# On $\aleph_0$-categorical weakly $o$-minimal structures

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**Abstract**

$\aleph_0$-categorical $o$-minimal structures were completely described in [4], and are essentially built up from copies of the rationals as an ordered set by 'cutting and copying'. Here we investigate the possible structures which an $\aleph_0$-categorical *weakly* $o$-minimal set may carry, and find that there are some rather more interesting (and not $o$-minimal) examples. We show that even here the possibilities are limited. We subdivide our study into the following principal cases: the structure is 1-indiscernible, in which case all possibilities are classified up to binary structure; the structure is 2-indiscernible, classified up to ternary structure; the structure is 3-indiscernible, in which case we show that it is $k$-indiscernible for every finite $k$. We also make some remarks about the possible structures of higher arities which an $\aleph_0$-categorical weakly $o$-minimal structure may carry.

## 1 Introduction

For any linearly ordered set $(X, <)$ an *open interval* is a subset of the form $(a, b) = \{x \in X : a < x < b\}$ for some $a, b \in X \cup \{\pm\infty\}$. Similarly we may define *closed* or *semi-closed* intervals, so that for instance a point of $X$ is itself a (trivial) closed interval. A subset $C$ of $X$ is *convex* if $x < z < y \wedge x, y \in C \Rightarrow z \in C$. Thus a set is convex if it is an interval whose endpoints are now allowed to lie in the order-completion $\overline{X}$ of $X$.

We recall that a structure $\mathcal{M}$ is said to be *o-minimal* if there is a relation $<$ in its signature which linearly orders $M$ (the domain of $\mathcal{M}$), and such that every definable subset of $M$ is a finite union of intervals, and it is *weakly o-minimal* if every definable subset of $M$ is a finite union of convex subsets. The easiest example of an $o$-minimal structure is the ordered set of rational numbers, and this is also $\aleph_0$-categorical. An important and non-$\aleph_0$-categorical example is afforded by the field of real numbers. It was shown in [4] that all $\aleph_0$-categorical

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$o$-minimal sets can be built up in a fairly straightforward way (by cutting, duplicating, and taking suitable disjoint unions) from $(\mathbb{Q}, <)$.

If we relax the requirement of $o$-minimality to weak $o$-minimality, but still require $\aleph_0$-categoricity, then as we shall see, the class of structures is somewhat broadened, but it is still possible to give substantial structural information without too much trouble. Our discussion proceeds by way of the degree of indiscernibility possessed by the structure. By the Ryll-Nardzewski Theorem, if $\mathcal{M}$ is $\aleph_0$-categorical then for each $k$ it has only a finite number of $k$-types of increasing $k$-tuples, and saying it is *$k$-indiscernible* just means that this number is 1.

Any $\aleph_0$-categorical structure is a finite union of 1-indiscernible pieces (the realizations of the 1-types), so it is natural to begin our investigations by assuming 1-indiscernibility. Here the easiest example of a weakly $o$-minimal structure which is not also $o$-minimal has domain consisting of $\mathbb{Q}$ copies of $\mathbb{Q}$, (that is, $\mathbb{Q} \times \mathbb{Q}$ ordered lexicographically), with an equivalence relation $\sim$ with convex classes saying that elements lie in the same copy of $\mathbb{Q}$. Because $\sim$ is in the language, each equivalence class is definable (with just one parameter), but its endpoints do not lie in the structure, and so $o$-minimality fails. It is almost as easy to see that this structure is weakly $o$-minimal and $\aleph_0$-categorical. The construction immediately generalizes to the lexicographic ordering on $\mathbb{Q}^n$ with a chain of equivalence relations. The main result of §2 is that these are essentially the only possible 1-indiscernible $\aleph_0$-categorical weakly $o$-minimal structures, at any rate if we restrict attention to the binary theory. One can build more general $\aleph_0$-categorical weakly $o$-minimal structures from these much as in [4] for the $o$-minimal case, and it should be possible to describe all possibilities (up to binary theory). We do not go into details here, which will be more involved than in [4], since there are $\aleph_0$ possible 'building blocks' (the $\mathbb{Q}^n$ with the relations mentioned above), rather than just one. The method we adopt throughout most of the paper for looking at these structures is to view them as ultrametric spaces with a finite set of possible distances, and indeed they may be viewed as having been constructed by a Fraïssé-style method among ultrametric spaces with specified finite set of possible distances. An alternative and essentially equivalent method analyses the structure of the cuts in the order-completion of the structure defined by the 2-atoms of increasing pairs.

In §§3 and 4 we carry out a similar analysis for the next stage up, constructing examples of 2-indiscernible $\aleph_0$-categorical weakly $o$-minimal structures, and classifying them up to ternary theory. Once again the two methods of visualization used in the previous case can be applied. Here we also use the key notion of a $C$-relation from [1]. In a similar way that a general 1-indiscernible $\aleph_0$-categorical weakly $o$-minimal structure is constructed from a chain of equivalence relations, here a general structure is constructed from a chain of $C$-relations, or more exactly, 'ordered' $C$-relations. Along with any $C$-relation comes a notion of two points being 'close' with respect to a third, and so a family of ultrametrics also naturally occurs. Since, by [1], $C$-relations may be viewed as branches through a tree under a natural ternary relation, it is not surprising that the construction actually proceeds by way of a suitable tree, with the branches carrying the structure of a 1-indiscernible $\aleph_0$-categorical weakly $o$-minimal set. In principle it would be possible to carry out the classification proof for these structures by showing that one can interpret the tree used in their construction. In fact we use a different method, which involves formulating a list of axioms, which we show are true in any 2-indiscernible $\aleph_0$-categorical weakly $o$-minimal structure, and which characterize the structure uniquely up to isomorphism (assuming countability).

In the remaining sections we show that if an $\aleph_0$-categorical weakly $o$-minimal structure is 3-indiscernible, then it is $k$-indiscernible for all $k$, so that its automorphism group is isomorphic as a permutation group to $\mathrm{Aut}(\mathbb{Q}, <)$ (§5), and that the structures we have constructed may be modified by including a wide variety of relations of higher arities (§6). In particular, there is an example which does not admit a finite language, in the sense that its automorphism group is not the automorphism group of a structure over a finite language, and we show that there are $2^{\aleph_0}$ structures like this.

# 2 $\aleph_0$-categorical, weakly $o$-minimal, 1-indiscernible structures

Let $\mathbb{Q}^n$ be the set of $n$-tuples $x = (x_0, \ldots, x_{n-1})$ of rational numbers, ordered lexicographically by $<$ where $x < y$ if $\exists i(x_i < y_i \wedge (\forall j < i)(x_j = y_j))$, and let the equivalence relation $E_i$ be given by $E_i(x, y) \Leftrightarrow (\forall j < n - i)(x_j = y_j)$. Then for each $i$ the equivalence classes of $E_i$ are convex subsets of $\mathbb{Q}^n$. Moreover, $E_0$ is the identity relation, $E_n$ the universal relation, and $E_{i-1}$ refines $E_i$ for each $i > 0$.

Using a back-and-forth argument one sees that $\mathcal{M} = (\mathbb{Q}^n; =, <, (E_i)_{0 \le i \le n})$ is $\aleph_0$-categorical, admits elimination of quantifiers, and is also weakly $o$-minimal and 1-indiscernible. We define an ultrametric on $\mathbb{Q}^n$ by

$$d(x, y) = i \Leftrightarrow E_i(x, y) \wedge \neg E_{i-1}(x, y).$$

(We recall that a metric $d$ is said to be an *ultrametric* if it satisfies the 'ultrametric inequality' $d(x, z) \le \max(d(x, y), d(y, z))$ for all $x, y$, and $z$. Note that our ultrametrics are unusual in that the 'distances' only take values in the finite set $\{0, 1, \ldots, n\}$.) We define $R_i$ for $1 \le i \le n$ by $R_i(x, y) \Leftrightarrow (x < y \wedge d(x, y) = i)$. Then the $R_i$ are all the 2-atoms in $\mathcal{M}$ of pairs $(x, y)$ with $x < y$. The structures $(\mathbb{Q}^n; =, <, (E_i)_{0 \le i \le n})$ and $(\mathbb{Q}^n; =, <, d)$ and $(\mathbb{Q}^n; =, <, (R_i)_{1 \le i \le n})$ are all inter-definable. (Actually $<$ can be defined in $(\mathbb{Q}^n; =, (R_i)_{1 \le i \le n})$ by $x < y \Leftrightarrow (\exists i) R_i(x, y)$, but we do not need to bother with this.) In a formal treatment, the binary function $d$ into $\{0, \ldots, n\}$ would be represented by $n$ binary relations.

We shall see that, up to binary structure, $(\mathbb{Q}^n; =, <, (R_i)_{1\le i\le n})$ is the only $\aleph_0$-categorical, weakly $o$-minimal, and 1-indiscernible structure having $n$ 2-atoms of increasing pairs.

Let us suppose therefore that $(M; =, <, \ldots)$ is $\aleph_0$-categorical, weakly $o$-minimal, and 1-indiscernible. By $\aleph_0$-categoricity there are finitely many 2-atoms of pairs $(x, y)$ with $x < y$. Let these be $R_1, \ldots, R_n$, indexed so that for all $x, y, y' \in M$ and for all $i, j$ with $1 \le i < j \le n$, $x < y, y' \wedge R_i(x, y) \wedge R_j(x, y') \Rightarrow y < y'$. This is possible since for fixed $x$, the $R_i(x, y)$ are the 1-atoms of the induced weakly $o$-minimal structure on $(x, \infty)^M$, and therefore define convex sets.

Instead of $R_i(x, y)$ we write $d_{\rightarrow}(x, y) = i$. Then the following hold:

- $x < y \wedge d_{\rightarrow}(x, y) = i$ isolates a complete 2-type,
- $x < y < z \Rightarrow d_{\rightarrow}(x, y) \le d_{\rightarrow}(x, z)$.

Now re-index $\{R_i : 1 \le i \le n\}$ as $\{S_i : 1 \le i \le n\}$ in such a way that for all $x, x', y \in M$ and for all $i, j$ with $1 \le i < j \le n$, $x, x' < y \wedge S_i(x, y) \wedge S_j(x', y) \Rightarrow x' < x$. If we write $S_i(x, y)$ as $d_{\leftarrow}(x, y) = i$, then $\{d_{\leftarrow}(x, y) = i : 1 \le i \le n\}$ are just the same formulae as $\{d_{\rightarrow}(x, y) = i : 1 \le i \le n\}$, perhaps in a different order, and

- $x < y \wedge d_{\leftarrow}(x, y) = i$ isolates a complete 2-type,
- $x < y < z \Rightarrow d_{\leftarrow}(y, z) \le d_{\leftarrow}(x, z)$.

We shall show that $d_{\rightarrow} = d_{\leftarrow}$, and is an ultrametric on $M$.

**Lemma 2.1** *For $m \le n + 1$ we have*
*(i) $d_{\rightarrow}(x, y) = m \Leftrightarrow d_{\leftarrow}(x, y) = m$,*
*(ii) if $x < y < z$ and $d_{\rightarrow}(x, y) \le m$ then $d_{\rightarrow}(y, z) = m \Rightarrow d_{\rightarrow}(x, z) = m$,*
*(iii) if $x < y < z$ and $d_{\rightarrow}(x, y) < m$ then $d_{\rightarrow}(y, z) = m \Leftrightarrow d_{\rightarrow}(x, z) = m$.*

**Proof** We prove the lemma by downwards induction on $m$.
**Basis**: $m = n + 1$. All three statements are vacuously fulfilled.
**Induction step**: Assume the result for $m + 1 \le n + 1$.

(i) It suffices to find $x' < y'$ such that $d_{\rightarrow}(x', y') = m$ and $d_{\leftarrow}(x', y') = m$ since both formulae isolate a complete type. Choose $x < y < z$ such that $d_{\rightarrow}(x, y) = m$ and $d_{\leftarrow}(y, z) = m$. Thus $d_{\rightarrow}(x, z) \ge m$ and $d_{\leftarrow}(x, z) \ge m$. We show that $d_{\rightarrow}(x, z) = d_{\leftarrow}(x, z) = m$. If one of $d_{\rightarrow}(x, z)$, $d_{\leftarrow}(x, z)$ is bigger than $m$ then by the induction hypothesis the other is too. But if $d_{\rightarrow}(x, z) > m$, then by (iii) (for values greater than $m$), $d_{\rightarrow}(y, z) = d_{\rightarrow}(x, z) > m$, and by (i) again, $d_{\leftarrow}(y, z) > m$, which is not the case.

(ii) Let $x < y < z$ and $d_{\rightarrow}(x, y) \le m$ and $d_{\rightarrow}(y, z) = m$. By (i) we have $d_{\leftarrow}(y, z) = m$ and therefore $d_{\leftarrow}(x, z) \ge m$ and $d_{\rightarrow}(x, z) \ge m$. If $d_{\rightarrow}(x, z) > m$ then by (iii) (for values greater than $m$), $d_{\rightarrow}(y, z) > m$. Thus $d_{\rightarrow}(x, z) = m$.

(iii) Let $x < y < z$ and $d_{\to}(x,y) < m$. We have to show that $d_{\to}(x,z) = m \Rightarrow d_{\to}(y,z) = m$ as the other direction follows from (ii). Suppose $d_{\to}(x,z) = m$. We choose $x < w_1 < w_2$ such that $d_{\to}(x,w_1) = m$ and $d_{\to}(w_1,w_2) = m$. By (ii) $d_{\to}(x,w_2) = m$. Now for all $y'$ if $d_{\to}(x,y') < m$ then $y' < w_1$ and by (i) $m = d_{\leftarrow}(x,w_2) \geq d_{\leftarrow}(y',w_2) \geq d_{\leftarrow}(w_1,w_2) = m$. Using (i) again, $d_{\to}(y',w_2) = m$. So there is $w'$ (namely $w_2$) with $d_{\to}(x,w') = m$ such that for all $y'$, $(d_{\to}(x,y') < m \Rightarrow d_{\to}(y',w') = m)$. As the formula $d_{\to}(x,w') = m$ isolates a complete 2-type, this holds for *every* $w'$ such that $d_{\to}(x,w') = m$, in particular for $z$, and so we deduce from $d_{\to}(x,z) = m$ and $d_{\to}(x,y) < m$ that $d_{\to}(y,z) = m$. $\square$

Now we define

$$d(x,y) = \begin{cases} d_{\to}(x,y) & \text{if } x < y \\ d_{\to}(y,x) & \text{if } y < x \\ 0 & \text{if } y = x. \end{cases}$$

Then $d$ induces an ultrametric. To see this we have to check the following statements:

- For $x < y < z$, $d(x,y) \leq d(y,z) \Rightarrow d(x,z) = d(y,z)$,
- For $x < y < z$, $d(z,y) \leq d(y,x) \Rightarrow d(z,x) = d(y,x)$.

The first statement is a reformulation of (ii) of Lemma 2.1 and the second statement follows from the same lemma applied to the reversed order. We further observe that $d$ is *compatible* with the ordering $<$ in the sense that for every $x$ and $i$ the set $\{y : d(x,y) \leq i\}$ is convex.

The following theorem sums up what we have shown:

**Theorem 2.2** *If $(M; =, <, \ldots)$ is $\aleph_0$-categorical, weakly o-minimal, and 1-indiscernible having $n$ 2-atoms of pairs $(x,y)$ with $x < y$, then there is an ultrametric $d$ on $M$ taking values in $\{0,\ldots,n\}$ compatible with $<$ such that the $n$ 2-atoms on increasing pairs $(x,y)$ are given by $d(x,y) = i$ (for $1 \leq i \leq n$). The ultrametric $d$ is uniquely determined by these properties.*

**Theorem 2.3** *The 0-definable equivalence relations on $M$ are precisely $E_i$ for $0 \leq i \leq n$ given by $E_i(x,y) \Leftrightarrow d(x,y) \leq i$.*

**Proof** Let $E$ be a 0-definable equivalence relation, $(x)_E$ the $E$-class containing $x$. If the classes of $E$ had more than one convex component, then the formula saying that $x$ is in the leftmost component of $(x)_E$ would be true of some but not all elements of $M$, contrary to 1-indiscernibility.

The set $\{(x,y) : xEy\}$ is a union of some of the 2-atoms and because the equivalence classes are convex there are $i,j$ such that $xEy \Leftrightarrow (y \leq x \wedge d(y,x) \leq i) \vee (x \leq y \wedge d(x,y) \leq j)$. By symmetry of $E$ we get $i = j$. $\square$

**Theorem 2.4** *For each $i < n$ and $x \in M$, $X = \{(y)_{E_i} : y \in (x)_{E_{i+1}}\}$ is countable and densely linearly ordered without endpoints.*

**Proof** Pick $y, z \in (x)_{E_{i+1}}$ such that $y < x < z$ and $d(y,x) = i+1 = d(x,z)$. We have $(y)_{E_i} < (x)_{E_i} < (z)_{E_i}$ and as $x$ was arbitrary $X$ has no endpoints. For density let $y', z' \in (x)_{E_{i+1}}$ with $(y')_{E_i} < (z')_{E_i}$, which just means that $d(y', z') = i+1$. Thus there is an automorphism $g$ sending $(y,z)$ to $(y', z')$, and then $(y')_{E_{i+1}} < (x^g)_{E_{i+1}} < (z')_{E_{i+1}}$. □

Theorem 2.4 just says that $(M; =, <, d)$ is universal and homogeneous (in the sense of Fraïssé) among all countable ultrametric spaces with possible distances in $\{0, 1, \ldots, n\}$ which are compatible with the ordering.

**Corollary 2.5** *If $(M; =, <, \ldots)$ is $\aleph_0$-categorical, weakly o-minimal, and 1-indiscernible having $n$ 2-atoms of pairs $(x,y)$ with $x < y$, then*

$$(M; =, <, (E_i)_{0 \leq i \leq n}) \cong (\mathbb{Q}^n; =, <, (E_i)_{0 \leq i \leq n}).$$

A variant of this corollary expresses things in terms of the 2-atoms rather than the 0-definable equivalence relations.

**Corollary 2.6** *If $(M; =, <, \ldots)$ is $\aleph_0$-categorical, weakly o-minimal, and 1-indiscernible having $n$ 2-atoms $R_i$ of pairs $(x,y)$ with $x < y$, indexed so that for $x < y, y'$ and $i < j$, $R_i(x,y) \wedge R_j(x,y') \Rightarrow y < y'$, then*

$$(M; =, <, (R_i)_{1 \leq i \leq n}) \cong (\mathbb{Q}^n; =, <, (R_i)_{1 \leq i \leq n}).$$

# 3 $\aleph_0$-categorical, weakly $o$-minimal, 2-indiscernible structures

In this section we carry out a similar classification of the structures in the 2-indiscernible case, this time up to ternary structure. We first construct examples of $\aleph_0$-categorical, weakly $o$-minimal, 2-indiscernible structures, using the structures in §2 as auxiliary, and then show that up to ternary structure these are the only ones. The construction gives a tree, and the weakly $o$-minimal structure we seek arises as a set of branches through the tree. (By *tree* we understand a partially ordered set for which any two elements have a common lower bound, and any two incomparable elements have no common upper bound.) So, not surprisingly, the resulting structures carry '$C$-relations' on them (see [1]). Here they are actually 'ordered $C$-relations', where this means that there is natural compatibility with $<$.

We start with $\mathbb{Q}^m$ ordered lexicographically and with $d$ defined as in §2. Fix a function $e : \mathbb{Q}^m \to 2$ $(= \{0,1\})$ such that $e^{-1}(0), e^{-1}(1)$ are both dense in $\mathbb{Q}^m$. We let $M$ be the set of maps $a$ from $\mathbb{Q}^m$ to 2 with $\{x : a(x) \neq e(x)\}$ finite, and $T$ be the family of initial segments of elements of $M$. Thus $T = \{a|(-\infty, x) : a \in M, x \in \mathbb{Q}^m\}$.

We view $M \cup T$ as a structure. Here $T$ is the 'scaffolding' used to build $M$, which may be discarded when the construction is complete. The basic relations on $M \cup T$ are as follows:

- a unary predicate picking out $M$,
- the lexicographic ordering $<$ on $M$,
- 'extension' $\preceq$ on $M \cup T$, given by $a \preceq b$ if, as functions, $b$ extends $a$.

If $a \preceq b \in T$ or $b \preceq a \in T$ we write $d(a,b)$ for $d(\sup \operatorname{dom} a, \sup \operatorname{dom} b)$.

Now for any $a < b$ in $M$ there is a least $x \in \mathbb{Q}^m$ such that $a(x) \neq b(x)$, and for this $x, a(x) < b(x)$. We define $t(a,b)$ to be $a|(-\infty, x)$. This suggests a general method for interpreting $T$ in $M$ (though we do not actually do things this way).

The ternary structure induced on $M$ by these relations is as follows:

- $T_i(x,y,z)$ if $x < y < z$ in $M$ and $t(x,y) \prec t(y,z)$ and $d(t(x,y), t(y,z)) = i$,
- $U_i(x,y,z)$ if $x < y < z$ in $M$ and $t(y,z) \prec t(x,y)$ and $d(t(y,z), t(x,y)) = i$.

Notice that if $x < y < z$, since $T$ is a *binary* tree, we cannot have $t(x,y) = t(y,z)$, so either $t(x,y) \prec t(y,z)$ or $t(y,z) \prec t(x,y)$, and for some $i$, $T_i(x,y,z)$ or $U_i(x,y,z)$.

The structure $\mathcal{M}_{2m}$ is then defined by

$$\mathcal{M}_{2m} = (M; =, <, T_1, \ldots, T_m, U_1, \ldots, U_m).$$

A slight modification yields $\mathcal{M}_{2m+1}$. This time we take $M$ to consist of all maps from $\mathbb{Q}^m$ to $\mathbb{Q}$ such that $\operatorname{supp} a = \{x : a(x) \neq 0\}$ is finite. We define $T_i, U_i$ as before, and in addition let $V(x,y,z)$ if $x < y < z$ in $M$ and $t(x,y) = t(y,z)$. Then $\mathcal{M}_{2m+1} = (M; =, <, T_1, \ldots, T_m, V, U_1, \ldots, U_m)$.

**Remarks 3.1** (1) By definition of 'lexicographic', if $x < y < z$ then $t(x,y) \prec t(y,z) \Rightarrow t(x,y) = t(x,z)$ and $t(x,y) \succ t(y,z) \Rightarrow t(y,z) = t(x,z)$. The converse implications also hold in $\mathcal{M}_n$ for $n$ even. If $x < y < z$ and $t(x,y) = t(y,z)$ in $\mathcal{M}_n$ then $n$ must be odd, and $t(x,y) = t(x,z) = t(y,z)$.

(2) The desired weakly $o$-minimal structures are the $\mathcal{M}_n$ (see Theorem 3.2 below). The analysis of a general 2-indiscernible, weakly $o$-minimal, $\aleph_0$-categorical structure may proceed by interpreting $T$ in $\mathcal{M}_n$ via equivalence classes of ordered pairs and the map $t$ above, though the method we use is to show that a certain list of axioms, which are true in $\mathcal{M}_n$, and which characterize

a countable structure uniquely up to isomorphism, hold in any 2-indiscernible, weakly $o$-minimal, $\aleph_0$-categorical structure.

(3) It will turn out that $T_i, U_i$ (and $V$ if $n$ is odd) are the atoms of $\mathcal{M}_n$ on increasing triples. From the tree representation it is easy to see that they come in the following order, as $z$ increases, for fixed $x$ and $y$:
$z < x < y : T_m(z,x,y), \ldots, T_1(z,x,y), V(z,x,y), U_1(z,x,y), \ldots, U_m(z,x,y)$,
$x < z < y : U_m(x,z,y), \ldots, U_1(x,z,y), V(x,z,y), T_1(x,z,y), \ldots, T_m(x,z,y)$,
$x < y < z : T_m(x,y,z), \ldots, T_1(x,y,z), V(x,y,z), U_1(x,y,z), \ldots, U_m(x,y,z)$,
(where $V$ just applies if $n$ is odd). Thus the order of the terms is reversed as we pass through a parameter, as in §2.

(4) The structures $\mathcal{M}_n$ can be described in terms of ultrametrics which are parametrized by elements of $\mathcal{M}_n$: If $x < y < z$ we write $d^r_x(y,z) = i$ (for $1 \leq i \leq n$) if $(x,y,z)$ lies in the $i$th 3-atom in increasing order in the above enumeration (regarding $x$ as fixed, $d^r_x$ measures distances between points to the *right* of $x$), and similarly for $d^l_x$ (distances between points on the *left* of $x$). So if $1 \leq i \leq m$ and $x < y < z$ then

$$\begin{aligned}
d^r_x(y,z) = i \quad &\Leftrightarrow \quad T_{m+1-i}(x,y,z) \\
&\Leftrightarrow \quad t(x,y) \prec t(y,z) \wedge d(t(x,y), t(y,z)) = m+1-i \\
&\Leftrightarrow \quad d^l_z(x,y) = n+1-i, \\
d^r_x(y,z) = n+1-i \quad &\Leftrightarrow \quad U_{m+1-i}(x,y,z) \\
&\Leftrightarrow \quad t(y,z) \prec t(x,y) \wedge d(t(y,z), t(x,y)) = m+1-i \\
&\Leftrightarrow \quad d^l_z(x,y) = i.
\end{aligned}$$

If $n = 2m+1$ is odd then we have in addition

$$d^r_x(y,z) = m+1 \Leftrightarrow V(x,y,z) \Leftrightarrow t(x,y) = t(y,z) \Leftrightarrow d^l_z(x,y) = m+1.$$

To make $d^r_x$ into an ultrametric on $(x, \infty)$ we have to symmetrize the function $d^r_x$ defined above and add: if $y > x$ then $d^r_x(y,y) = 0$. The structure $\mathcal{M}_n$ is interdefinable with $(M; =, <, d^r_x)$, where the functions $d^r_x$ are represented by means of ternary relations. For the proof that $d^r_x$ is an ultrametric, see Theorem 4.5.

(5) The construction given generalizes one which may be given using a '$C$-relation'. We say that a *$C$-relation* on a set $X$ is a ternary relation $C$ (written with a semicolon between the first two arguments, to emphasize the symmetry between the second and third) such that

(C0) $x \neq y \rightarrow C(x; y, y)$,
(C1) $C(x; y, z) \rightarrow C(x; z, y)$,
(C2) $C(x; y, z) \rightarrow \neg C(y; x, z)$,
(C3) $C(x; y, z) \wedge \neg C(w; y, z) \rightarrow C(x; w, z)$.

The definition given in [1] has two further axioms, but those we give are the universal axioms, and the treatment goes more smoothly with just these four.

The other ($\forall\exists$) axioms will in any case hold in the generic model. In addition, when there is a linear ordering on the structure, we say that the $C$-relation is *order-compatible* if $C(x;y,z) \wedge y < w < z \Rightarrow C(x;y,w)$. This is the same as saying that when we fix $x$, the equivalence relation given by $y \sim z$ if $C(x;y,z)$ has convex classes.

If in the present context we define $C_i$ for $1 \leq i \leq m$ by

$$C_i(x;y,z) \text{ if } (y,z \in (x,\infty) \wedge d^r_x(y,z) \leq i) \vee (y,z \in (-\infty,x) \wedge d^l_x(y,z) \leq i)$$

then $C_i$ is indeed an order-compatible $C$-relation, and moreover $T_i$ and $U_i$ can be recovered from $\{C_j\}_{j \leq m}$ by

$$\begin{aligned} T_i(x,y,z) \quad &\Leftrightarrow \quad x < y < z \wedge C_{m+1-i}(x;y,z) \wedge \neg C_{m-i}(x;y,z) \\ \text{and } U_i(x,y,z) \quad &\Leftrightarrow \quad x < y < z \wedge C_{m+1-i}(z;x,y) \wedge \neg C_{m-i}(z;x,y), \end{aligned}$$

and for $n$ odd $V$ can also be recovered by

$$V(x,y,z) \Leftrightarrow x < y < z \wedge \neg C_m(x;y,z) \wedge \neg C_m(z;x,y).$$

Thus the situation is analogous to that in §2, where the models could be specified by the 2-atoms $R_i$, the equivalence relations $E_i$, or the ultrametric $d$. Here the $C$-relations play the role of the equivalence relations; in fact, they may be regarded as indexed families of equivalence relations. If $C$ is one of the $C$-relations mentioned above, we get corresponding equivalence relations $\sim_x$ on $\{y \in M : x \neq y\}$ by letting $y \sim_x z$ if $C(x;y,z)$.

Since $T_i$, $U_i$, and (for $n$ odd) $V$ can be defined from the $C_i$ it follows that so can the ultrametrics $d^r_x$ and $d^l_x$ described in (4). Spelling this out in more detail we have, for $x < y < z$:

$$\begin{aligned} \text{for } 1 \leq i \leq m: \quad & d^r_x(y,z) = i \quad && \text{if } C_i(x;y,z) \wedge \neg C_{i-1}(x;y,z), \\ \text{for } n+1-m \leq i \leq n: \quad & d^r_x(y,z) = i \quad && \text{if } C_{n+1-i}(z;x,y) \wedge \neg C_{n-i}(z;x,y), \\ \text{for} i = m+1: \quad & d^r_x(y,z) = m \quad && \text{if } \neg C_m(x;y,z) \wedge \neg C_m(z;x,y), \end{aligned}$$

where the third clause is omitted if $n = 2m$ is even (which corresponds to the condition $x < y < z \Rightarrow C_m(x;y,z) \vee C_m(z;x,y)$), and $d^l_z(x,y) = n+1-d^r_x(y,z)$, and $C_0(x;y,z)$ just means $x \neq y$ and $y = z$.

The idea of a homogeneous ordered $C$-relation is implicit in [2]. For instance, $\mathcal{M}_2$ is $\partial PT_3$ of that paper.

(6) The structure $\mathcal{M}_3$ is a reduct of one of the weakly $o$-minimal structures first considered (by Dickmann, see [3], where the notion of weak $o$-minimality was originally defined). Let $(R;=,<,+,\cdot)$ be a countable non-archimedean real-closed field, $V$ the valuation ring consisting of the finite elements, $\Gamma$ the value group, and $v : R \to \Gamma$ the corresponding valuation. Define $C$ on $R$ by $C(x;y,z)$ if $v(y-x) < v(y-z)$. Then $\mathcal{M}_3$ is essentially the structure $(R;=,<,C)$ (that is, $T, V, U$ are interdefinable with $C$). Also $C$ is definable in $(R;=,+,\cdot,v)$, and the latter is weakly $o$-minimal by [3] (though of course it is not $\aleph_0$-categorical).

**Theorem 3.2** *For each $n \geq 1$, $\mathcal{M}_n$ is an $\aleph_0$-categorical, weakly o-minimal, 2-indiscernible structure having exactly $n$ atoms of strictly increasing triples given by $T_i$, $U_i$ (and, for odd $n$, $V$).*

**Proof** This may be seen directly, but it will also follow from the work of the next section. There we give a list of axioms which we shall show hold in $\mathcal{M}_n$, and in any $\aleph_0$-categorical weakly $o$-minimal 2-indiscernible structure with $n$ 3-atoms of increasing triples. The back-and-forth proof establishes quantifier elimination, and the other properties follow from this. For instance, to check weak $o$-minimality one only needs to show that sets defined (using parameters) from $<$, or one of the $T_i$, $U_i$, $V$ are convex, and this is straightforward. □

# 4 An axiomatization in terms of ultrametrics

In this section we give a list of axioms, prove that they hold in $\mathcal{M}_n$, and in any $\aleph_0$-categorical, weakly $o$-minimal, 2-indiscernible structure with $n$ 3-atoms of increasing triples, and that they are $\aleph_0$-categorical. This will conclude the classification (up to ternary structure) in the 2-indiscernible case. The reader may wish to begin by reading the statement and proof of Theorem 4.6 for an outline.

The axioms to describe such a structure $M$ are formulated in a language containing $<$ and binary functions $d_a^r$, $d_a^l$ to $\{0, 1, \ldots, n\}$ for each $a \in M$ (which are formally represented by means of ternary relations), and are as follows, where by 'dual' we understand the statement obtained by reversing all orderings on elements of $M$, and interchanging $d^r$ and $d^l$:

**Universal axioms**

I $<$ is a linear ordering,

II $d_x^r$ is an ultrametric on $(x, \infty)$, and $d_x^l$ is an ultrametric on $(-\infty, x)$,

III for $x < y$, $\{z > x : d_x^r(y, z) \leq i\}$ and $\{z < y : d_y^l(x, z) \leq i\}$ are convex,

IV for $x < y < z$, $d_x^r(y, z) + d_z^l(x, y) = n + 1$,

V for $x < y < z < w$ and $i \leq n/2$, $d_x^r(y, z) \leq i \rightarrow (d_x^r(z, w) \leq i \vee d_w^l(y, z) \leq i)$ and dual,

**$\forall\exists$ axioms**

VI for $i \leq n$, $\forall x \forall y (x < y \rightarrow (\exists z > y)(d_x^r(y, z) = i))$ and dual,

VII for $i \leq n$, $\forall x \forall y (x < y \rightarrow \exists z (x < z < y \wedge d_x^r(z, y) = i))$,

VIII for $i \leq n/2$ and $i \leq j \leq n + 1 - i$, $\forall x \forall y \forall z (x < y < z \wedge d_x^r(y, z) = i$ $\rightarrow \exists w (x < w < y \wedge d_x^r(w, y) = i \wedge d_w^r(y, z) = j))$ and dual,

IX for $j, k_1, k_2 \leq n/2, \min(k_1, k_2) = j$, $\forall x \forall y \forall z (x < y < z \wedge d_x^r(y, z) = j$ $\rightarrow \exists w (z < w \wedge d_w^l(y, z) = k_1 \wedge d_x^r(z, w) = k_2))$, and dual.

Once again, the $\forall\exists$ axioms VI-IX will be true in the (Fraïssé-) generic structure.

We first draw some consequences from the universal axioms and point out that they just axiomatize refining order-compatible $C$-relations.

**Lemma 4.1** *Let $M$ be a structure satisfying axioms I to V. Then the following statements and their duals hold:*

*(i) If $x < y < z, w$ then $d^r_y(z,w) \geq d^r_x(z,w)$.(For this only I-IV are required.)*

*(ii) Suppose that $x < y < z < w$ and $i \leq n$. If $d^r_x(y,z) \leq i$ then $d^r_x(z,w) \leq i$ or $d^l_w(y,z) \leq i$.*

*(iii) If $x < y < z < w$, then $\min(d^r_x(z,w), d^l_w(y,z)) \leq d^r_x(y,z)$.*

*(iv) If $i \leq n/2$, $i \leq j \leq n+1-i$, $x < y < z < w$, $d^r_x(y,w) = i$ and $d^r_y(z,w) = j$ then $d^r_x(y,z) = d^r_x(z,w) = i$.*

*(v) If $x < y < z < w$, $d^r_x(y,z) \leq n/2$ and $d^l_w(x,y) \leq n/2$, then $d^l_w(y,z) = \min(d^l_w(x,y), d^r_x(y,z))$.*

**Proof** (i) Suppose without loss of generality that $z < w$. By axiom III $d^l_w(y,z) \leq d^l_w(x,z)$, and applying axiom IV we get $d^r_y(z,w) = n+1-d^l_w(y,z) \geq n+1-d^l_w(x,z) = d^r_x(z,w)$.

(ii) Let $i > n/2$, as the case $i \leq n/2$ is axiom V. Suppose that $d^r_x(y,z) \leq i$ but $d^l_w(y,z) \geq i+1$. By (i), $d^r_x(z,w) \leq d^r_y(z,w)$ and by axiom IV, $d^r_y(z,w) = n+1-d^l_w(y,z) \leq n+1-(i+1) \leq n-n/2 = n/2 \leq i$.

(iii) This is a simple reformulation of (ii).

(iv) By axiom IV we have $i \leq d^l_w(y,z) \leq n+1-i$. If $d^r_x(y,z) < i$ then by axiom V either $d^l_w(y,z) < i$ or $d^r_x(z,w) < i$, and the latter implies $d^r_x(y,w) < i$, so both cases are impossible. As on the other hand $d^r_x(y,z) \leq d^r_x(y,w) = i$ we must have $d^r_x(y,z) = i$.

¿From $d^l_w(y,z) \leq n+1-i$ and $d^l_w(x,y) = n+1-i$ it follows by axiom II that $d^l_w(x,z) = n+1-i$, which gives $d^r_x(z,w) = i$.

The fact that $d^r_x(y,w) = i \leq n/2$ corresponds to the fact that the branches to $y$ and $w$ meet on 'level $i$', above where the branch to $x$ meets each of them. That $i \leq j = d^r_y(z,w) \leq n+1-i$ indicates that the branch to $z$ meets those to $y$ and $w$ on level $i$ or above. This is all illustrated in Fig.1(a), where we have not shown the branch to $z$, since it may join the branch to $y$ either above or below where it joins the branch to $w$. In Figure 3 below, the lack of a line to vertex $w$ has a similar interpretation. (The diagram is to assist the reader, but does not of course form part of the proof.)

(v) Let $d^l_w(x,y) = i \leq n/2$. Then $d^r_x(y,w) = n+1-i$ by axiom IV. Since $d^r_x(y,z) \leq n/2 < n+1-i$, axiom II gives $d^r_x(z,w) = n+1-i$. Using axiom IV again, $d^l_w(x,z) = i$. By axiom III, $d^l_w(y,z) \leq i$, thus $d^l_w(y,z) \leq d^l_w(x,y)$.

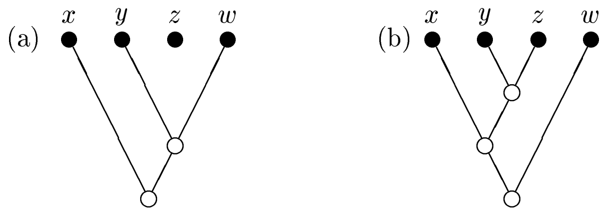



Figure 1: Diagrams illustrating Lemma 4.1(iv) and (v)

Let $d^r_x(y,z) = j \leq n/2$. By axiom II, $d^l_w(x,z) \leq \max(d^l_w(x,y), d^l_w(y,z)) = d^l_w(x,y) \leq n/2$, from which it follows that $d^r_x(z,w) > n/2 \geq j$. By axiom V, we get $d^l_w(y,z) \leq j = d^r_x(y,z)$.

So we have $d^l_w(y,z) \leq \min(d^l_w(x,y), d^r_x(y,z))$. By the dual of (iii), $\min(d^l_w(x,y), d^r_x(y,z)) \leq d^l_w(y,z)$. The configuration here is similarly illustrated in Fig.1(b). This time, since we know where the branches meet, all possible lines have been shown. □

Axiom V is crucial. It ensures that the relation $(d^r_x(y,z) \leq i \vee d^l_x(y,z) \leq i)$ defines a $C$-relation $C_i$ for $i \leq n/2$. In fact axioms I-V assert that $(C_i : 1 \leq i \leq n/2)$ is a family of refining $C$-relations:

**Lemma 4.2** *Let $(M; =, <)$ be a linearly ordered structure.*

*(i) Suppose $C_1, \ldots, C_m$ is a sequence of order-compatible $C$-relations with $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_m$. Let us write in addition $C_0(x;y,z)$ for $x \neq y = z$, and for $x \in M$ let the ultrametrics $d^r_x$ and $d^l_x$ be given as at the end of Remark 3.1 (5) (where there are two possible cases corresponding to $n$ odd or even, the latter distinguished by $x < y < z \Rightarrow C_m(x;y,z) \vee C_m(z;x,y)$). Then axioms I-V hold in $(M; =, <)$ expanded by the $d^r_x$, $d^l_x$.*

*(ii) Suppose that $d^r_x$ and $d^l_x$ are ultrametrics for which axioms I-V hold. For $m$ the integer part of $n/2$ and $1 \leq i \leq m$ we define:*

$$C_i(x;y,z) \iff d^r_x(y,z) \leq i \text{ or } d^l_x(y,z) \leq i.$$

*Then each $C_i$ is an order-compatible $C$-relation, and $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_m$.*

**Proof** The proof is fairly routine. Let us only point out (in the proof of (ii)) that axiom V only states one of three cases of the $C$-relation axiom (C3). The other two cases already follow from the first four axioms. For example suppose that $x < w < y < z$ and $C_i(x;y,z)$ holds, which means that $d^r_x(y,z) \leq i$. We have to show $d^r_x(w,z) \leq i$ or $d^l_z(w,y) \leq i$. But $d^r_x(y,z) \leq i \Rightarrow d^l_z(x,y) \geq n+1-i \Rightarrow d^l_z(x,w) \geq n+1-i$ or $d^l_z(w,y) \geq n+1-i$. The former implies $d^r_x(w,z) \leq i$ and the latter $d^r_x(y,z) \leq d^r_w(y,z) \leq i$ by Lemma 4.1(i). □

Next we want to verify the axioms in a general 2-indiscernible $\aleph_0$-categorical weakly $o$-minimal structure having $n = 2m$ or $2m+1$ 3-atoms $R_1, \ldots, R_n$ of increasing triples and we first say how $d^r$ and $d^l$ are given. We fix $a$ in the structure $M$ and look at the ordered structure $M_a$ with domain $(a, \infty)$ and the binary relation $R_{i,a} = \{(x,y) \in (a,\infty) : R_i(a,x,y)\}$ for every 3-atom $R_i$. Then $M_a$ is weakly o-minimal and 1-indiscernible and has $n$ 2-atoms on increasing pairs, so by Theorem 2.2 there is an ultrametric $d^r_a$ on $M_a$ such that the 2-atoms on increasing pairs $M_a$ are given by $d^r_a(x,y) = i$ for $i = 1, \ldots n$. In the same way we may consider the structure with domain $(-\infty, a)$ and get an ultrametric $d^l_a$ on it. In each case the ultrametric is uniquely determined.

**Theorem 4.3** *Let $\mathcal{N}$ be a countable, weakly o-minimal, 2-indiscernible structure with domain $N$, ordering $<$, and $n$ atoms of strictly increasing triples. Then axioms I-IX all hold in $\mathcal{N}$.*

**Proof** II and III These follow from Theorem 2.2.

IV Fix $a < b$. By 2-indiscernibility we can pick $x_1 < x_2 < \ldots < x_n$ with $a < x_i < b$ such that $(a, x_i, b)$ for $1 \le i \le n$ realize all the 3-atoms. Then $n \ge d^r_a(x_1, b) > \ldots > d^r_a(x_n, b) \ge 1$ and $1 \le d^l_b(a, x_1) < \ldots < d^l_b(a, x_n) \le n$ and so for $1 \le i \le n$ we have $d^r_a(x_i, b) = n+1-i$ and $d^l_b(a, x_i) = i$. As all the formulae of the form $d^r_a(x,b) = j$ and $d^l_b(a,x) = j$ isolate complete 3-types it follows that $d^l_b(a,x) = i \Leftrightarrow d^r_a(x,b) = n+1-i$ which is equivalent to the equality we wanted to prove.

V To prove this axiom, we introduce some notation. For $0 \le i \le n/2$ and $a < b$ we let

$E^{r,i}_a(b) = \{x : a < x \wedge d^r_a(x,b) \le i\}$,

$E^{r,i}_a(b)^+ = \{x \in E^{r,i}_a(b) : x > b\}$,

$E^{r,i}_a(b)^- = \{x \in E^{r,i}_a(b) : x < b\}$,

and similarly

$E^{l,i}_b(a) = \{x : x < b \wedge d^l_b(x,a) \le i\}$,

$E^{l,i}_b(a)^+ = \{x \in E^{l,i}_b(a) : x > a\}$, and

$E^{l,i}_b(a)^- = \{x \in E^{l,i}_b(a) : x < a\}$.

**Lemma 4.4** *Let $i \le n/2$, $a < x < b$, $i < d^r_a(x,b) < n+1-i$. Then $E^{r,i}_a(x) = E^{l,i}_b(x)$.*

**Proof** We prove by induction on $i$ that $E^{r,i}_a(x)^+ = E^{l,i}_b(x)^+$. The case $i = 0$ is obvious.

As both sets are convex with infimum $x$, either $E^{r,i}_a(x)^+ \subseteq E^{l,i}_b(x)^+$ or $E^{l,i}_b(x)^+ \subseteq E^{r,i}_a(x)^+$. Suppose for a contradiction that $E^{l,i}_b(x)^+ \subset E^{r,i}_a(x)^+$ (the case $E^{r,i}_a(x)^+ \subset E^{l,i}_b(x)^+$ being similar; note that we use $\subset$ to stand for *proper* subset throughout the paper). Let $k = d^r_a(x,b)$, so that $i < k < n+1-i$. As $(d^r_a(x,b) = k) \wedge (a < x < b)$ isolates a complete type in $b$, we also have that $b' > x \wedge d^r_a(x,b') = k \Rightarrow E^{l,i}_{b'}(x)^+ \subset E^{r,i}_a(x)^+$. Note that by the dual of Lemma 4.1(i), $a < x < b_1 < b_2$ implies $E^{l,i}_{b_1}(x)^+ \subseteq E^{l,i}_{b_2}(x)^+$.

We let $U_x = \bigcap\{E^{l,i}_y(x)^+ : x < y \wedge d^r_a(x,y) = k\}$. Then

- $U_x$ is definable over $a, x$,
- $U_x \subset E^{r,i}_a(x)^+$,
- $E^{r,i-1}_a(x)^+ \subset U_x$.

To prove the last statement, pick $c > x$ with $d^r_a(x,c) = k-1$. For every $b' > x$ with $d^r_a(x,b') = k$ we have $c < b'$ and $E^{l,i}_c(x)^+ \subseteq E^{l,i}_{b'}(x)^+$ and thus $E^{l,i}_c(x)^+ \subseteq U_x$ which implies $E^{l,i-1}_c(x)^+ \subset U_x$. Now the induction yields $E^{l,i-1}_c(x)^+ = E^{r,i-1}_a(x)^+$.

Therefore $E^{r,i-1}_a(x)^+ \subset U_x \subset E^{r,i}_a(x)^+$ which is impossible since every point in $E^{r,i}_a(x)^+ - E^{r,i-1}_a(x)^+$ has the same type over $a, x$.

Similarly $E^{r,i}_a(x)^- = E^{l,i}_a(x)^-$, and the result follows. $\square$

Now we prove V. Let $x < y < z < w$ and $d^r_x(y,z) \le i \le n/2$. Suppose $d^r_x(z,w) > i$. If $d^r_x(z,w) < n+1-i$ then, by Lemma 4.4, $E^{r,i}_x(z) = E^{l,i}_w(z)$, which implies that $d^l_w(y,z) \le i$. If $d^r_x(z,w) \ge n+1-i$ then we have $d^l_w(x,z) \le i$ which also implies $d^l_w(y,z) \le i$.

The dual of axiom V follows as easily from the lemma.

VI Pick $x_1 < y_1 < z_1$ such that $d^r_{x_1}(y_1,z_1) = i$. By 2-indiscernibility we may suppose that $x_1 = x$ and $y_1 = y$, and for the desired $z$ we take $z = z_1$.

VII This is established by a similar 2-indiscernibility argument.

VIII We choose $w, y, z$ with $w < y < z$ and $d^r_w(y,z) = j$ and use the dual of VI, and IV, to choose $x < w$ with $d^r_x(w,z) = i$. By Lemma 4.1(iv) we have $d^r_x(w,y) = i$ and $d^r_x(y,z) = i$.

We have thus found $x, y, z$ with $x < y < z$ and $d^r_x(y,z) = i$, satisfying $\exists w(x < w < y \wedge d^r_x(w,y) = i \wedge d^r_w(y,z) = j))$. As $x < y < z \wedge d^r_x(y,z) = i$ isolates a complete 3-type, this applies to the $x, y, z$ we started with.

IX We choose $y < z < w$ with $d^l_w(y,z) = k_1$ and $x < y$ with $d^r_x(y,w) = k_2$. Hence as $d^r_x(z,w) \le d^r_x(y,w)$, we have $d^r_x(z,w) \le n/2$. By the dual of Lemma 4.1(v), $d^r_x(y,z) = \min(d^r_x(z,w), d^l_w(y,z))$. As $d^r_x(y,z) \le d^r_x(z,w)$ we have $d^r_x(y,w) = d^r_x(z,w)$ and therefore $d^r_x(y,z) = \min(k_1,k_2) = j$.

As $x < y < z \wedge d^r_x(y,z) = j$ isolates a complete type, axiom IX follows. $\square$

**Theorem 4.5** *Axioms I-IX all hold in $\mathcal{M}_n$ as defined in §3.*

**Proof** We shall use the facts that $M$ is linearly ordered, and that $\prec$ defines a tree structure on $T$. This means for instance that for any $x < y < z$, one of $t(x,y)$ and $t(y,z)$ is equal to $t(x,z)$.

**II** To show that $d^r_x$ is an ultrametric it is just necessary to verify the ultrametric inequality. Let $x < y < z < w$. We illustrate four possible cases in Fig. 2. We

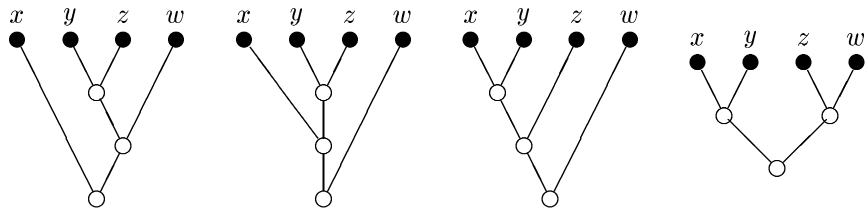



Figure 2: Typical cases arising in the verification of II

have to show that the two greatest members of $d_x^r(y,z), d_x^r(y,w)$, and $d_x^r(z,w)$ are equal. We consider $t(x,y)$, $t(y,z)$, and $t(y,w)$. Clearly $t(y,w) \preceq t(y,z)$.

**Case 1**: $t(x,y) \preceq t(y,w)$. Then $d_x^r(y,z) = m+1-d(t(x,y),t(y,z))$, $d_x^r(y,w) = m+1-d(t(x,y),t(y,w))$, and $d_x^r(z,w) = m+1-d(t(x,z),t(z,w))$. Since $T$ is a tree, the two smallest members of $t(y,z), t(y,w)$, and $t(z,w)$ are equal, and hence the two greatest members of $d_x^r(y,z), d_x^r(y,w)$, and $d_x^r(z,w)$ are also equal, as required.

**Case 2**: $t(y,w) \prec t(x,y) \preceq t(y,z)$. Then $d_x^r(y,z) = m+1-d(t(x,y),t(y,z))$, $d_x^r(y,w) = n-m+d(t(y,w),t(x,y))$, and $d_x^r(z,w) = n-m+d(t(z,w),t(x,z))$, of which the latter two are the largest (being at least $m+1$). But $t(x,y) = t(x,z)$ and $t(y,w) = t(z,w)$, and so these two largest are equal.

**Case 3**: $t(y,w) \prec t(y,z) \prec t(x,y)$. Then $d_x^r(y,z) = n-m+d(t(y,z),t(x,y))$, $d_x^r(y,w) = n-m+d(t(y,w),t(x,y))$, and $d_x^r(z,w) = n-m+d(t(x,w),t(x,z))$. This time $t(x,z) = t(y,z)$ and $t(x,w) = t(y,w)$, and the result follows from the ultrametric inequality for $d$.

**Case 4**: $t(y,w) = t(y,z) \prec t(x,y)$. Then $d_x^r(y,z) = n-m+d(t(y,z),t(x,y))$, $d_x^r(y,w) = n-m+d(t(y,w),t(x,y))$, and $d_x^r(z,w) = m+1-d(t(x,z),t(z,w))$. As $t(y,z) = t(y,w)$, the result follows.

**III** To show $\{z > x : d_x^r(y,z) \leq i\}$ is convex, let $z_1 \leq w \leq z_2$ where $d_x^r(y,z_1), d_x^r(y,z_2) \leq i$. Then also $d_x^r(z_1,z_2) \leq i$. We first show that $d_x^r(z_1,w) \leq i$.

If $d_x^r(z_1,z_2) \leq \frac{n+1}{2}$ then $t(x,z_1) \preceq t(z_1,z_2) \preceq t(z_1,w)$, so $d_x^r(z_1,w) = m+1-d(t(x,z_1),t(z_1,w)) \leq m+1-d(t(x,z_1),t(z_1,z_2)) \leq i$. Otherwise $d_x^r(z_1,z_2) > \frac{n+1}{2}$. If $t(x,z_1) \preceq t(z_1,w)$ then $d_x^r(z_1,w) \leq m+1 \leq i$ is immediate. If $t(z_1,w) \preceq t(x,z_1)$ then $d_x^r(z_1,w) = n-m+d(t(z_1,w),t(x,z_1)) \leq n-m+d(t(z_1,z_2),t(x,z_1))$ (since $t(z_1,z_2) \preceq t(z_1,w)$) $\leq i$.

We deduce that $d_x^r(y,w) \leq \max(d_x^r(y,z_1), d_x^r(z_1,w)) \leq i$ as required.

**IV** is immediate from the definition.

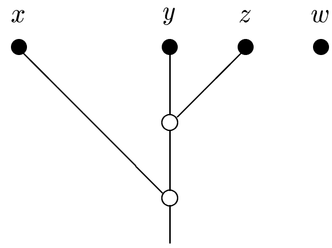


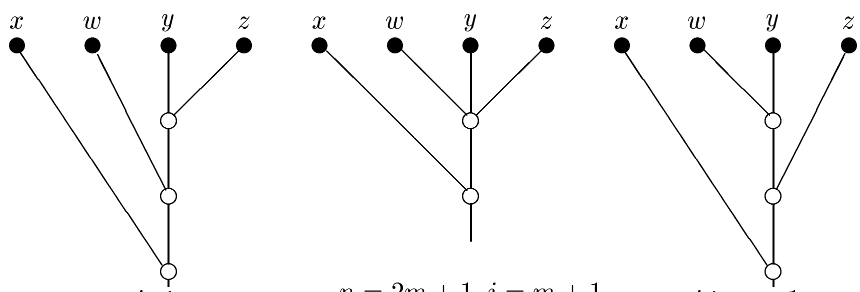

Figure 3: Verification of V

**V** Suppose $d^r_x(y,z) \leq i \leq n/2$. Then $t(x,y) \prec t(y,z)$ and $d(t(x,y),t(y,z)) \geq m+1-i$. Hence $t(x,z) = t(x,y)$. See Fig. 3.
**Case 1**: $t(y,z) \preceq t(z,w)$. Then $t(x,z) \prec t(z,w)$ and $d(t(x,z),t(z,w)) \geq m+1-i$, so $d^r_x(z,w) \leq i$.
**Case 2**: $t(z,w) \preceq t(x,z)$. Then $t(z,w) \prec t(y,z)$ and $d(t(z,w),t(y,z)) \geq m+1-i$, so $d^l_w(y,z) \leq i$.
**Case 3**: $t(x,z) \prec t(z,w) \prec t(y,z)$. Since $d(t(x,y),t(y,z)) \geq m+1-i$, either $d(t(x,z),t(z,w)) \geq m+1-i$ or $d(t(z,w),t(y,z)) \geq m+1-i$, again giving $d^r_x(z,w) \leq i$ or $d^l_w(y,z) \leq i$.

**VI** First suppose $i \leq m$. Pick $t \in T$ with $t(x,y) \prec t \prec y$ such that $d(t(x,y),t) = m+1-i$ and $y(q) = 0$ where $(-\infty,q) = \operatorname{dom} t$. Choose $z \in M$ so that $z|(-\infty,q) = t$, and $z(q) = 1$. Then $y < z$ and $d^r_x(y,z) = i$.

If $i \geq n+1-m$ we use a similar argument with $t \prec t(x,y)$. If $n = 2m+1$ and $i = m+1$ we let $z|(-\infty,q) = y|(-\infty,q)$ where $(-\infty,q) = \operatorname{dom} t(x,y)$ and $z(q) = y(q)+1$.

**VII** The argument is similar to that in VI. If $i \leq m$, choose $t$ so that $t(x,y) \prec t \prec y$ and $z$ branching to the left of $y$ at $t$; if $i \geq n+1-m$, choose $t$ so that




$j \leq m$ $\qquad$ $n = 2m+1, j = m+1$ $\qquad$ $j \geq n+1-m$

Figure 4: Verification of VIII

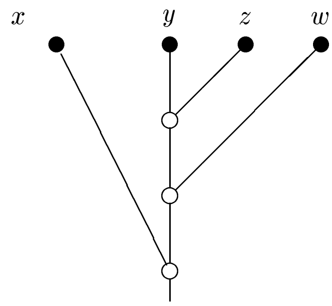



Figure 5: Verification of IX

$t(x,y) \prec t \prec x$ and $z$ branching to the right of $x$ at $t$; and if $i = m+1$ ($n$ odd) let $t = t(x,y)$ and $z$ branch to the right of $x$ and the left of $y$ at $t$.

**VIII** The hypotheses ensure that $t(x,y) \prec t(y,z)$ and $d(t(x,y),t(y,z)) = m+1-i$. If $j \leq m$ choose $t$ so that $t(x,y) \prec t \prec t(y,z)$, $d(t(x,y),t) = m+1-i$, $d(t,t(y,z)) = m+1-j$, and if $n$ is even and $\operatorname{dom} t = (-\infty,q)$, then $y(q) = 1$. Take $w \in M$ so that $w|(-\infty,q) = y|(-\infty,q)$ and $w(q) = y(q)-1$. If $j \geq n+1-m$ choose $t$ with $t(y,z) \prec t \prec y$ and if $n = 2m+1$ and $j = m+1$ let $t = t(y,z)$. See Fig. 4.

**IX** We are given that $t(x,y) \prec t(y,z)$ and $d(t(x,y),t(y,z)) = m+1-j = \max(m+1-k_1, m+1-k_2)$. Choose $t$ with domain $(-\infty,q)$ satisfying $t(x,y) \prec t \prec t(y,z), d(t(x,y),t) = m+1-k_1$, $d(t,t(y,z)) = m+1-k_2$, and $y(q) = 0$. Let $w \in M$ be such that $w|(-\infty,q) = y|(-\infty,q)$ and $w(q) = y(q)+1$. See Fig. 5. □

**Theorem 4.6** *Any two countable models of the list of axioms are isomorphic.*

We prepare for the proof of Theorem 4.6 by proving a series of lemmas that follow from the universal axioms I-V. Dual versions also hold.

**Lemma 4.7** *Suppose that $x < y < z < w$ and $d_x^r(y,z) > n/2$. Then*
*(i) $d_x^r(z,w) = d_y^r(z,w)$,*
*(ii) $d_w^l(y,z) \geq d_w^l(x,y)$.*

**Proof** (i) By Lemma 4.1(i) we have $d_x^r(z,w) \leq d_y^r(z,w)$, so we assume for a contradiction that $d_x^r(z,w) < d_y^r(z,w)$, and hence that $d_w^l(x,z) > d_w^l(y,z)$. By axiom II, $d_w^l(x,z) = d_w^l(x,y)$. We consider two cases:

**Case 1**: $d_w^l(x,z) \leq (n+1)/2$. Then $d_w^l(y,z) \leq n/2$, so by the dual of Lemma 4.1(iii), $\min(d_w^l(x,y), d_x^r(y,z)) \leq d_w^l(y,z) \leq n/2$. But $d_w^l(x,y) > d_w^l(y,z)$ and $d_x^r(y,z) > n/2$, so we have a contradiction.

**Case 2**: $d^l_w(x,z) > (n+1)/2$. Then $d^l_w(x,y) > (n+1)/2$, so $d^r_x(y,w) < n+1-(n+1)/2 = (n+1)/2$. Thus by axiom III $d^r_x(y,z) \le d^r_x(y,w) \le n/2$, contradicting the hypothesis.

(ii) By (i) we have $d^l_w(y,z) = d^l_w(x,z) \ge d^l_w(x,y)$. □

**Lemma 4.8** *Let $A$ be a finite model of axioms I-V and let $a < b$ be adjacent elements of $A$. Consider two models $A \cup \{x_1\}$ and $A \cup \{x_2\}$ of axioms I-V where $a < x_1 < b$ and $a < x_2 < b$, and $d^r_a(x_1,b) = d^r_a(x_2,b) > n/2$. Then for every $c > b$, $d^l_c(x_1,b) = d^l_c(x_2,b)$.*

**Proof** By Lemma 4.7(i) we have $d^r_a(b,c) = d^r_{x_1}(b,c)$ and $d^r_a(b,c) = d^r_{x_2}(b,c)$, so $d^r_{x_1}(b,c) = d^r_{x_2}(b,c)$. This implies $d^l_c(x_1,b) = d^l_c(x_2,b)$. □

**Lemma 4.9** *Suppose that $x < y < z < w$.*

*(i) If $d^l_w(x,y) < d^l_w(y,z)$ then $d^l_z(x,y) = d^l_w(x,y)$.*

*(ii) If $d^l_w(x,y) > d^l_w(y,z)$ then $d^r_x(y,z) = \min(d^l_w(y,z), d^r_x(y,w))$.*

**Proof** (i) By Lemma 4.1(i) $d^l_z(x,y) \ge d^l_w(x,y)$, so we assume for a contradiction that $d^l_z(x,y) > d^l_w(x,y)$. But then $d^r_x(y,z) < d^r_x(y,w)$, so by axiom II $d^r_x(y,w) = d^r_x(z,w)$, and hence $d^l_w(x,y) = d^l_w(x,z)$. Again using axiom II, $d^l_w(x,y) \ge d^l_w(y,z)$, contradicting the hypothesis.

(ii) By axiom III we have $d^r_x(y,z) \le d^r_x(y,w)$. The dual version of Lemma 4.1(ii) says that either $d^r_x(y,z) \le d^l_w(y,z)$ or $d^l_w(x,y) \le d^l_w(y,z)$. Since $d^l_w(x,y) > d^l_w(y,z)$, we must have $d^r_x(y,z) \le d^l_w(y,z)$. Thus $d^r_x(y,z) \le \min(d^l_w(y,z), d^r_x(y,w))$. On the other hand, Lemma 4.1(ii) says that either $d^r_x(z,w) \le d^r_x(y,z)$ or $d^l_w(y,z) \le d^r_x(y,z)$. In the former case, $d^r_x(y,w) = d^r_x(y,z)$ by axiom II, so $d^r_x(y,z) \ge \min(d^l_w(y,z), d^r_x(y,w))$. In the latter case, the desired conclusion is immediate. □

**Lemma 4.10** *Let $A$ be a finite model of axioms I-V and let $a < b$ be adjacent elements of $A$. Consider two models $A \cup \{x_1\}$ and $A \cup \{x_2\}$ of axioms I-V where $a < x_1 < b$ and $a < x_2 < b$ and $d^l_b(a,x_1) = d^l_b(a,x_2) = j \le n/2$. For every $a_1 < a$, if $d^l_b(a_1,a) \ne j$ then $d^r_{a_1}(a,x_1) = d^r_{a_1}(a,x_2)$.*

**Proof** If $d^l_b(a_1,a) < j$ then by Lemma 4.9(i) we have $d^l_{x_1}(a_1,a) = d^l_b(a_1,a) = d^l_{x_2}(a_1,a)$. This implies the desired conclusion by axiom IV. On the other hand, if $d^l_b(a_1,a) > j$ then by Lemma 4.9(ii) we have $d^r_{a_1}(a,x_1) = \min(d^l_b(a,x_1), d^r_{a_1}(a,b))$ $= d^r_{a_1}(a,x_2)$. □

**Lemma 4.11** *Let $A$ be a finite model of axioms I-V and let $h < a$ be elements of $A$. Consider two models $A \cup \{x_1\}$ and $A \cup \{x_2\}$ of axioms I-V where $a < x_1$ and $a < x_2$.*

*(i) If $d^l_{x_1}(h,a) = d^l_{x_2}(h,a) \le \frac{n+1}{2}$ and $h \le h_1 < a$ then $d^r_{h_1}(a,x_1) = d^r_{h_1}(a,x_2)$*

*(ii) If $d^r_h(a,x_1) = d^r_h(a,x_2) \le \frac{n+1}{2}$ and $h_1 \le h$ then $d^r_{h_1}(a,x_1) = d^r_{h_1}(a,x_2)$.*

**Proof** (i) We distinguish two cases:

**Case 1**: Suppose $d^r_h(h_1, a) > n/2$. By Lemma 4.7 $d^r_{h_1}(a, x_1) = d^r_h(a, x_1)$ and $d^r_{h_1}(a, x_2) = d^r_h(a, x_2)$, so the result follows by axiom IV.

**Case 2**: Suppose $d^r_h(h_1, a) \leq n/2$. Since $d^l_{x_1}(h, a) = d^l_{x_2}(h, a) \leq \frac{n+1}{2}$, we have $d^r_h(a, x_1) = d^r_h(a, x_2) \geq n + 1 - \frac{n+1}{2} > d^r_h(h_1, a)$. Then the dual form of Lemma 4.9(ii) can be applied (twice) to give $d^l_{x_1}(h_1, a) = \min(d^r_h(h_1, a), d^l_{x_1}(h, a))$ $= \min(d^r_h(h_1, a), d^l_{x_2}(h, a)) = d^l_{x_2}(h_1, a)$, and we again use axiom IV.

(ii) Since $d^l_{x_1}(h, a) = d^l_{x_2}(h, a) > n/2$, the dual of Lemma 4.7(i) gives $d^l_{x_1}(h_1, h) = d^l_a(h_1, h) = d^l_{x_2}(h_1, h)$. As $d^l_{x_1}(h, a) = d^l_{x_2}(h, a)$ we have $d^l_{x_1}(h_1, a) = d^l_{x_2}(h_1, a)$ by the ultrametric inequality. □

**Lemma 4.12** *Suppose that $A \cup \{x_1\}$ and $A \cup \{x_2\}$ are two finite models of axioms I-V. Let $a < b$ be consecutive elements of $A$ with $a < x_1 < b$ and $a < x_2 < b$ and $d^r_a(x_1, b) = d^r_a(x_2, b)$. Then $A \cup \{x_1\} \cong A \cup \{x_2\}$ if and only if for every $a_1 < a$ and $b_1 > b$, $d^r_{a_1}(a, x_1) = d^r_{a_1}(a, x_2)$ and $d^l_{b_1}(x_1, b) = d^l_{b_1}(x_2, b)$.*

**Proof** The only possible isomorphism is the identity on $A$ extended by mapping $x_1$ to $x_2$. For each $y \in A$, the functions $d^r_y$ and $d^l_y$ are determined by their values on pairs of consecutive arguments, using axioms I-III. In a first step using $d^r_a(x_1, b) = d^r_a(x_2, b)$ we see that $d^r_a$ coincides on $A \cup \{x_1\}$ and $A \cup \{x_2\}$ and similarly for $d^l_b$. In a second step we consider $d^r_c$ for $c < a$ and $d^l_d$ for $d > b$. The only pairs of arguments we need to check are those involving $x_1$ or $x_2$. But these are all covered by the hypotheses and the first step (using axiom IV). □

**Lemma 4.13** *Suppose $h < a < x < b$ and $d^l_b(h, a) = d^l_b(a, x) = i \leq n/2$. Then $i \leq d^l_x(h, a) \leq n + 1 - i$.*

**Proof** By the dual of Lemma 4.1(i), $d^l_x(h, a) \geq d^l_b(h, a) = i$. Suppose for a contradiction that $d^l_x(h, a) > n + 1 - i$. Then $d^r_h(a, x) < i$, so by axiom V either $d^r_h(x, b) < i$ or $d^l_b(a, x) < i$. The latter contradicts the hypothesis. The former gives $d^l_b(h, x) > n + 1 - i > n/2 \geq i$, but $d^l_b(h, x) = i$ by axiom II. □

**Lemma 4.14** *Suppose $h_2 < h_1 < a < x < b$ and $d^l_b(h_1, a) = d^l_b(h_2, a) = i \leq n/2$. Then*

*(i) $d^r_{h_2}(h_1, a) \geq i$,*

*(ii) if $k_1 = d^l_x(h_1, a)$, $k_2 = d^r_{h_2}(a, x)$, $k = d^r_{h_2}(h_1, a)$, and $k_1, k_2 \leq n/2$, then $k = \min(k_1, k_2)$.*

**Proof** (i) Suppose for a contradiction that $d^r_{h_2}(h_1, a) < i$. By axiom V, either $d^l_b(h_1, a) < i$ or $d^r_{h_2}(a, b) < i$. The former possibility is contrary to the hypothesis, and the latter implies $d^l_b(h_2, a) > n + 1 - i > n/2 \geq i$, also contrary to hypothesis.

(ii) This is just the dual of Lemma 4.1(v). □

**Proof of Theorem 4.6** We can now conclude the proof of the theorem using Lemmas 4.8 and 4.10-4.14. We use a back-and-forth argument, showing that if $p$ is an isomorphism from a finite substructure $A$ of a model $M$ onto a substructure $A'$ of a model $M'$, then for every $x \in M$ there is $x' \in M'$ such that the extension of $p$ carrying $x$ to $x'$ is a partial isomorphism. If $|A|, |A'| \leq 1$ this is immediate (since all relations except for $<$ are ternary).

We note first that we may assume that $a < x < b$ for some $a, b \in A$. For if not, we may make that true by first extending $p$ as follows. For example, if $x$ is less than every element of $A$, use the dual of axiom VI to choose $a \in A$ and $a' \in A'$ such that $d_a^r(c, d) = 1 = d_{a'}^r(c', d')$, where $c, c'$ are the smallest and $d, d'$ are the largest elements of $A \cup \{x\}, A'$. It then follows that these equalities hold for every $c, d \in A$ and $c', d' \in A'$. Extending $p$ by sending $a$ to $a'$ gives a partial isomorphism for which $a < x < b$. A similar argument using axiom VI handles the case where $x$ exceeds every element of $A$.

Given $x \in M - A$, let $a < b$ be consecutive elements of $A$ with $a < x < b$ and let $a', b'$ be their images under $p$.

We may suppose that $d_b^l(a, x) \leq (n + 1)/2$, for otherwise we would have $d_a^r(x, b) \leq (n + 1)/2$ by axiom IV and the dual of the following argument would apply.

**Case 1**: Suppose that $n = 2m + 1$ and $d_b^l(a, x) = m + 1$. Then also $d_a^r(x, b) = m+1$. By axiom VII we select $x' \in M'$ with $a' < x' < b'$ and $d_{a'}^r(x', b') = m+1$. We also adjoin a new element $x'' \notin M'$ to $A'$ such that the extension of $p$ mapping $x$ to $x''$ is an isomorphism from $A \cup \{x\}$ onto $A' \cup \{x''\}$. We show that $A' \cup \{x'\} \cong A' \cup \{x''\}$. Since $d_{a'}^r(x', b') = m+1$ and $d_{a'}^r(x'', b') = d_a^r(x, b) = m+1$, Lemma 4.8 applies to give for every $b_1' > b'$ $d_{b_1'}^l(x', b') = d_{b_1'}^l(x'', b')$. Similarly, the dual of Lemma 4.8 applies (because $n+1-(m+1) = m+1$) to give for every $a_1' < a'$ $d_{a_1'}^r(a', x') = d_{a_1'}^r(a', x'')$. By Lemma 4.12, this is enough to establish the claimed isomorphism.

**Case 2**: Suppose that $d_b^l(a, x) = i \leq n/2$. We distinguish further subcases.
**Case 2a**: Suppose that for every $h \in A$ if $h < a$ then $d_b^l(h, a) \neq i$. By axiom VII we select $x' \in M'$ such that $d_{b'}^l(a', x') = i$. As in Case 1, we choose $x''$ such that the extension of $p$ mapping $x$ to $x''$ is an isomorphism from $A \cup \{x\}$ onto $A' \cup \{x''\}$ and show that $A' \cup \{x'\} \cong A' \cup \{x''\}$. Lemma 4.8 tells us that, for every $b_1' \geq b'$, $d_{b_1'}^l(x', b') = d_{b_1'}^l(x'', b')$, and Lemma 4.10 that, for all $a_1' < a'$, $d_{a_1'}^r(a', x') = d_{a_1'}^r(a', x'')$. By Lemma 4.12 this suffices.

**Case 2b**: Suppose that there exists $h < a$ such that $d_b^l(h, a) = i$, $d_x^l(h, a) = m + 1$, and $n = 2m + 1$. Write $h'$ for the image of $h$ under $p$. By the dual of axiom VIII, taking $j = m + 1$, we can select $x' \in M'$ such that $a' < x' < b'$, $d_{b'}^l(x', a') = i$ and $d_{x'}^l(a', h') = m + 1$. Choose $x''$ such that $A \cup \{x\} \cong A' \cup \{x''\}$ via the extension of $p$ mapping $x$ to $x''$. Lemma 4.8 gives us that for every $b_1' \geq b'$, $d_{b_1'}^l(x', b') = d_{b_1'}^l(x'', b')$. Let $a_1' < a'$. If $a_1' \geq h'$, then Lemma 4.11(i) applies (with $a_1'$ here playing the role of $h_1$ in 4.11) to give $d^r_{a_1'}(a',x') = d^r_{a_1'}(a',x'')$. If $a_1' \le h'$, then Lemma 4.11(ii) applies to give $d^r_{a_1'}(a',x') = d^r_{a_1'}(a',x'')$. By Lemma 4.12 this suffices.

**Case 2c**: Suppose again that there is $h < a$ such that $d^l_b(h,a) = i$, but this time that for every such $h$ we have either $d^l_x(h,a) \le n/2$ or $d^r_h(a,x) \le n/2$ (no longer assuming that $n = 2m+1$). Now Lemma 4.13 says that $i \le d^l_x(h,a)$ and $i \le d^r_h(a,x)$, so the hypothesis of this case immediately gives either $i \le d^l_x(h,a) \le n/2$ or $i \le d^r_h(a,x) \le n/2$. We use this for a further case division, according to whether all $h < a$ satisfy the first inequality, all satisfy the second inequality, or some satisfy one and some satisfy the other.

**Case 2c(i)**: Suppose that for all $h < a$ with $d^l_b(h,a) = i$ we have $i \le d^l_x(h,a) \le n/2$. We choose $h < a$ minimal such that $d^l_b(h,a) = i$ and let $j = d^l_x(h,a)$. Write $h'$ for the image of $h$ under $p$. Then by the dual of axiom VIII we may choose $x' \in M'$ such that $a' < x' < b'$, $d^l_{b'}(a',x') = i$ and $d^l_{x'}(h',a') = j$. Choose $x''$ such that $A \cup \{x\} \cong A' \cup \{x''\}$ via the extension of $p$ mapping $x$ to $x''$. Lemma 4.8 implies that for all $b_1' > b'$, $d^l_{b_1'}(x',b') = d^l_{b_1'}(x'',b')$. Now let $a_1' < a'$. If $h' \le a_1'$ then by Lemma 4.11(i) we get $d^r_{a_1'}(a',x') = d^r_{a_1'}(a',x'')$. If $a_1' < h'$ then by the minimality of $h$ we have $d^l_{b'}(a_1',a') \ne i$, so Lemma 4.10 applies to give $d^r_{a_1'}(a',x') = d^r_{a_1'}(a',x'')$. Again we finish by Lemma 4.12.

**Case 2c(ii)**: Suppose that for all $h < a$ with $d^l_b(h,a) = i$ we have $i \le d^r_h(a,x) \le n/2$. We choose $h < a$ maximal such that $d^l_b(h,a) = i$. Let $k = d^r_h(a,x)$ and $j = n+1-k = d^l_x(h,a)$. Write $h'$ for the image of $h$ under $p$. By the dual of axiom VIII we may choose $x' \in M'$ such that $a' < x' < b'$, $d^l_{b'}(a',x') = i$, and $d^l_{x'}(h',a') = j$. Then $d^r_{h'}(a',x') = k \le n/2$. We choose $x''$ such that $A \cup \{x\} \cong A' \cup \{x''\}$ via the extension of $p$ mapping $x$ to $x''$. Lemma 4.8 implies that for all $b_1' > b'$, $d^l_{b_1'}(x',b') = d^l_{b_1'}(x'',b')$. Now let $a_1' < a'$. If $a_1' \le h'$, then by Lemma 4.11(ii) we get $d^r_{a_1'}(a',x') = d^r_{a_1'}(a',x'')$. If $h' < a_1'$ then by the maximality of $h$ we have $d^l_{b'}(a_1',a') \ne i$, so Lemma 4.10 applies to give $d^r_{a_1'}(a',x') = d^r_{a_1'}(a',x'')$. Again we finish by Lemma 4.12.

**Case 2c(iii)**: What remains is the case where there are $h_1, h_2 < a$ such that $d^l_b(h_1,a) = d^l_b(h_2,a) = i$, $i \le d^l_x(h_1,a) \le n/2$ and $i \le d^r_{h_2}(a,x) \le n/2$. Choose $h_1$ minimal and $h_2$ maximal satisfying these conditions. As $d^l_x(h_2,a) \ge n/2 + 1 > d^l_x(h_1,a)$, we have $h_2 < h_1$. Furthermore, $h_2$ and $h_1$ are adjacent, for it follows from the minimality of $h_1$ and the maximality of $h_2$ that any $h$ between them would satisfy $d^l_b(h,a) = i$ and $d^l_x(h,a), d^r_h(a,x) > n/2$, contrary to the assumption of Case 2c.

Write $k_1 = d^l_x(h_1,a)$, $k_2 = d^r_{h_2}(a,x)$ and $k = d^r_{h_2}(h_1,a)$. By Lemma 4.14(ii), $k = \min(k_1,k_2)$.

Write $h_1'$ and $h_2'$ for the images of $h_1$ and $h_2$ under $p$. Now by axiom IX we can find $x' \in M'$ such that $a' < x'$ and $d^r_{h'_2}(a', x') = k_1$ and $d^l_{x'}(h'_1, a') = k_2$. Because $d^l_{b'}(h'_2, a') = i \leq m$, we have $d^r_{h'_2}(a', b') = n + 1 - i > n/2 \geq k_1 = d^r_{h'_2}(a', x')$, so $x' < b'$.

By Lemma 4.1(v), $d^l_{b'}(a', x') \leq d^l_{b'}(h'_2, a') = i$ and by Lemma 4.7(ii), $d^l_{b'}(a', x') \geq d^l_{b'}(h'_1, a') = i$. Therefore $d^l_{b'}(a', x') = i$.

Choose $x''$ such that $A \cup \{x\} \cong A' \cup \{x''\}$ via the extension of $p$ mapping $x$ to $x''$. By Lemma 4.8, for all $b'_1 > b'$, $d^l_{b'_1}(x', b') = d^l_{b'_1}(x'', b')$. Let $a'_1 < a'$. If $h'_1 \leq a'_1$, then by Lemma 4.11(i), $d^r_{a'_1}(a', x') = d^r_{a'_1}(a', x'')$. If $a'_1 \leq h'_2$ then by Lemma 4.11(ii) we have $d^r_{a'_1}(a', x') = d^r_{a'_1}(a', x'')$. By Lemma 4.12, this finishes the proof. □

**Theorem 4.15** *Let $\mathcal{N}$ be a countable, weakly o-minimal, 2-indiscernible structure with domain $N$, ordering $<$, and $n$ atoms of strictly increasing triples. Then those atoms may be listed so that*

$$(N; =, <, (R_i)_{1 \leq i \leq n}) \cong \mathcal{M}_n.$$

**Proof** This follows at once from Theorems 4.3, 4.5, and 4.6. □

# 5 $\aleph_0$-categorical, weakly $o$-minimal, 3-indiscernible structures

The main result of this section is that such structures are in fact $k$-indiscernible for all finite $k$, which means that they are essentially equal to $(\mathbb{Q}, <)$ (and hence are $o$-minimal). For this it suffices to show that they are 4-indiscernible, since we may then use induction.

**Theorem 5.1** *Any $\aleph_0$-categorical weakly o-minimal 3-indiscernible structure is 4-indiscernible.*

**Proof** Let the 4-atoms $\{R_i\}_{1 \leq i \leq n}$ of increasing quadruples be listed, as in previous sections, in such a way that $R_i(x, y, z, t) \wedge R_j(x, y, z, t') \wedge i < j \Rightarrow t < t'$, and assume for a contradiction that $n > 1$. This time we get an ultrametric if we fix *two* points; in fact we get three ultrametrics, on the left, right, and in the middle. These are determined by $R_i(x, y, z, t) \Leftrightarrow d^r_{xy}(z, t) = i \Leftrightarrow d^m_{xt}(y, z) = n + 1 - i \Leftrightarrow d^l_{zt}(x, y) = i$, and $d^r_{xy}, d^m_{xt}$, and $d^l_{zt}$ are then ultrametrics on $(y, \infty), (x, t)$, and $(-\infty, z)$ respectively. The formula $d^r_{xy}(z, t) = n + 1 - d^m_{x,t}(y, z) = d^l_{zt}(x, y)$ holds.

**Lemma 5.2** *There are $a < b < x < y < z < c < d$ such that*

$$d^r_{ab}(x, y) = d^l_{cd}(y, z) = 1, d^r_{ab}(y, z) = d^l_{cd}(x, y) = n.$$

**Proof** There are $a_1 < b_1 < x_1 < y_1 < z_1$ such that $R_1(a_1, b_1, x_1, y_1)$ and $R_n(a_1, b_1, y_1, z_1)$, as one sees by starting with any $a_1 < b_1 < y_1$ and then choosing appropriate $x_1$ and $z_1$. Hence $d^r_{a_1 b_1}(x_1, y_1) = 1$ and $d^r_{a_1 b_1}(y_1, z_1) = n$. By 3-indiscernibility for any $x < y < z$ there are $a < b < x$ such that $d^r_{ab}(x, y) = 1$ and $d^r_{ab}(y, z) = n$. Similarly there are $d > c > z$ such that $d^l_{cd}(y, z) = 1$ and $d^l_{cd}(x, y) = n$. □

**Lemma 5.3** *If $a < b < x < y < z < c < d$ are as in Lemma 5.2 then also for every $x' \in (b, c], d^r_{ab}(x', d) = n$.*

**Proof** $d^r_{ab}(y, z) = n \Rightarrow d^r_{ab}(y, d) = n \Rightarrow d^m_{ad}(b, y) = 1$ and $d^l_{cd}(x, y) = n \Rightarrow d^l_{cd}(a, y) = n \Rightarrow d^m_{ad}(y, c) = 1$ and therefore $d^m_{ad}(b, c) = 1$. This implies $d^r_{ab}(c, d) = n$. As for every $x' \in (b, c]$ $d^r_{ab}(x', d) \geq d^r_{ab}(c, d)$ we have $d^r_{ab}(x', d) = n$. □

Now continuing with the main proof, we may define various equivalence relations from the ultrametrics. The two we wish to concentrate on are defined from two parameters $a < b$: $E^r_{ab}(x, y)$ for $b < x, y$ if $d^r_{ab}(x, y) \leq 1$ ($x$ and $y$ are $ab$-equivalent on the *right*), and $E^l_{ab}(x, y)$ for $x, y < a$ if $d^l_{ab}(x, y) \leq 1$ ($x$ and $y$ are $ab$-equivalent on the *left*). If $a < b < c < d$ as in Lemma 5.2 we can therefore discuss the equivalence of points of $(b, c)$ under either $E^r_{ab}$ or $E^l_{cd}$, but there is no obvious reason why there should be any relationship between the two. The idea of our proof is to exploit the possible interaction. We write $E^r_{ab}(x)$ or $E^l_{ab}(x)$ for the $E^r_{ab}$-class ($E^l_{ab}$-class respectively) of $x$.

In what follows we shall keep $a < b < d$ fixed, and allow $c$ to vary in $I = \{x \in (b, d) : d^r_{ab}(x, d) = n\}$ (which is a 1-type over $\{a, b, d\}$). If $E^r_{ab}(x) \subseteq (b, c)$ then a *right $c$-companion* of $E^r_{ab}(x)$ is an $E^l_{cd}$-class $R$ such that

$$\inf E^r_{ab}(x) < \inf R < \sup E^r_{ab}(x) < \sup R.$$

A *left $c$-companion* of $E^r_{ab}(x)$ is an $E^l_{cd}$-class $L$ such that

$$\inf L < \inf E^r_{ab}(x) < \sup L < \sup E^r_{ab}(x).$$

**Lemma 5.4** *(i) Any $E^r_{ab}$-class can have at most one right $c$-companion.*

*(ii) If $R$ is a right $c$-companion of $E^r_{ab}(x)$ then there is $y \in R$ such that $E^r_{ab}(x) < E^r_{ab}(y) \subseteq R$.*

*(iii) For every $c \in I$ there are at least one and at most finitely many $E^r_{ab}$-classes having a right $c$-companion.*

**Proof** (i) Two right $c$-companions of $E^r_{ab}(x)$ must contain a non-empty final segment of $E^r_{ab}(x)$, and as they are equivalence classes of $E^l_{cd}$, must be equal.

(ii) By the results of §4 we know the ternary structure of $(a, \infty)$, which in particular implies that $<$ induces on $(a, \infty)$ and the $E^r_{ab}$-classes a dense linear ordering. Choose $y' \in R - E^r_{ab}(x)$, and $y$ so that $E^r_{ab}(x) < E^r_{ab}(y) < E^r_{ab}(y')$, and then $E^r_{ab}(y) \subseteq (\sup E^r_{ab}(x), y') \subseteq R$.

(iii) Since $c \in I$, by Lemma 5.3 it has the same type over $\{a, b, d\}$ as in Lemma 5.2, so $x, y, z$ as given by that lemma exist. Then $E^r_{ab}(y)$ has $E^l_{cd}(y)$ as its right companion since $E^r_{ab}(x, y) \wedge \neg E^r_{ab}(y, z)$ but $\neg E^l_{cd}(x, y) \wedge E^l_{cd}(y, z)$. (It is at this point that we have used $n > 1$. We know that $d^r_{ab}(y, z) = n$ and as $n > 1$ it follows that $d^r_{ab}(y, z) \neq 1$.)

If infinitely many $E^r_{ab}$-classes have a right $c$-companion, then by weak $o$-minimality there is an interval $J \subseteq (b, c)$ containing infinitely many $E^r_{ab}$-classes, such that for all $x \in J$, $E^r_{ab}(x)$ has a right $c$-companion. Choose $x_1, x_2 \in J$ with $E^r_{ab}(x_1) < E^r_{ab}(x_2)$, and let $R$ be the right $c$-companion of $E^r_{ab}(x_1)$. By (ii) and density there is $y \in R$ such that $E^r_{ab}(x_1) < E^r_{ab}(y) < E^r_{ab}(x_2)$ and $E^r_{ab}(y) \subseteq R$. Since the only $E^l_{cd}$-class meeting $E^r_{ab}(y)$ is $R$, $E^r_{ab}(y)$ cannot have a right $c$-companion, contradiction. $\square$

This lemma justifies us in writing $L_c$ for the least $E^r_{ab}$-class having a right $c$-companion, $R_c$ for that right $c$-companion, and $T_c$ for $L_c \cup R_c$.

**Lemma 5.5** *Suppose that $c_1 \leq c_2 < d$ and that $E_1, E_2$ are $E^l_{c_1 d}, E^l_{c_2 d}$-classes with non-trivial intersection. Then $E_1 \subseteq E_2$.*

**Proof** For $x < y < c_1$ we have $d^m_{xd}(y, c_1) \leq d^m_{xd}(y, c_2)$. This implies $d^l_{c_1 d}(x, y) \geq d^l_{c_2 d}(x, y)$. Thus $E_1 \subseteq E_2$. $\square$

**Lemma 5.6** *If $c_1, c_2 \in I$ and $R_{c_1} \subset R_{c_2}$ or $T_{c_1} \subseteq R_{c_2}$ then $c_1 < c_2$.*

**Proof** If $c_2 \leq c_1$ then by Lemma 5.5 we would have $R_{c_2} \subseteq R_{c_1}$. But $R_{c_1}$ is a proper subset of $R_{c_2}$. $\square$

**Lemma 5.7** *For any $c \in I$ there are $c', c'' \in I$ such that $T_{c'} \cap T_{c''} = \emptyset$ and $T_{c'} \cup T_{c''} \subseteq R_c - L_c$.*

**Proof** Let $\varphi(x)$ be the formula (with parameters $a, b, d$) which asserts

$$(\exists c' \in I)(E^r_{ab}(x) \text{ has a left } c'\text{-companion}).$$

Since $I$ is a 1-type over $\{a, b, d\}$, either $(\forall x \in I)\varphi(x)$ or $(\forall x \in I)\neg\varphi(x)$.

**Case 1**: $(\forall x \in I)\varphi(x)$: By Lemma 5.4(ii) there are $w_1, w_2 \in R_c$ such that $L_c < E^r_{ab}(w_1) < E^r_{ab}(w_2)$ and $E^r_{ab}(w_i) \subseteq R_c$. Since $\varphi(w_1)$ holds, there is $c' \in I$ and a left $c'$-companion $F$ of $E^r_{ab}(w_1)$. Arguing as in Lemma 5.4(ii) there is $w_0 \in F$ such that $L_c < E^r_{ab}(w_0) < E^r_{ab}(w_1)$ and $E^r_{ab}(w_0) \subseteq F$. As $w_0, w_2 \in I$ (and $L_c \subseteq I$) there are $c_0, c_2 \in I$ such that $E^r_{ab}(w_0) = L_{c_0}$ and $E^r_{ab}(w_2) = L_{c_2}$. Now $L_{c_0} \subseteq F$ and so $R_{c_0} \cap F \neq \emptyset$. By Lemma 5.5 $R_{c_0} \subseteq F$ or $F \subseteq R_{c_0}$. But the latter is impossible since it would imply $L_{c_0} \subseteq R_{c_0}$, and we must have $R_{c_0} \subseteq F$. A similar argument shows that as $L_{c_2} \subseteq R_c$ we must have $R_{c_2} \subseteq R_c$. We may therefore let $c' = c_0$ and $c'' = c_2$.

**Case 2**: $(\forall x \in I)\neg\varphi(x)$: Choose $w_1 \in R_c$ such that $L_c < E^r_{ab}(w_1) \subseteq R_c$ and $c_1 \in I$ such that $E^r_{ab}(w_1) = L_{c_1}$. Then $R_{c_1} \subset R_c$. By Lemma 5.6 $c_1 < c$. Choose $y \in R_{c_1}$ and $x \in L_{c_1} - R_{c_1}$. Then $x < y$ and $x \in R_c - R_{c_1}$. As $x$ and $y$ lie in $R_c$, $d^l_{cd}(x,y) = 1$, and so $d^m_{xd}(y,c) = n$. As $x \notin R_{c_1}$, $d^l_{c_1 d}(x,y) > 1$ and so $d^m_{xd}(y,c_1) < n$. It follows that $d^m_{xd}(c_1,c) = n$ and hence $d^l_{cd}(x,c_1) = 1$. Thus $c_1 \in R_c$. Since $\neg\varphi(c_1)$, $R_{c_1} < E^r_{ab}(c_1)$, as otherwise $R_{c_1}$ would be a left $c_1$-companion of $E^r_{ab}(c_1)$. Similarly, $E^r_{ab}(c_1) \subset R_c$ (as otherwise $R_c$ would be a left $c$-companion of $E^r_{ab}(c_1)$. Now choose $c_2 \in I$ such that $E^r_{ab}(c_1) = L_{c_2}$. Then $R_{c_2} \subseteq R_c$, so this time we can let $c' = c_1$ and $c'' = c_2$. $\square$

We now conclude the proof of Theorem 5.1. Let $m$ be a positive integer greater than the greatest number of convex sets required to express an arbitrary subset of the model definable from 4 parameters as a union of convex sets. The assumptions of $\aleph_0$-categoricity and weak $o$-minimality ensure that such $m$ exists.

We let $c_{00} \in I$ be arbitrary, and choose $c_{i0}, c_{i1}$ by induction for $i \geq 1$. Assuming that $c_{i-1\,0}$ has been defined, we let $c_{i0} < c_{i1}$ in $I$ be as given by Lemma 5.7 so that $T_{c_{i0}}, T_{c_{i1}}$ are disjoint subsets of $R_{c_{i-1\,0}} - L_{c_{i-1\,0}}$. By Lemma 5.6, $c_{i1} < c_{i-1\,0}$, hence

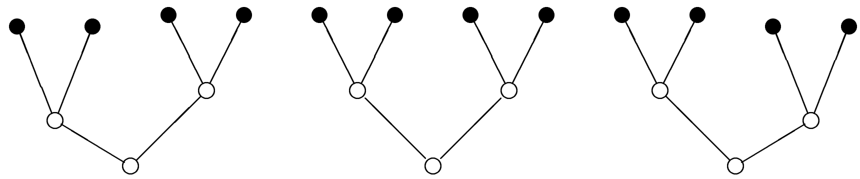
$$c_{m0} < c_{m1} < c_{m-1\,0} < c_{m-1\,1} < \ldots c_{10} < c_{11} < c_{00}.$$

Pick $e \in T_{c_{m0}}$. Then $e \in T_{c_{i0}}$ for each $i \leq m$ and $e \notin T_{c_{i1}}$ for each positive $i \leq m$. The set $A = \{x \in I : e \in T_x\}$ is thus a subset of the model which is definable over $\{a, b, e, d\}$ but which cannot be written as the union of fewer than $m$ convex sets, since $c_{i0} \in A$ and $c_{i1} \notin A$, contrary to the choice of $m$. $\square$

# 6 Higher arities

The structures we have classified have been restricted to the binary and ternary cases. It is possible that by considering higher arities we may obtain new and essentially richer structures. The purpose of this section is to give some examples illustrating that there is a wide variety of $\aleph_0$-categorical weakly $o$-minimal structures. We shall not do things in full generality (which in any case seems unrealistic), and so concentrate on one of the easier structures described in §3.

First we give an example of a 2-indiscernible but not 3-indiscernible $\aleph_0$-categorical weakly $o$-minimal structure for which 4 is the smallest number $k$ such that every formula true in the structure is logically equivalent to a boolean combination of formulae in $k$ or fewer variables. Its domain $M$ consists of all maps $a$ from $\mathbb{Q}$ to $\mathbb{Q}$ such that supp $a = \{q : a(q) \neq 0\}$ is finite, and $<$ is the lexicographic ordering. We let $l : \{(a,b) \in M^2 : a < b\} \to \mathbb{Q}$ be given by $l(a,b) =$ the least $q \in \mathbb{Q}$ such that $a(q) \neq b(q)$ (in which case necessarily $a(q) < b(q)$). If $x < y$ and $z < w$ in $M$ we let $R(x,y;z,w)$ if $l(x,y) < l(z,w)$. From this quaternary relation, which essentially allows us to refer to the set of levels of the tree $\{a|(-\infty,q) : a \in M, q \in \mathbb{Q}\}$ as a linearly ordered structure, we may easily define



$l(x,y) < l(z,w)$ $\qquad$ $l(x,y) = l(z,w)$ $\qquad$ $l(x,y) > l(z,w)$

Figure 6: Three increasing quadruples having distinct 4-types in $\mathcal{M}$

the relevant ternary relations introduced before, which are given by $T(x,y,z) \Leftrightarrow l(x,y) < l(y,z), V(x,y,z) \Leftrightarrow l(x,y) = l(y,z)$, and $U(x,y,z) \Leftrightarrow l(y,z) < l(x,y)$. Alternatively, in this case the ternary structure may be captured by a single $C$-relation given by $C(x;y,z) \Leftrightarrow T(x,y,z) \vee T(x,z,y) \vee U(y,z,x) \vee U(z,y,x)$. We take $\mathcal{M} = (M;=,<,T,V,U,R)$ (though $(M;=,<,R)$ would suffice). Note that there is a significant difference here between the function $l$ defining levels, and the function $t$ used earlier which defined greatest common initial segments, but which did not allow us to compare levels on different branches.

**Theorem 6.1** *$\mathcal{M}$ is a 2-indiscernible $\aleph_0$-categorical weakly o-minimal structure which admits elimination of quantifiers, it has 3 atoms of increasing triples, and 4 is the least number $k$ such that every formula true in $\mathcal{M}$ is equivalent to a boolean combination of formulae in $k$ or fewer variables.*

**Proof** Everything follows from the next theorem, except for the final assertion. By definition it is immediate (appealing to quantifier elimination) that 4 is such a number $k$. On the other hand, in $\mathcal{M}$ there are three different 4-types of increasing quadruples $(x,y,z,w)$ as portrayed in Fig. 6 (according as $l(x,y) <$, $=$, or $> l(z,w)$). But if we pick any 3-subtuple, the types agree in all three cases, which shows that 4 is the *smallest* $k$ which will serve. $\Box$

We now give a construction procedure based on the above model to give a wider class of examples. Let $\mathcal{P} = (\mathbb{Q};=,<,(P_i)_{i\in I})$ be $\aleph_0$-categorical, weakly $o$-minimal, and admit elimination of quantifiers, where $P_i$ has arity $r_i$. Let $\mathcal{M} = (M;=,<,T,V,U,R)$ be the structure described at the beginning of §6, and expand $\mathcal{M}$ to a structure $\mathcal{M}[\mathcal{P}]$ having additional relations $(\widehat{P}_i)_{i\in I}$, where $\widehat{P}_i$ has arity $2r_i$, and where, if $a_1,\ldots,a_{r_i},b_1,\ldots,b_{r_i} \in M$, and $a_j \neq b_j$ for each $j$, $\mathcal{M}[\mathcal{P}] \models \widehat{P}_i(a_1,b_1,\ldots,a_{r_i},b_{r_i})$ if and only if $\mathcal{P} \models P_i(l(a_1,b_1),\ldots,l(a_{r_i},b_{r_i}))$ (where $l$ is symmetrized to allow for both $a_j < b_j$ and $b_j < a_j$). Note that if $I = \emptyset$, that is, we do not add any extra structure, then the result will be the structure $\mathcal{M}$ of the previous theorem.

**Theorem 6.2** *$\mathcal{M}[\mathcal{P}]$ is $\aleph_0$-categorical, weakly o-minimal, and admits elimination of quantifiers.*

**Proof** The key point is to establish the natural back-and-forth property, and everything then follows quite easily. If $\overline{a} = (a_1, a_2, \ldots, a_n)$ is a strictly increasing sequence of members of $\mathcal{M}[\mathcal{P}]$ let us write $\overline{c}_{\overline{a}}$ for the sequence (of length $\frac{1}{2}n(n-1)$) with entries $l(a_i, a_j)$ for $i < j$ in some standard order. The main point is that if $\overline{a}$ and $\overline{b}$ are two such sequences, then they have the same quantifier-free type in $\mathcal{M}[\mathcal{P}]$ if and only if $\overline{c}_{\overline{a}}$ and $\overline{c}_{\overline{b}}$ have the same quantifier-free type in $\mathcal{P}$. This precisely corresponds to the way in which $\widehat{P}_i$ was defined from $P_i$.

Now let $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ be strictly increasing $n$-tuples of elements of $M$ having the same quantifier-free type in $\mathcal{M}[\mathcal{P}]$, and let $a \in M - \{a_1, \ldots, a_n\}$. We have to find $b \in M$ such that $\overline{a}a$ and $\overline{b}b$ have the same quantifier-free type in $\mathcal{M}[\mathcal{P}]$.

For ease we suppose that for some $i$, $a_i < a < a_{i+1}$ and we also suppose that $U(a_i, a, a_{i+1})$ or $V(a_i, a, a_{i+1})$ holds. (If $a < a_1$ or $a > a_n$ or $T(a_i, a, a_{i+1})$ a similar argument to the one given may be used.) Note that in this case the only possible element in the tuple $\overline{c}_{\overline{a}a}$ not in $\overline{c}_{\overline{a}}$ is $l(a_i, a)$, as for every $j \neq i$ we have $l(a_j, a) = l(a_j, a_i)$ or $l(a_j, a) = l(a_i, a)$. Of course $l(a_i, a)$ may already appear in $\overline{c}_{\overline{a}}$. We define $c$ to be $l(a_i, a)$.

By the remark above, $\overline{c}_{\overline{a}}$ and $\overline{c}_{\overline{b}}$ have the same quantifier-free type in $\mathcal{P}$, and by quantifier elimination in $\mathcal{P}$ there is $d$ in $\mathcal{P}$ so that $\overline{c}_{\overline{a}}c$ and $\overline{c}_{\overline{b}}d$ have the same quantifier-free type in $\mathcal{P}$. Choosing $b$ such that $b_i < b < b_{i+1}$ and $l(b_i, b) = d$, $\overline{a}a$ and $\overline{b}b$ then have the same quantifier-free type in $\mathcal{M}[\mathcal{P}]$, as required.

Next we verify weak $o$-minimality. For this it suffices to show that atomic formulae involving the new predicates $\widehat{P}_i$ define finite unions of convex sets. Let

$$S = \{x : \mathcal{M}[\mathcal{P}] \models \widehat{P}_i(a_1, b_1, \ldots, a_{r_i - 1}, b_{r_i - 1}, a_{r_i}, x)\}$$

be a set defined in this way, and let $l_j = l(a_j, b_j)$ for $1 \leq j \leq r_i - 1$. Then $\{l : \mathcal{P} \models P_i(l_1, \ldots, l_{r_i - 1}, l)\}$ is a finite union of convex sets in $\mathcal{P}$, $C_1 \cup \ldots \cup C_s$ say. Hence

$$S = \{x : x < a_{r_i} \wedge l(a_{r_i}, x) \in C_1\} \cup \ldots \cup \{x : x < a_{r_i} \wedge l(a_{r_i}, x) \in C_s\} \cup$$

$$\{x : x > a_{r_i} \wedge l(a_{r_i}, x) \in C_1\} \cup \ldots \cup \{x : x > a_{r_i} \wedge l(a_{r_i}, x) \in C_s\}$$

is also a finite union of convex sets. $\Box$

Next we show how the construction of Theorem 6.2 can be iterated. Let us define $\mathcal{N}_i$ inductively by $\mathcal{N}_0 = (\mathbb{Q} :=, <)$, $\mathcal{N}_{i+1} = \mathcal{M}[\mathcal{N}_i]$. Then the theorem immediately implies that each $\mathcal{N}_i$ is $\aleph_0$-categorical, weakly $o$-minimal, and admits elimination of quantifiers. Our task here is to show how $\mathcal{N}_{i+1}$ can be viewed as an expansion of $\mathcal{N}_i$ for each $i$ so that we may form a limit structure, and to investigate the structure of the $n$-types in these structures well enough to show that the limit structure is itself $\aleph_0$-categorical and weakly $o$-minimal, with quantifier elimination. The key is to show that although the expansions are proper, they only affect the theory in higher and higher arities.

Let us write $\mathcal{P} \sqsubseteq \mathcal{Q}$ to mean that $\mathcal{Q}$ is isomorphic to an expansion of $\mathcal{P}$. Saying that $\mathcal{Q}$ is an expansion of $\mathcal{P}$ (or that $\mathcal{P}$ is a *reduct* of $\mathcal{Q}$) means that $\mathcal{P}$ and $\mathcal{Q}$ have the same domain, and every relation of the language of $\mathcal{P}$ is 0-definable in $\mathcal{Q}$ (since we are just considering relational languages here). This is equivalent (in our $\aleph_0$-categorical setting) to saying that the automorphism group of $\mathcal{Q}$ is a subgroup of the automorphism group of $\mathcal{P}$. It is a *proper* expansion if $\mathrm{Aut}(\mathcal{Q})$ is a proper subgroup of $\mathrm{Aut}(\mathcal{P})$, written $\mathcal{P} \sqsubset \mathcal{Q}$.

Now since $\mathcal{N}_1$ is densely linearly ordered without endpoints, it is an expansion of $\mathcal{N}_0$ (proper in fact, as one sees by counting types of increasing triples). Clearly the construction of Theorem 6.2 has the property that if $\mathcal{P} \sqsubseteq \mathcal{Q}$ then $\mathcal{M}[\mathcal{P}] \sqsubseteq \mathcal{M}[\mathcal{Q}]$, and it follows that

$$\mathcal{N}_0 \sqsubseteq \mathcal{N}_1 \sqsubseteq \mathcal{N}_2 \ldots$$

We make a suitable identification of $\mathcal{N}_{i+1}$ with an expansion of $\mathcal{N}_i$ when required.

**Theorem 6.3** *(i) For each $i$ there are strictly increasing tuples $\overline{a}, \overline{b}$ of length $i+3$ which have equal types in $\mathcal{N}_i$ but not in $\mathcal{N}_{i+1}$.*

*(ii) If $\overline{a}$ and $\overline{b}$ are tuples of length $\leq i+2$ which have the same type in $\mathcal{N}_i$ then they have the same type in $\mathcal{N}_{i+1}$.*

**Proof** (i) The relevant tuples are defined inductively. When $i = 0$, we may take $\overline{a}$ and $\overline{b}$ to be increasing triples satisfying $T(\overline{a})$ and $U(\overline{b})$. Now assume that increasing $i+3$-tuples $\overline{a} = (a_0, \ldots, a_{i+2})$ and $\overline{b} = (b_0, \ldots, b_{i+2})$ have been chosen having different types in $\mathcal{N}_{i+1}$ but the same type in $\mathcal{N}_i$. Choose an arbitrary member $c$ of $\mathcal{N}_{i+1} = \mathcal{M}[\mathcal{N}_i]$, which is (by definition) a function from $\mathcal{N}_i$ to $\mathbb{Q}$. The $i+4$-tuples $\overline{c}$ and $\overline{d}$ are now given by $c_{i+3} = d_{i+3} = c$ and

$$c_j(q) = \begin{cases} c(q) & \text{if } q \neq a_j \\ c(q) - 1 & \text{if } q = a_j \end{cases}$$

and

$$d_j(q) = \begin{cases} c(q) & \text{if } q \neq b_j \\ c(q) - 1 & \text{if } q = b_j \end{cases}$$

Thus $c_j$ and $d_j$ branch off from $c$ to the left at the points $a_j$ and $b_j$ respectively, as shown in Fig.7.

Observe that for $0 \leq j < k \leq i+3$ we have $l(c_j, c_k) = a_j$ and $l(d_j, d_k) = b_j$. It now follows that the sequences $\overline{c}$ and $\overline{d}$ have equal types in $\mathcal{N}_{i+1}$ but not in $\mathcal{N}_{i+2}$ (using the same remarks as in the previous theorem about the relation between the types in these two structures), establishing the induction step.

(ii) We show by induction on $i$ that if $\overline{a}$ and $\overline{b}$ are sequences having at most $i+2$ distinct entries, and having the same type in $\mathcal{N}_i$, then they have the same type in $\mathcal{N}_{i+1}$. The basis case $i = 0$ is immediate by 2-indiscernibility of $\mathcal{N}_1$.

Now assume the result for $i$, and let $\overline{a}$ and $\overline{b}$ be strictly increasing sequences of elements $\mathcal{N}_{i+2}$ of length $i+3$ having the same type in $\mathcal{N}_{i+1}$. We may form

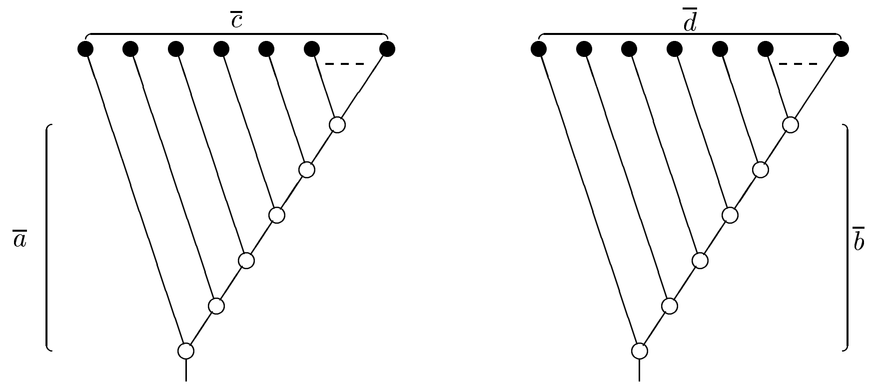



Figure 7: The construction in Theorem 6.3(i)

the sequences $\overline{c}_{\overline{a}}$ and $\overline{c}_{\overline{b}}$ in $\mathcal{N}_{i+1}$ as in the proof of Theorem 6.2, $\overline{c}_{\overline{a}}$ and $\overline{c}_{\overline{b}}$ have the same type in $\mathcal{N}_i$. The key point here is that these can have at most $i+2$ distinct entries. This may be easily proved by induction; the reason is that a finite graph-theoretical tree without vertices of valency 2 has at least one more leaf than internal node. We may now appeal to the induction hypothesis to deduce that $\overline{c}_{\overline{a}}$ and $\overline{c}_{\overline{b}}$ have the same type in $\mathcal{N}_{i+1}$, and hence that $\overline{a}$ and $\overline{b}$ have the same type in $\mathcal{N}_{i+2}$ as desired. □

Let $\mathcal{N}^*$ be the relational structure whose domain is $\mathbb{Q}$ and whose family of relations is the union of those of all the $\mathcal{N}_i$. By Theorem 6.3(ii) the $i$-types of $\mathcal{N}^*$ are the same as those of $\mathcal{N}_i$. Since $\mathcal{N}_i$ is $\aleph_0$-categorical this number is finite, and so $\mathcal{N}^*$ is also $\aleph_0$-categorical. In addition we deduce that $\mathcal{N}^*$ is weakly $o$-minimal and admits elimination of quantifiers. By Theorem 6.3(i) $\mathcal{N}_{i+1}$ is a proper expansion of $\mathcal{N}_i$ for each $i$, and so $\mathcal{N}^*$ is a proper expansion of each $\mathcal{N}_i$. Putting these remarks together it follows that $\mathcal{N}^*$ is not finitely axiomatizable, and, more strongly, does not admit a finite language, in the sense that its automorphism group is not equal to the automorphism group of any structure (with the same domain) having a finite language.

We conclude by showing that there are $2^{\aleph_0}$ essentially different structures of the kind just described, which may be constructed by suitably modifying the same method. Above we used the notation $\mathcal{M}[\mathcal{P}]$ to denote a structure obtained by using the domain $P$ of $\mathcal{P}$ to index the levels of a copy of $\mathcal{M}_3$. Here we use two such constructions which we can interlock in $2^{\aleph_0}$ ways, $\mathcal{M}_0[\mathcal{P}]$, which is the same as $\mathcal{M}[\mathcal{P}]$, and $\mathcal{M}_1[\mathcal{P}] = \mathcal{M}[\mathcal{P}^+]$ where $\mathcal{P}^+$ is obtained from $\mathcal{P}$ by 'naming a point'.

More precisely, if $\mathcal{P}$ is a 2-indiscernible ordered relational structure admitting elimination of quantifiers, we choose $a \in P$, and let $\mathcal{P}^+$ be the structure having new unary predicates $D, M, U$ (for 'down', 'middle', and 'up') standing for $(-\infty, a)$, $\{a\}$, and $(a, \infty)$ respectively, and for each relation symbol $R(x_1, \ldots, x_n)$ of $\mathcal{P}$, new relational symbols corresponding to each way of replacing one or more of the variables by $a$. (In fact $D, M$, and $U$ are of this form, but we wish to single them out particularly.) Then $\mathcal{P}^+$ is also relational and admits elimination of quantifiers. Since $\mathcal{P}$ is 1-indiscernible the resulting structure is independent of the choice of $a$, and as $\mathcal{P}$ is 2-indiscernible, $D$, $M$, and $U$ are the only 1-atoms of $\mathcal{P}^+$.

We now remark on the degree of indiscernibility possessed by $\mathcal{M}_0[\mathcal{P}]$ and $\mathcal{M}_1[\mathcal{P}]$. If $\mathcal{P}$ is 1-indiscernible then $\mathcal{M}_0[\mathcal{P}]$ is 2-indiscernible, since the ordered pairs $(a_1, a_2), (b_1, b_2)$ have the same type in $\mathcal{M}_0[\mathcal{P}]$ if and only if $\overline{c}_{(a_1,a_2)}$ and $\overline{c}_{(b_1,b_2)}$ have the same type in $\mathcal{P}$. On the other hand, if $\mathcal{P}$ is 2-indiscernible, then $\mathcal{M}_1[\mathcal{P}]$ is only 1-indiscernible, and has three 2-atoms of increasing pairs, given by $\widehat{D}$, $\widehat{M}$, and $\widehat{U}$, as again follows by considering the possible types of $\overline{c}_{(a_1,a_2)}$. So if we wish to use combinations of the processes $\mathcal{M}_0$ and $\mathcal{M}_1$ to iterate and form a limit structure, we must ensure that $\mathcal{M}_1$ is never applied twice consecutively.

For each infinite binary sequence $\mathbf{r} = (r_i : i \in \omega)$ the *derived* sequence $\mathbf{r}'$ is obtained by deleting the first entry (so that $r'_i = r_{i+1}$). We define $\mathcal{N}^{\mathbf{r}}_n$ over all $\mathbf{r}$ which do not contain two consecutive 1s (there are $2^{\aleph_0}$ such sequences) by induction on $i$ thus:

$$\mathcal{N}^{\mathbf{r}}_0 = (\mathbb{Q}; =, <), \quad \mathcal{N}^{\mathbf{r}}_{i+1} = \mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{r}'}_i].$$

So $\mathcal{N}^{\mathbf{r}}_i = \mathcal{M}_{r_0}[\mathcal{M}_{r_1}[\ldots \mathcal{M}_{r_{i-1}}[(\mathbb{Q}; =, <)] \ldots]]$. The fact that $\mathcal{N}^{\mathbf{r}}_i \sqsubseteq \mathcal{N}^{\mathbf{r}}_{i+1}$ follows easily by induction, and so we obtain a 'limit structure' $\mathcal{N}^{\mathbf{r}}$ having the same domain ($\mathbb{Q}$) as each $\mathcal{N}^{\mathbf{r}}_i$, but whose language is the union of all their languages. The fact that the expansions $\mathcal{N}^{\mathbf{r}}_{i+1} \sqsupset \mathcal{N}^{\mathbf{r}}_i$ are all proper follows by the argument given in Theorem 6.3(i), and so $\mathcal{N}^{\mathbf{r}}$ does not admit a finite language.

We next have to verify a version of Theorem 6.3(ii) for the new situation. We show by induction on $i$ that if there are $k$ 0s in $\mathbf{r}$ before the $i$th entry, and $\overline{a}$ and $\overline{b}$ are increasing sequences of length $k$ having the same type in $\mathcal{N}^{\mathbf{r}}_i$ then they also have the same type in $\mathcal{N}^{\mathbf{r}}_{i+1}$. For the induction step assume that $\overline{a}$ and $\overline{b}$ of length $k$ have the same type in $\mathcal{N}^{\mathbf{r}}_{i+1} = \mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{r}'}_i]$, and that there are at least $k$ 0s in $\mathbf{r}$ before the $(i+1)$th entry. If $r_0 = 0$ then there are at least $k-1$ 0s in $\mathbf{r}'$ before the $i$th entry and the induction goes through as in the original proof. If $r_0 = 1$ then there are at least $k$ 0s in $\mathbf{r}'$ before the $i$th entry. Also $\overline{c}_{\overline{a}}$ and $\overline{c}_{\overline{b}}$ have the same type in $(\mathcal{N}^{\mathbf{r}'}_i)^+$, and so if $a'$ is the named point, $\overline{c}_{\overline{a}}a'$ and $\overline{c}_{\overline{b}}a'$ have the same type in $\mathcal{N}^{\mathbf{r}'}_i$. Now $\overline{c}_{\overline{a}}a'$ and $\overline{c}_{\overline{b}}a'$ have at most $k$ distinct entries, so by the induction hypothesis they have the same type in $\mathcal{N}^{\mathbf{r}'}_{i+1}$, and the proof is concluded as before. Since every $\mathbf{r}$ we consider has infinitely many 0s, this is strong enough for us to be able to deduce that $\mathcal{N}^{\mathbf{r}}$ is $\aleph_0$-categorical, admits elimination of quantifiers, and is weakly $o$-minimal.

Now $\mathcal{N}^{\mathbf{r}} = \mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{r}'}]$, and $\mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{r}'}]$ has $2r_0 + 1$ 2-atoms on increasing pairs, so it follows that if $\mathcal{N}^{\mathbf{r}}$ and $\mathcal{N}^{\mathbf{s}}$ are 'isomorphic' (meaning that their

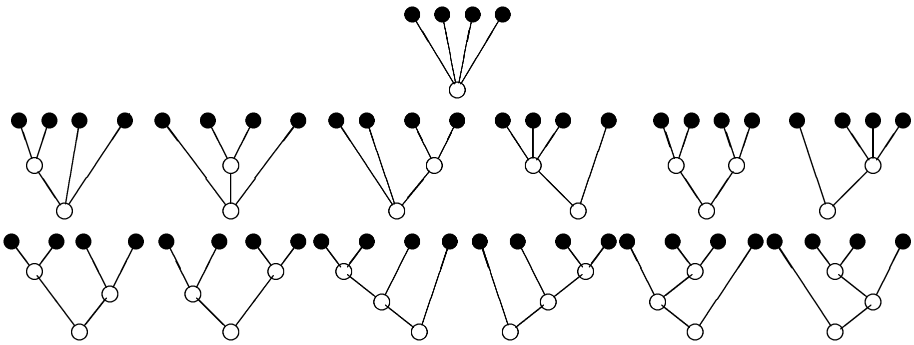


Figure 8: Possible levelled ordered trees having 4 leaves

automorphism groups are isomorphic as permutation groups) then $r_0 = s_0$. We want to show that $r_i = s_i$ for every $i$, and this will establish that we really have found $2^{\aleph_0}$ 'essentially distinct' examples. The following statement is sufficient:

$$(*)\left\{\begin{array}{l}\text{if for every } k,\ \mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{r}'}] \text{ and } \mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{s}'}] \text{ have the same number of}\\ \text{types of increasing } k\text{-tuples, then the same is true of } \mathcal{N}^{\mathbf{r}'} \text{ and } \mathcal{N}^{\mathbf{s}'}.\end{array}\right.$$

For this we show how to work out the number of types of increasing $k+1$-tuples of $\mathcal{M}_0[\mathcal{P}]$ and $\mathcal{M}_1[\mathcal{P}]$ in terms of the number of types of increasing $\leq k+1$-tuples of $\mathcal{P}$.

We say that a (finite) *levelled ordered tree* is a tree $(T, \preceq)$ together with a function $l$ from the set $R$ of non-leaves of $T$ onto a linearly ordered set $(L, \leq)$ (of 'levels') such that if $x < y$ in $R$ then $l(x) < l(y)$ in $L$, and a linear ordering of the set of immediate successors of each member of $R$. Let $t_{k,n}$ be the number up to isomorphism of levelled ordered trees having $n$ levels and $k$ leaves. (In Fig. 8 we show the levelled ordered trees with 4 leaves, illustrating that $t_{4,1} = 1$ and $t_{4,2} = t_{4,3} = 6$.) We can then see that if $\mathcal{P}$ has $n_i$ types of increasing $i$-tuples for $i < k$ then $\mathcal{M}_0[\mathcal{P}]$ and $\mathcal{M}_1[\mathcal{P}]$ have $\sum_{i<k} n_i t_{k,i}$ and $\sum_{i<k}(in_i + (i+1)n_{i+1})t_{k,i}$ types respectively of increasing $k$-tuples. The proof of this uses the link between a $k$-tuple $\overline{a}$ in $\mathcal{M}_0[\mathcal{P}]$ and $\overline{c_{\overline{a}}}$ in $\mathcal{P}$ remarked on several times before, and in addition the observation that the number of types of increasing $i$-tuples in $\mathcal{P}^+$ is equal to

$i\times$ the number of types of increasing $i$-tuples in $\mathcal{P}$ (those $i$-tuples containing the named point $a$ as an entry in one of $i$ possible positions)

$+(i+1)\times$ the number of types of increasing $(i+1)$-tuples in $\mathcal{P}$ (corresponding to those $i$-tuples not containing $a$ as an entry, with $i+1$ possible positions).

We are now in a position to verify the truth of $(*)$. We show by induction that $\mathcal{N}^{\mathbf{r}'}$ and $\mathcal{N}^{\mathbf{s}'}$ have the same number of types of increasing $k$-tuples. Let $m_i$ and $n_i$ be the numbers of types of increasing $k$-tuples of $\mathcal{N}^{\mathbf{r}'}$ and $\mathcal{N}^{\mathbf{s}'}$ respectively, and assume inductively that $m_i = n_i$ for $i < k$. Then by the above calculation if $r_0 = 0$ the numbers of types of increasing $k+1$-tuples of $\mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{r}'}]$ and $\mathcal{M}_{r_0}[\mathcal{N}^{\mathbf{s}'}]$ are $\sum_{i \leq k} m_i t_{k+1,i}$ and $\sum_{i \leq k} n_i t_{k+1,i}$ respectively, and these are known to be equal. Applying the induction hypothesis we deduce, since $t_{k+1,k} \neq 0$, that $m_k = n_k$, as desired. If $r_0 = 1$ we consider instead the numbers of types of increasing $k$-tuples, which are $\sum_{i<k}(im_i + (i+1)m_{i+1})t_{k,i}$ and $\sum_{i<k}(in_i + (i+1)n_{i+1})t_{k,i}$ respectively. Since these are known to be equal, and $t_{k,k-1} \neq 0$, we again deduce that $m_k = n_k$.

# References


[1] S.Adeleke and Peter M. Neumann, Relations related to betweenness: their structure and automorphisms, Memoirs of the American Mathematical Society 623 (1998).

[2] Peter J. Cameron, Some treelike objects, Quarterly Journal of Mathematics, (2) 38 (1987), 155-183.

[3] M.A.Dickmann, Elimination of quantifiers for ordered valuation rings, Proceedings of 3rd Easter Model Theory Conference at Gross Köris, Seminarberichte, Humboldt University (1985).

[4] A.Pillay and C.Steinhorn, Definable sets in ordered structures I, Transactions of the American Mathematical Society 295 (1986), 565-592.