrightarrow} B_1$, with $k(x) = b^*$. From the preceding lemma, we see that the elements of $C - A$ are mapped by k to elements of B^*, so there is no link containing an element of $C - A$ and an element of $A - D$. This implies that C is based on D (see the remark following Definition 5.3).

To conclude, we see that

$$\mathcal{E}_B(b/D) \subseteq \mathcal{E}_{B_1}(b^*/A) \subseteq \mathcal{E}_B(b/A)$$

The first inclusion comes from the fact that if $C \in \mathcal{E}_B(b/D)$ and $k : C \underset{w,A}{\longrightarrow} B$ with $k(x) = b$, then the map k' from C to B defined by: $k'(y) = y$ for $y \in A$ and $k'(y) = (k(y))^*$ for $y \in C - A$ is a weak A-homomorphism and $k'(x) = b^*$. The second inclusion is true because there exists $h : B_1 \underset{w,A}{\longrightarrow} B$ with $h(b^*) = b$. Now it is easy to check that $t_B(b/D) = t_{B_1}(b^*/D)$.

♡

Let $t = t_B(b/D) = (\Gamma, \mathcal{E})$ be a type based on $D \subseteq A$, and let p be a strong partial automorphism of A whose domain contains D. Then we define $p(t)$ to be the pair $(p(\Gamma), p(\mathcal{E}))$, where $p(\mathcal{E})$ is the set $\{p(B) \ ; \ B \in \mathcal{E}\}$. It is easy to prove that if $b \in D$, then $p(t_B(b/D)) = t_B(p(b)/D')$. It is also important to notice:

Lemma 5.21 *With these notations, $p(t)$ is a type.*

Proof. Let $D' = p[D]$. First we may assume that $b \notin D$ (if not, $p(t) = t_B(p(b)/D')$). Let B^* be, as above, an isomorphic disjoint copy of B. Let $B_1 = B \cup B^*$ and h be the map from $B^* \cup D'$ to B such that: for $a \in B$, $h(a^*) = a$ and for $a \in D'$, $h(a) = p^{-1}(a)$. Now, we define another structure on B_1: if R is a n-ary predicate symbol and $a_1, a_2, \ldots, a_n$ are elements of B_1, $B_1 \models Ra_1a_2\ldots a_n$ if and only if either all the a_i belong to B and $B \models Ra_1a_2\ldots a_n$; or all the a_i belong to $B^* \cup D'$ and $B \models R\ h(a_1)\ h(a_2)\ \ldots\ h(a_n)$. Our aim is to prove $t_{B_1}(b^*/D') = p(t)$.

To prove $A \subseteq_s B_1$, by Lemma 5.4, it suffices to prove that $B \subseteq_s B_1$, and, again by Lemma 5.4, that, if C is a small structure of B_1, then there exists $k : C \underset{w,B\cap C}{\longrightarrow} B$. Let $A_0 = C \cap D'$, $A_1 = C \cap B$, $A_2 = C \cap (B^* \cup D')$. By definition of B_1, there is no link containing an element of $A_1 - A_0$ and an element of $A_2 - A_0$. Thus $C = A_1 *_{A_0} A_2$ and it suffices to find $k_1 : A_2 \underset{w,A_0}{\longrightarrow} B$. It is clear, from the definition of B_1 again, that h is a weak homomorphism. Let $A_3 = h[A_2]$ and k_2 be the restriction of h to A_2. Because p is strong, we know that there exists a weak homomorphism k_3 from A_3 to B extending p on $A_3 \cap D$. Set $k_1 = k_3 \circ k_2$. Then $k_1 : A_2 \underset{w,A_0}{\longrightarrow} B$ as required.

As a matter of fact, a similar argument proves that $\mathcal{E}_{B_1}(b^*/D') \subseteq p(\mathcal{E}_B(b/D))$. Conversely, let C be a short extension of A based on D and $k : C \underset{w,A}{\longrightarrow} B$ such that $k(x) = b$. Set $C' = p(C)$. Since C' is based on D', there is no link of C' containing an element of $C' - A$ and an element of $A - D'$. We let g be the bijection from $C - A$ to $C' - A$ witnessing $p(C) = C'$. Thus the map k_1 defined by: $k_1(y) = y$ if $y \in A$ and $k_1(y) = (k(g^{-1}(y)))^*$ if $y \in C' - A$ is a weak A-homomorphism from C' to B_1 and $k_1(x) = b^*$. Consequently, $p(\mathcal{E}_B(b/D)) \subseteq \mathcal{E}_{B_1}(b^*/D')$. Lemma 5.10 and Lemma 5.14 now imply $p(t_B(b/D)) = t_{B_1}(b^*/D')$.

♡

As in Lemma 5.11, we see that $t_B(b/D)$ carries enough information:

Lemma 5.22 *Let p be a strong partial automorphism of A with domain D and image D'. Let B be a strong extension of A and let a and a' be elements of U_1^B, $a \notin D$ and $a' \notin D'$. Define a map q with domain $D \cup \{a\}$ by: $q_{|D} = p$ and $q(a) = a'$. Then q is a strong isomorphism if and only if $t_B(a'/D') = p(t_B(a/D))$.*

Proof. One direction is clear: if q is strong, then, by Lemma 5.11, $\mathcal{E}_B(a'/D') = p(\mathcal{E}_B(a/D))$, and $t_B(a'/D') = t_B(a/D)$. Conversely, we have to prove that $\mathcal{E}_B(a'/D') = p(\mathcal{E}_B(a/D))$, assuming that $\mathcal{E}_B^{irr}(a'/D') = p(\mathcal{E}_B^{irr}(a/D))$. But this is a direct consequence of Lemma 5.16.

♡

We are now ready to state the next reduction.

Proposition 5.23 *Let A be a finite structure and let $p_1, p_2, \ldots, p_n \in \mathrm{Part}(A, A)$, and suppose that the p_i are strong in A. Write D_i for the domain of p_i and D'_i for its image. There exists a finite strong extension B of A such that, for every i, $1 \le i \le n$, and for every type t based on D_i, the sets $\{b \in U_1^B; t_B(b/D_i) = t\}$ and $\{b \in U_1^B; t_B(b/D'_i) = p_i(t)\}$ have the same cardinality.*

Proof. Fix i, $1 \le i \le n$. We first remark that for every type t based on D_i, the sets $\{b \in U_1^B - D_i; t_B(b/D_i) = t\}$ and $\{b \in U_1^B - D'_i; t_B(b/D'_i) = p_i(t)\}$ have the same cardinality. Thus, it is possible to extend p_i to an injective map g_i from $D_i \cup U_1^B$ onto $D'_i \cup U_1^B$ such that, for all $b \in U_1^B$, $t_B(g_i(b)/D'_i) = p_i(t_B(b/D_i))$. By Lemma 5.22, for every $b \in U_1^B$ we have that $(h_i)_{|D_i \cup \{b\}}$ is a strong partial automorphism of B. But this easily implies that h_i is a strong partial automorphism of B as every small structure contains at most one point in U_1.

♡

5.4 The weight of a type

Now, we must face the following problem. We want to find a strong extension B of A satisfying the conclusion of Proposition 5.23. Suppose, for example, that the cardinality of the set $\{b \in U_1^A\ ;\ t_A(b/D_i) = t\}$, where t is a type based on D_i, is smaller than the cardinality of $\{b \in U_1^A\ ;\ t_A(b/D'_i) = p_i(t)\}$. It is fairly easy to increase the cardinality of the set $\{b \in U_1^A\ ;\ t_A(b/D_i) = t\}$ by one, and eventually to get a strong extension where the cardinality of these two sets are the same. But, we cannot perform such an operation simultaneously for all types and for all partial automorphisms, because when taking care of another partial automorphism p_j, we may destroy what we have done for p_i. It is to control this phenomenon that we need the notion of weight.

Let C be a tiny extension A. We define:

- $n(C)$ the number of links of C which are not links of A.
- If $a \in C - A$, we define the height of a inductively: $h(a) = 0$ if and only if $a \in U_1^C$; $h(a) = n+1$ if there is a link containing a and an element $b \in C - A$ such that $h(b) = n$ (and the height of a has not been already defined).

Because C is irreducible over A, we know that every element of $C - A$ has a height. Now we define

- the height of C, $h(C) = \max(h(a); a \in C - A)$

- $h_0 = \max(h(C); C$ is a tiny extension of $A)$.
- For i, $0 \leq i \leq h_0$, $w_i(C)$ is the number of elements $a \in A$ such that there exist $b \in C - A$ with $h(b) \leq i$ and a link containing a and b.
- The weight of C is the sequence $w(C) = ((w_0(C), w_1(C), \ldots, w_{h_0}(C), \mathrm{card}(C - A), n(C))$ ordered lexicographically.

Lemma 5.24 *1) Assume that $C \leq B$. Then $w(C) \leq w(B)$ and if $w(C) = w(B)$, then C and B are isomorphic.*

2) If B is based on $D \subseteq A$ and p is a partial automorphism of A of domain D, then $w(B) = w(p(B))$.

Proof. 1.) Let $k : C \underset{w,A}{\longrightarrow} B$ such that $k(u(C)) = u(B)$. If $a \in A$ is linked to $u(C)$ (in C), it is linked (in B) to $u(B)$. Thus $w_0(C) \leq w_0(B)$. Moreover, suppose that $c \in C - A$ has height 1 and $k(c) \in A$. Then $w_0(C) < w_0(B)$. Indeed, there is a link between $u(C)$ and c (because c has height 1), thus there is a link between $u(B)$ and $k(c)$. But since c and $k(c)$ belong to the same U_i, there is no link between $u(C)$ and $k(c)$. Thus there is at least one element of A which is linked to $u(B)$ (in B), but not to $u(C)$.

Thus, if $w_0(C) = w_0(B)$ and $c \in C - A$ has height 1, $k(c) \in B - A$, and of course, has height 1. We can continue and prove in the same way that $w_1(B) \leq w_1(C)$ and if $w_1(B) = w_1(C)$ and $c \in C - A$ has height 2, then $k(c) \in B - A$ and $h(k(c)) \leq 2$. After h_0 steps, we get that either

$$(w_0(C), w_1(C), \ldots, w_{h_0}(C)) < (w_0(B), w_1(B), \ldots, w_{h_0}(B))$$

or

$$(w_0(C), w_1(C), \ldots, w_{h_0}(C)) = (w_0(B), w_1(B), \ldots, w_{h_0}(B))$$

and that k is injective. The first part follows easily.

2) is clear from the definitions.

♡

Definition 5.25 *Suppose $t = (\Gamma, \mathcal{E})$ and $t' = (\Gamma', \mathcal{E}')$ are two types. Then:*

$t \leq t'$ if the identity map from Γ to Γ' is a weak homomorphism and, for every $C \in \mathcal{E}$, there exists $C' \in \mathcal{E}'$ such that $C \leq C'$.

We will need the following easy facts.

Lemma 5.26 *Let $D \subseteq A \subseteq_s B$, t and t' be types based on D.*

1) $t \leq t'$ and $t' \leq t$ implies $t = t'$.

2) For $b \in B$ we have: $t_B(b/A) \geq t$ if and only if $t_B(b/D) \geq t$.

The next definition will give an order homomorphism from the partial order of all types over A into a total order.

Definition 5.27 *Let $t = (\Gamma, \mathcal{E})$ be a type. The weight of t is the sequence $w(t) = (n_0, n_1, \ldots, n_m)$ where: n_0 is the number of links of the structure Γ, and $(n_1, n_2, \ldots, n_m)$ is the weakly decreasing sequence (that is such that $n_1 \geq n_2 \geq \ldots \geq n_m$) of the form $(w(C_1), w(C_2), \ldots, w(C_m))$, where $\mathcal{E} = \{C_1, C_2, \ldots, C_m\}$. These sequences are ordered lexicographically by $\leq$.*

Lemma 5.28 *Let t and t' be two types.*

1) Assume that $t \leq t'$. Then $w(t) \leq w(t')$ and if $w(t) = w(t')$, then $t = t'$.

2) If t is based on $D \subseteq A$ and p is a partial automorphism with domain D, then $w(t) = w(p(t))$.

Proof. Again, 2) follows immediately from the definitions. Set $t = (\Gamma, \mathcal{E})$, $t' = (\Gamma', \mathcal{E}')$. The identity map from Γ to Γ' is a bijective weak A-isomorphism, and, if it not an isomorphism, there are strictly more links in Γ than in Γ', and $w(t) < w(t')$.

So, suppose that $\Gamma = \Gamma'$. Let $\mathcal{E} = \{C_1, C_2, \ldots, C_m\}$, $\mathcal{E}' = \{C'_1, C'_2, \ldots, C'_{m'}\}$, and suppose that $(w(C_1), w(C_2), \ldots, w(C_m))$ and $(w(C'_1), w(C'_2), \ldots, w(C'_{m'}))$ are both weakly decreasing. We know that, for all i, $1 \leq i \leq m$, there exist an integer, say $k(i)$ such that $1 \leq k(i) \leq m'$ and $C_i \leq C'_{k(i)}$. So, $w(C_1) \leq w(C'_{k(1)}) \leq w(C'_1)$ and either $w(t) < w(t')$ or $w(C_1) = w(C'_1)$ and C_1 is equal to $C'_{k(1)}$. By reordering $\{C'_1, \ldots, C'_{m'}\}$ we can suppose $k(1) = 1$. In this case, $k(2) \neq 1$ (because, otherwise, $C_2 \leq C'_1 = C_1$, and two elements of $\mathcal{E}$ cannot be comparable) and $k(2) \geq 2$. Again $w(C_2) \leq w(C'_{k(2)}) \leq w(C'_2)$, and either $w(t) < w(t')$ or $C_2 = C'_{k(2)}$. Going on, we reach the results that, either $w(t) < w(t')$ or k is injective. If k is surjective, this means that $t = t'$, and if not, this implies that $w(t) < w(t')$.

♡

We can now state our last reduction.

Proposition 5.29 *Let A be a finite structure. Then there exists a strong extension B of A such that, for every type t and t', if $w(t) = w(t')$, then the set $\{b \in U_1^B; t_B(b/A) \geq t\}$ and $\{b \in U_1^B; t_B(b/A) \geq t'\}$ have the same number of elements.*

Proof. Let B be as in the conclusion of Proposition 5.29. By Lemma 5.26

$$\{b \in U_1^B; t_B(b/D) = t\} = \{b \in U_1^B; t_B(b/A) \geq t\} - \bigcup_{t'>t,\ t' \text{ is based on } D} \{b \in U_1^B; t_B(b/D) = t'\}$$

We argue by downward induction: assume that we have proved that the sets $\{b \in U_1^B; t_B(b/D_i) = t\}$ and $\{b \in U_1^B; t_B(b/D'_i) = p_i(t)\}$ have the same cardinality for every type t based on D_i of weight strictly bigger than a given value ℓ, and let u be a type based on D_i of weight ℓ. We have

$$\{b \in U_1^B; t_B(b/D_i) = u\} = \{b \in U_1^B; t_B(b/A) \geq u\} - \bigcup_{u'>u,\ u' \text{ is based on } D_i} \{b \in U_1^B; t_B(b/D_i) = u'\}$$

and, analogously,

$$\{b \in U_1^B; t_B(b/D'_i) = p_i(u)\} = \{b \in U_1^B; t_B(b/A) \geq p_i(u)\} - \bigcup_{u'>p_i(u),\ u' \text{ is based on } D'_i} \{b \in U_1^B; t_B(b/D'_i) = u'\}$$

by our previous remark. The map p_i induces a one to one correspondence between the set $\{u' > u, u' \text{ is based on } D_i\}$ and the set $\{u' > p_i(u), u' \text{ is based on } D'_i\}$. Moreover, by induction hypothesis, for each type $u' > u$, u' based on D_i,

$$\text{card}(\{b \in U_1^B; t_B(b/D_i) = u'\}) = \text{card}\,\{b \in U_1^B; t_B(b/D'_i) = p_i(u')\})$$

From these facts,we deduce that

$$\mathrm{card}(\{b \in U_1^B; t_B(b/D_i) = u\}) = \mathrm{card}(\{b \in U_1^B; t_B(b/D_i') = p_i(u)\}).$$

♡

We have now finished the reductions, and it remains to prove Proposition 5.29. Let $n_0 = \max(w(t); t \text{ is a type})$ (here, by abuse of language, we identify that the weight of a type with an integer). We construct by downward induction a chain of strong extensions $(B_n, 0 \leq n \leq n_0 + 1)$ such that, for every types t and t', if $w(t) = w(t') \geq n$, then the set $\{b \in U_1^{B_n}; t_{B_n}(b/A) \geq t\}$ and $\{b \in U_1^{B_n}; t_{B_n}(b/A) \geq t'\}$ have the same number of elements. We start with $B_{n_0+1} = A$, and we end with B_0, the structure that we need.

We show how to get B_n from B_{n+1}. For every type t, let $s(t) = \mathrm{card}(\{b \in B_{n+1}; t_{B_{n+1}}(b/A) \geq t\})$ and $r = \max(s(t); t \text{ is a type of weight } n)$. Assume that for a given type t of weight n, $s(t) < r$. We are going to construct a strong extension B' of B_{n+1} such that $U_1^{B'} = U_1^{B_{n+1}} \cup \{a\}$, where $t_{B'}(a/A) = t$. Thus for no t' with $w(t') \geq w(t)$ and $t' \neq t$ we have $t_B(b/A) \geq t'$. Repeating this process $r - s(t)$ times, and then doing this for all types of weight n, we will get the structure B_n with the desired properties.

By the proof of Lemma 5.18, we know that there exists a strong extension A_1 of A and an element $a \in A_1 - A$ such that $t_{A_1}(a/A) = t$. Let A_2 be the substructure of A_1 whose universe is

$$A_1 - \left\{b; b \in U_1^{A_1} - U_1^A \text{ and } b \neq a\right\}$$

By Lemma 5.4, A_2 is a strong extension of A, and it is immediate to check that $t_{A_2}(a/A) = t$. Now let $B' = B_{n+1} *_A A_2$. By Lemma 5.4 $B_{n+1} \subset_s B'$ and $A_2 \subseteq_s B'$ and therefore also $t_{B'}(a/A) = t$. Thus B' has the desired properties.

♡

6 Final comments.

We wish to conclude this article by some comments and one open question.

Just before the difficult writing of this paper was over, we became aware (thanks to J.E. Pin and P. Weil) of the work of Almeida [1] and of Almeida and Delgado [2]. It seems that starting from a theorem of Ash ([3]) they have proved a theorem which can be seen to be equivalent to our Theorem 3.3. See [2] for more details.

The result in this paper improves the results in [6]. Let us recall some notation from [6]. Suppose $\mathcal{R}$ is a set of link structures and $\mathcal{F}$ is a finite set of structures. We denote by $\mathcal{K}_{\mathcal{R}\mathcal{F}}$ the class of all $\mathcal{F}$-free structures of link type $\mathcal{R}$. If $\mathcal{K}_{\mathcal{R}\mathcal{F}}$ has the amalgamation property (AP) (where we allow the common part to be empty), then there exists a countable homogeneous universal structure in $\mathcal{K}_{\mathcal{R}\mathcal{F}}$, which we call $M_{\mathcal{R}\mathcal{F}}$ and which is uniquely determined up to isomorphism. In [6] the property EPPA was called (WEP). The property (EP) is the following property for a class $\mathcal{C}$: For all finite M_0 and $P \subseteq \mathrm{Part}(M_0, M_0)$ the $(M_0, P, \mathcal{C})$-extension problem has a finite solution.

Theorem 6.1 *Let $\mathcal{L}$ be a finite relational language. Let $\mathcal{R}$ be a set of link structures and $\mathcal{F}$ be a finite set of $\mathcal{L}$-structures.*
a) $\mathcal{K}_{\mathcal{R}\mathcal{F}}$ has EPPA.
b) If $\mathcal{K}_{\mathcal{R}\mathcal{F}}$ has (AP) then $\mathcal{K}_{\mathcal{R}\mathcal{F}}$ has (EP) and $M_{\mathcal{R}\mathcal{F}}$ satisfies the small index property.

See [6] or [7] for the definition of the small index property.

Proof. a) follows from Theorem 3.2. Use the remark after Definition 2.1 to get a slim solution and observe that a slim solution of the $(M_0, P, \mathcal{C})$-extension problem has the same link type as the structure M_0 you started with.
b) If $\mathcal{K}_{\mathcal{RF}}$ has (AP) then $M_{\mathcal{RF}}$ provides an infinite solution for every $(M_0, P, \mathcal{K}_{\mathcal{RF}})$-extension problem, where M_0 is finite: Embed M_0 into $M_{\mathcal{RF}}$, which you can do by the universality of $M_{\mathcal{RF}}$ and by the homogeneity you can extend every partial isomorphism from P to an automorphism of $M_{\mathcal{RF}}$. So in this case EPPA implies (EP). Now use the method of [7] to prove the small index property: Like in the proof of Lemma 14 in [6] deduce that $\mathcal{K}_{\mathcal{RF}}$ actually satisfies the free amalgamation property. This implies that also the classes $\mathcal{K}_{\mathcal{RF}}^{+n}$ (for $n \in \omega$) satisfy (AP). Now Theorem 11 in [6] states that $M_{\mathcal{RF}}$ has the small index property. ♡

The case of the class of tournaments raises an interesting problem. Recall the definition of a tournament: it is a directed irreflexive graph Γ such that, for every two distinct points a and b in Γ, either there is an arrow from a to b, or there is an arrow from b to a, but not both. There is a countable tournament Γ_0 which is universal (every finite tournament can be embedded in it) homogeneous (every isomorphism between two finite tournaments included in Γ_0 can be extended to an automorphism of Γ_0), which in fact is determined up to isomorphism by these properties and which we shall call the generic tournament. The following question is open:

(1) Has the generic tournament the small index property?

But this question would be settled affirmatively if we could prove:

(2) The class of all tournaments has the EPPA.

This last question turns out to be equivalent to a rather natural question about free groups. Let $F(P)$ be the free group generated by the finite set P. Consider the topology on $F(P)$ for which a basis of open sets is

$$\{f \cdot H; f \in F(P), H \text{ is a normal subgroup of } F(P) \text{ of finite odd index}\}.$$

We shall call this topology the odd-adic topology. Then, (2) is equivalent to the following assertion

(3) let H be a finitely generated subgroup of $F(P)$. Then the 2 following properties are equivalent:

(i) H is closed for the odd-adic topology.

(ii) For all $a \in F(P)$, if $a^2 \in H$, then $a \in H$.

We sketch the proof that (2) and (3) are equivalent. We first remark that, in any case, (3 i) implies (3 ii). Assume, toward a contradiction, that H is a subgroup of $F(P)$ which is closed for the odd-adic topology, that $a \notin H$ and $a^2 \in H$. Because H is closed for the odd-adic topology, there exists a homomorphism φ from $F(P)$ onto a finite group G of odd order, such that, if we set $H' = \varphi[H]$ and $a' = \varphi(a)$, then $a'^2 \in H'$ but $a' \notin H'$. Now, if we consider $G_1 = \{g \in G; gH' = H' \text{ and } ga'H' = a'H'\}$ and $G_2 = \{g \in G; \{gH', ga'H'\} = \{H', a'H'\}\}$, then we see that G_1 is a subgroup of order 2 of G_2, and this is impossible since G_1 has odd order, as subgroup of a group of odd order.

Now assume (2). We show that (3 ii) implies (3 i). So, let H be a finitely generated subgroup of $F(P)$ satisfying (3 ii) and α an element of $F(P)$ not in H.

We will first define a tournament T, whose base set is $F(P)/H$ in such a way that, for all $f \in F(P)$, the left multiplication by f is an automorphism of T. To do that, consider, for each pair (aH, bH) in $F(P)/H$ with $aH \neq bH$, the orbit of (aH, bH) under the action of $F(P)$, that is $X_{a,b} = \{(faH, fbH); f \in F(P)\}$. Such an orbit cannot contain both a pair (cH, dH) and (dH, cH).

Otherwise, for some $f \in F(P)$, $fcH = dH$ and $fdH = cH$, so $f^2dH = dH$, and $(d^{-1}fd)^2H = H$. By condition (3 ii), this implies that $(d^{-1}fd) \in H$, so $cH = dH$, which is impossible.

Consider now the set $\{\{aH, bH\}\,; aH \neq bH\}$ and partition it into orbits for the action of $F(P)$. Each orbit O is equal to the set $\{\{faH, fbH\}\,;\ f \in F(P)\}$, where a and b are two elements of $F(P)$ and $aH \neq bH$. The subset $X(O) = \{(cH, dH); \{cH, dH\} \in O\}$ of $(F(P)/H)^2$ is the disjoint union of exactly two orbits of $(F(P)/H)^2$ under the action of $F(P)$. Choose one of them, say $Y(O)$, and define the tournament structure on $F(P)/H$ by deciding that there is an arrow from aH to bH, if and only if $(aH, bH) \in Y(O)$ for some orbit O.

We are now ready to use our machinery. Let X_0 be a finite subset of F(P) containing α, the generators of H and closed under initial segments, as in section 2. Let T_0 consist of the cosets modulo this subset, considered as a subtournament of T. To each element $p \in P$ corresponds a partial automorphism $\widehat{p}$ of T_0. By (2), there exists a tournament T_1 containing T_0 and automorphisms $\widetilde{p}$ of T_1 extending $\widehat{p}$. This allows us to define an action of $F(P)$ on T_1 that is a homomorphism $f \longmapsto \widetilde{f}$ from $F(P)$ into $\mathrm{Aut}(T_1)$, the automorphism group of T_1. Because we have put enough elements in T_0, the stabiliser H' of H for this action includes H, does not contain α and, of course, is of finite index. We want to show that H' is open.

The kernel of this action, that is $K = \{f \in F(P); \widetilde{f} = 1\}$, is a normal subgroup of $F(P)$ of finite index contained in H', and we will be done if we prove that its index is odd. But $F(P)/K$ is isomorphic to a subgroup of $\mathrm{Aut}(T_1)$, and $\mathrm{Aut}(T_1)$ has odd order, because it cannot contain an involution.

Let us now prove that (3) implies (2). We start from a finite tournament T and a set P of partial automorphisms of T. We consider T as a directed graph, and we choose an element x_0 in T. We may assume that for all $x \in T$, there exists $p \in P$ such that $x = p(x_0)$. So, there is a correspondence between the subgroups H of $F(P)$ containing the set

$$X_0 = \{p^{-1} \cdot q; p(x_0) = q(x_0)\} \cup \{r^{-1} \cdot p \cdot q; r(x_0) = p \circ q(x_0)\}$$

and disjoint from the set

$$X_1 = \{p^{-1} \cdot q; p(x_0) \neq q(x_0)\}$$

and the tuple $(U, \widetilde{p}; p \in A)$ where $T \subset U$ and for all $p \in A$, $\widetilde{p}$ is a permutation of U extending p. Let H_0 be the subgroup of $F(P)$ generated by X_0. In fact we have $H_0 = \{p_1p_2\dots p_n\,; p_1 \circ p_2 \circ \dots \circ p_n(x_0) = x_0\}$. We show that H_0 is closed for the odd-adic topology, by showing, by (3), that if $g^2 \in H_0$ then $g \in H_0$.

Say that $g = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, where n is an integer and the p_i are elements of $P \cup P^{-1}$ and this representation is reduced. It is easy to construct a finite tournament T_1 extending T and, for each $p \in P$ a partial automorphism p' of T_1 with the following properties: 1) p' extends p; 2) $p'_1 \circ p'_2 \circ \ldots \circ p'_n(x_0)$ is defined, and if $p_1 \circ p_2 \circ \ldots \circ p_n(x_0)$ is not defined, then $p'_1 \circ p'_2 \circ \ldots \circ p'_n(x_0) \neq x_0$. We may now embed T_1 into the generic tournament T_0 and extend the maps p' (for $p \in P$) into automorphisms $\widetilde{p}$ of T_0. As usual, this will provide us with a homomorphism $f \longmapsto \widetilde{f}$ from $F(P)$ into $\mathrm{Aut}(T_0)$. We see that $\widetilde{g}(x_0) = p'_1 \circ p'_2 \circ \ldots \circ p'_n(x_0)$, and thanks to our hypothesis on T_1, if $g \notin H_0$, then $\widetilde{g}(x_0) \neq x_0$. Since $g^2 \in H_0$, we see that $\widetilde{g^2}(x_0) = \widetilde{g}^2(x_0) = x_0$, and since $\widetilde{g}$ cannot switch 2 different points, $\widetilde{g}(x_0) = x_0$, and $g \in H_0$.

Now, we apply our technique: we find a finite set Γ_1 extending T and a homomorphism $f \longmapsto \widetilde{f}$ from $F(P)$ into $\mathrm{Sym}(\Gamma_1)$ such that, for all $p \in A$, $\widetilde{p}$ extends p. From the fact that H_0 is closed for the odd-adic topology, we may demand that the kernel of this homomorphism is of odd index. Thus, the subgroup G of $\mathrm{Sym}(\Gamma_1)$ generated by $\{\widetilde{p}; p \in P\}$ is of odd order. We can suppose that G acts transitively on Γ_1 and as usual we define for $\alpha, \beta \in \Gamma_1$: there is an arrow from α to β if there exists $a, b \in \Gamma$ and $g \in F(P)$ such that there is an arrow from a to b and $\widetilde{g}(a) = \alpha$ and $\widetilde{g}(b) = \beta$. As G is odd this defines the structure of a directed graph on Γ_1 and Γ is a substructure of Γ_1.

It just remains to add arrows in Γ_1 to turn it into a tournament, in such a way that the $\widetilde{p}$ for $p \in P$ remain automorphisms. We do that exactly as above, when we have defined a tournament structure on the set $F(P)/H$.

References

[1] Jorge Almeida : Hyperdecidable pseudovarieties and the calculation of semidirect products, submitted.

[2] Jorge Almeida & Manuel Delgado : Sur les système d'équations avec contraintes dans un groupe libre, preprint.

[3] Chris J. Ash : Inevitable graphs: a proof of type II conjecture and some related decision procedures, Int. J. Algebra and Computation 1 (1991), pp. 127-146.

[4] Marshall Hall : A topology for free groups and related groups, Ann. of Maths 52 (1950), pp. 127-139.

[5] Bernhard Herwig : Extending partial isomorphisms on finite structures, Combinatorica 15, (1995), pp. 365-371.

[6] Bernhard Herwig : Extending partial isomorphisms for the small index property of many ω-categorical structures, preprint.

[7] Wilfrid Hodges, Ian Hodkinson, Daniel Lascar & Saharon Shelah : The small index property for ω-categorical ω-stable structures and for the random graph, Journal of the London Math. Society (2), 48 (1993), 204-218.

[8] Ehud Hrushovski : Extending partial isomorphisms of graphs, Combinatorica 12, (1992), pp. 204-218.

[9] Luis Ribes & Pavel.A. Zalesskiì, On the profinite topology on a free group. Bull. London Math. Soc. 25 (1993), pp. 37.43.

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