Fibered Multiderivators and (co)homological descent

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Abstract

The theory of derivators enhances and simplifies the theory of triangulated categories. In this article a notion of fibered (multi-)derivator is developed, which similarly enhances fibrations of (monoidal) triangulated categories. We present a theory of cohomological as well as homological descent in this language. The main motivation is a descent theory for Grothendieck’s six operations.

Introduction

Grothendieck’s six functors and descent

Let $S$ be a category, for instance a suitable category of schemes, topological spaces, analytic manifolds, etc. A Grothendieck six functor formalism on $S$ consists of a collection of (derived) categories $D_S$, one for each “base space” $S$ in $S$ with the following six types of operations:

\[
\begin{align*}
  f^* & \quad f_* \quad \text{for each } f \text{ in } \text{Mor}(S) \\
  f! & \quad f^! \quad \text{for each } f \text{ in } \text{Mor}(S) \\
  \otimes & \quad \text{HOM} \quad \text{in each fibre } D_S
\end{align*}
\]

The fibre $D_S$ is, in general, a derived category of “sheaves” over $S$, for example coherent sheaves, $l$-adic sheaves, abelian sheaves, $D$-modules, motives, etc. The functors on the left hand side are left adjoints of the functors on the right hand side. $f!$ and its right adjoint $f^!$ are called “push-forward with proper support”, and “exceptional pull-back”, respectively. The six functors come along with a bunch of compatibility isomorphisms between them (cf. A.2.12) and it is not easy to make their axioms really precise. In an appendix to this article, we explain that one quite simple precise definition is the following:
Definition [A.2.9] A Grothendieck six-functor-formalism on a category $S$ (with fibre products) is a bifibration of symmetric multicategories

$$p : D \to S^{\text{cor}}$$

where $S^{\text{cor}}$ is the symmetric multicategory of correspondences in $S$ (cf. Definition [A.2.8]).

From such a bifibration we obtain the operations $f_*, f^*$ (resp. $f^!, f_!$) as pull-back and push-forward along the correspondences

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & & Y'
\end{array}
$$

and

$$
\begin{array}{ccc}
X & \xleftarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & & Y'
\end{array}
$$

respectively. We get $\mathcal{E} \otimes \mathcal{F}$ for objects $\mathcal{E}, \mathcal{F}$ above $X$ as the target of a Cartesian 2-ary multimorphism from the pair $\mathcal{E}, \mathcal{F}$ over the correspondence

$$
\begin{array}{ccc}
X & \xleftarrow{f} & X \\
\downarrow & & \downarrow \\
X & & X
\end{array}
$$

Given a simplicial resolution $\pi : U_\bullet \to S$ of a space $S \in S$ (for example arising from a Čech cover w.r.t. a suitable Grothendieck topology)

$$
\cdots \rightarrow U_2 \rightarrow U_1 \rightarrow U_0,
$$

and given an object $\mathcal{E}$ in $D_S$, one can construct complexes in the category $D_S$:

$$
\cdots \rightarrow \pi_2 \ast \pi_2^! \mathcal{E} \rightarrow \pi_1 \ast \pi_1^! \mathcal{E} \rightarrow \pi_0 \ast \pi_0^! \mathcal{E} \rightarrow 0
$$

The first question of homological (resp. cohomological) descent is whether the hyper(co)homology of these complexes recovers the homology (resp. cohomology) of $\mathcal{E}$. Without a suitable enhancement of the situation, this question, however, does not make sense because a double complex, once considered as a complex in the derived category, looses the information of the homology of its total complex. There are several remedies for this problem. Classically, if at least the $\pi^*$ are derived functors and $\mathcal{E}$ is acyclic w.r.t. them, one can derive the whole construction to get a coherent double complex. This does not work, however, with the functors $f_!, f^!$ which are often only constructed on the derived category. One possibility is to consider enhancements of the triangulated categories in question as DG-categories or $\infty$-categories. In this article, we have worked out a different approach based on Grothendieck’s idea of derivators which is, perhaps, conceptually even simpler. It is sufficiently powerful to glue the six functors and define them for morphisms between stacks, or even higher stacks. Like the $\infty$-category approach, it is also very general, not being restricted to the stable case.
Fibered multiderivators

The notion of triangulated category developed by Grothendieck and Verdier in the 1960’s, as successful as it has been, is not sufficient for many purposes, for both practical reasons (certain natural constructions cannot be performed) as well as for theoretical reasons (the axioms are rather involved and lack conceptual clarity). Grothendieck much later [10], with the notion of derivator, proposed a marvelously simple remedy to both deficiencies. The basic observation is that all problems mentioned above are based on the following fact: Consider a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ (quasi-isomorphisms, weak equivalences, etc.) which one would like to become isomorphisms. Then homotopy limits and colimits w.r.t. $(\mathcal{C}, \mathcal{W})$ cannot be reconstructed once passed to the homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ (for example a derived category, or the homotopy category of a model category). In particular, the cone (required to exist in a triangulated category in a brute-force way, but not functorially!) or the total complex of a complex of complexes (totally lost in the derived category) are instances of homotopy (co)limits. Grothendieck also observed that very basic and intuitive properties of homotopy limits and colimits (and more general Kan extensions) not only determine the additional structure (triangles, shift functors) on a triangulated category but also imply all of its rather involved axioms. This idea has been successfully worked out by Cisinski, Groth, Grothendieck, Heller, Maltsiniotis, and others. We refer to the introductory article [7] for an overview.

The purpose of this article is to propose a notion of fibered (multi-)derivator which enhances the notion of a fibration of (monoidal) triangulated categories in the same way as the notion of usual derivator enhances the notion of triangulated category. We emphasize that this new context is very well suited to reformulate (and reprove the theorems of) the classical theory of cohomological descent and to establish a completely dual theory of homological descent which should be satisfied by the $f_!, f^!$-functors.

(Co)homological descent with fibered derivators

Pursuing the idea of derivators, there is a neat conceptual solution to the problem of (co)homological descent mentioned above: Analogously to a derivator which associates a (derived) category with each diagram shape $I$, we should consider a (derived) category for each diagram $F : I \to S^{\text{cor}}$ (denoted by $(I, F)$). Then, given a simplicial resolution $\pi : U_\bullet \to S$ as before, considered as a morphism $p : (\Delta^{\text{op}}, U_\bullet) \to (\cdot, S)$ of diagrams in $S^{\text{cor}}$, resp. $i : (\Delta, (U_\bullet)^{\text{op}}) \to (\cdot, S)$ in a dual diagram category (cf. [1.6.2]), the question becomes:

Q1: Does the corresponding pull-back $i^*$ have a right adjoint $i_*$, respectively does $p^*$ have a left adjoint $p_!$ (a straightforward generalization of the question of existence of homotopy (co)limits in usual derivators!) and is the corresponding unit $\text{id} \to i_*i^*$ (resp. counit $pp^* \to \text{id}$) an isomorphism?

Instead of, however, taking an association $(I, F) \mapsto \mathbb{D}(I, F)$ as the fundamental datum, we propose to take a morphism of pre-derivators $p : \mathbb{D} \to S$ (or even pre-multiderivators) as the fundamental datum, the $\mathbb{D}(I, F)$ being reconstructed as its fibers $\mathbb{D}(I)_F$, if $S$ is the pre-derivator associated with a category. This allows also more general situations, where $S$ is not associated with an ordinary category. For example, in a forthcoming article we will define (and give examples of) a derivator version of a Grothendieck six-functor formalism, that is, a symmetric fibered multiderivator

$$p : \mathbb{D} \to S^{\text{cor}},$$
where $S^{\text{cor}}$ is the symmetric pre-multiderivator of correspondences in $S$. Here $S^{\text{cor}}$ is not the symmetric pre-multiderivator associated with the symmetric multicategory $S^{\text{cor}}$ but it is associated with the dendroidal class of correspondences in $S$ whose homotopy multicategory is the symmetric multicategory $S^{\text{cor}}$.

Q2: More generally, we may consider Cartesian (resp. coCartesian) objects in the fibre over a diagram $(\Delta^{\text{op}}, U_\bullet)$ (resp. $(\Delta, (U_\bullet)^{\text{op}})$), and ask whether these categories depend only on $U_\bullet$ up to taking (finite) hypercovers w.r.t. a fixed Grothendieck topology on $S$. This would allow to define the six operations, for example, if the simplicial objects $U_\bullet$ are presentations of stacks.

Overview

In section 1 we give the general definition of a left (resp. right) fibered multiderivator $p : D \to S$. The axioms are basically a straightforward generalization of those of a left, resp. right derivator. To give a priori some conceptual evidence that these axioms are indeed reasonable, we prove that the notion of fibered multiderivator is transitive (1.4), and that it gives rise to a pseudo-functor from ‘diagrams in $S$’ to categories, for which a neat base-change formula holds (1.6).

In section 2 a theory of (co)homological descent for fibered derivators is developed (the monoidal, i.e. multi-, aspect does not play any role here). We propose a definition of fundamental localizer in the category of diagrams in $S$ which is a generalization of Grothendieck’s notion of fundamental localizer in categories. The latter gives a nice combinatorial description of weak equivalences of categories in terms of the condition of Quillen’s theorem A. In our more general setting the notion of fundamental localizer depends on the choice of a Grothendieck (pre-)topology on $S$. In section 2.2 we show purely abstractly that a finite hypercover, considered as a morphism of simplicial diagrams, lies in any fundamental localizer. Thus this more general notion of fundamental localizer has a similar relation to weak equivalences of simplicial pre-sheaves, although we will not yet give any precise statement in this direction.

In sections 2.3 and 2.4 this notion of fundamental localizer is tied to the theory of fibered derivators. We introduce two notions of (co)homological descent for a fibered derivator $p : D \to S$: weak and strong $D$-equivalences. The notion of weak $D$-equivalences (related to Q1 above) is a straightforward generalization of Cisinski’s notion of $D$-equivalence for usual derivators. In our relative context, both notions of $D$-equivalence come in a cohomological as well as in a homological flavour (a phenomenon which does not occur for usual derivators).

Whenever the fibered derivator is (co)local w.r.t. to the Grothendieck pre-(co)topology — a rather weak and obviously necessary condition (see section 1.5) — then the Main Theorem 2.4.4 (resp. 2.4.5) of this article states that weak $D$-equivalences form a fundamental localizer under very general conditions (the easier case) and that also strong $D$-equivalences (related to Q2 above) form a fundamental localizer, at least in the case of fibered derivators with stable, compactly generated fibers. The proof uses results from the theory of triangulated categories due to Neeman and Krause (centering around Brown representability type theorems). The link to our theory of fibered (multi-)derivators is explained in section 3.

In section 4 we introduce the notion of fibration of multi-model-categories. This is the most favorable standard context in which a fibered multi-derivator (whose base is associated with a usual multicategory) can be constructed. We will present more general methods of constructing fibered multiderivators in a forthcoming article, in particular, those encoding a full six-functor-formalism.
Notation

We denote by \( \mathcal{C}AT \) the 2-"category" of categories, by \((\mathcal{S})MCAT\) the 2-"category" of (symmetric) multicategories, and by \( \text{Cat} \) the 2-category of small categories. We consider a partially ordered set (poset) \( X \) as a small category by considering the relation \( x \leq y \) to be equivalent to the existence of a unique morphism \( x \rightarrow y \). We denote the natural numbers (resp. including 0) by \( \mathbb{N} \) (resp. \( \mathbb{N}_0 \)). The ordered sets \( \{0, \ldots, n\} \subset \mathbb{N}_0 \) considered as a small category are denoted by \( \Delta_n \). We denote by \( \text{Mor}(\mathcal{D}) \), resp. \( \text{Iso}(\mathcal{D}) \), the class of morphisms (resp. isomorphisms) in a category \( \mathcal{D} \). The final category (which consists of only one object and its identity) is denoted by \( \cdot \) or \( \Delta_0 \). The same notation is also used for the final multi-category, i.e. that with one object and precisely one \( n \)-ary morphism for any \( n \). The conventions about multicategories and fibered (multi-)categories that we are using are summarized in an appendix \( \text{A} \).

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1 where "category" has classes replaced by 2-classes (or, if the reader prefers, is constructed w.r.t. a larger universe).
1 Fibered derivators

1.1 Categories of diagrams

Definition 1.1.1. A diagram category is a full sub-2-category $\Dia \subset \Cat$, satisfying the following axioms:

(Dia1) The empty category $\emptyset$, the final category $\cdot$ (or $\Delta_0$), and $\Delta_1$ are objects of $\Dia$.

(Dia2) $\Dia$ is stable under taking finite coproducts and fibered products.

(Dia3) For each functor $\alpha : I \to J$ in $\Dia$ and object $j \in J$ the slice categories $I \times_J j$ and $j \times_J I$ are in $\Dia$.

A diagram category $\Dia$ is called self-dual, if it satisfies in addition

(Dia4) If $I \in \Dia$ then $I^{op} \in \Dia$.

A diagram category $\Dia$ is called infinite, if it satisfies in addition

(Dia5) $\Dia$ is stable under taking arbitrary coproducts.

In the following we mean by a diagram a small category.

Example 1.1.2. We have the following diagram categories:

Cat category of all diagrams. It is self-dual.

Inv category of inverse diagrams $C$, i.e. small categories $C$ such that there exists a functor $C \to \mathbb{N}_0$ with the property that the preimage of an identity consists of identities. An example is the injective simplex category $\Delta^\circ$:

\[ \cdots \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square 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1.2 Pre-(multi-)derivators

Definition 1.2.1. A pre-derivator of domain Dia is a contravariant (strict) 2-functor
\[ D : \text{Dia}^{1-\text{op}} \to \text{CAT}. \]

A pre-multiderivator of domain Dia is a contravariant (strict) 2-functor
\[ D : \text{Dia}^{1-\text{op}} \to \text{MCAT} \]
into the 2-“category” of multicategories. A morphism of pre-derivators is a natural transformation.

For a morphism \( \alpha : I \to J \) in Dia the corresponding functor
\[ D(\alpha) : D(J) \to D(I) \]
will be denoted by \( \alpha^* \).

We call a pre-multiderivator symmetric (resp. braided), if its images are symmetric (resp. braided), and the morphisms \( \alpha^* \) are compatible with the actions of the symmetric (resp. braid) groups.

1.2.2. The pre-derivator associated with a category: Let \( S \) be a category. We associate with it the pre-derivator
\[ S : I \mapsto \text{Hom}(I, S). \]
The pullback \( \alpha^* \) is defined as composition with \( \alpha \). A 2-morphism \( \kappa : \alpha \to \beta \) induces a natural 2-morphism \( S(\kappa) : \alpha^* \to \beta^* \).

1.2.3. The pre-derivator associated with a simplicial class (in particular with an \( \infty \)-category): Let \( S \) be a simplicial class, i.e. a functor \( S : \Delta \to \text{CLASS} \) into the “category” of classes. We associate with it the pre-derivator
\[ S : \Delta \to \text{CLASS} \]
into the “category” of classes. We associate with it the pre-derivator
\[ S : I \mapsto \text{Ho}(\text{Hom}(N(I), S)), \]
where \( N(I) \) is the nerve of \( I \) and Ho is the left adjoint of \( N \). In detail this means the objects of the category \( S(I) \) are morphisms \( \alpha : N(I) \to S \), the class of morphisms in \( S(I) \) is freely generated by morphisms \( \mu : N(I \times \Delta_1) \to S \) considered to be a morphism from its restriction to \( N(I \times \{0\}) \) to its restriction to \( N(I \times \{1\}) \) modulo the relations given by morphisms \( \nu : N(I \times \Delta_2) \to S \), i.e. if \( \nu_1, \nu_2 \) and \( \nu_3 \) are its restrictions to the 3 faces of \( \Delta_2 \) then we have \( \mu_3 = \mu_2 \circ \mu_1 \). The pullback \( \alpha^* \) is defined as composition with the morphism \( N(\alpha) : N(I) \to N(J) \). A 2-morphism \( \kappa : \alpha \to \beta \) can be given as a functor \( I \times \Delta_1 \to J \) which yields (applying \( N \) and composing) a natural transformation which we call \( S(\kappa) \).

1.2.4. More generally, consider the full subcategory \( \Delta \subset \text{MCAT} \) of all finite connected multicategories \( M \) that are freely generated by a finite set of multimorphisms \( f_1, \ldots, f_n \) such that each object of \( M \) occurs at most once as a source and at most once as the target of one of the \( f_i \). Similarly consider the full subcategory \( T \subset \text{SMCAT} \) which is obtained from \( \Delta \) adding images under the operations of the symmetric groups. (This category is usually called the symmetric tree category.) With a functor
\[ S : \Delta \to \text{CLASS}, \quad S : T \to \text{CLASS} \]
we associate the pre-multiderivator
\[ S : I \to Ho(Hom(N(I), S)), \]
where \( N : \mathcal{MCAT} \to \mathcal{CLASS}^{M\Delta} \) (resp. \( N : \mathcal{SMCAT} \to \mathcal{CLASS}^T \)) is the nerve, \( I \) is considered to be a multicategory without any \( n \)-ary morphisms for \( n \geq 2 \), and \( Ho \) is the left adjoint of \( N \). Objects in \( \mathcal{SET}^T \) are called dendroidal sets in [17].

1.3 Fibered (multi-)derivators

1.3.1. Let \( p : D \to S \) be a morphism of pre-derivators with domain Dia and \( \alpha : I \to J \) a morphism in Dia. Consider an object \( S \in S(J) \). The functor \( \alpha^* \) induces a morphism between fibers (denoted the same way)
\[ \alpha^* : D(J)_S \to D(I)_{\alpha^* S}. \]
We are interested in the case that the latter has a left adjoint \( \alpha^*_S \), resp. right adjoint \( \alpha^*_S \). These will be called relative left/right homotopy Kan extension functors with base \( S \). For better readability we often omit the base from the notation. Though the base is not determined by the argument of \( \alpha \), it will often be understood from the context. cf. also [1.3.21].

1.3.2. We are interested in the case in which all morphisms
\[ p(I) : D(I) \to S(I) \]
are Grothendieck fibrations, resp. opfibrations [A.1] or, more generally, (op)fibrations of multi-categories [A.2]. We always assume that the maps \( \alpha^* := D(\alpha) \) map (co)Cartesian morphisms to (co)Cartesian morphisms. Then we will choose an associated pseudo-functor, i.e. for each \( f : S \to T \) in \( S(I) \) a functor
\[ f_* : D(I)_S \to D(I)_T, \]
resp. a functor
\[ f^* : D(I)_T \to D(I)_S. \]

1.3.3. For a diagram of categories
\[
\begin{array}{ccc}
I & & \\
| & \alpha | & \\
K & \beta \downarrow & J \\
\end{array}
\]
the slice category \( K \times_J I \) is the category of triplets \( (k, i, \mu) \) of \( k \in K, i \in I \) and \( \mu : \alpha(i) \to \beta(k) \). It sits in a corresponding 2-commutative square:
\[
\begin{array}{ccc}
K \times_J I & \overset{B}{\to} & I \\
\downarrow & \swarrow_{\beta^\mu} & \downarrow \alpha \\
K & \overset{\beta}{\to} & J \\
\end{array}
\]
which is universal w.r.t. such squares. This construction is associative, but of course not commutative unless \( J \) is a groupoid. The projection \( K \times_J I \to K \) is a Grothendieck fibration and the projection \( K \times_J I \to I \) is a Grothendieck opfibration (see [A.1]). There is an adjunction
\[ I \times_J J \overset{\cong}{\to} I. \]
Consider an arbitrary 2-commutative square:

\[
\begin{array}{ccc}
L & B \\
A & \downarrow \alpha \downarrow \\
K & J
\end{array}
\]

(1)

and let \( S \in \mathbb{S}(J) \) be an object and \( E \) a preimage in \( \mathbb{D}(J) \) w.r.t. \( p \). The 2-morphism (natural transformation) \( \mu \) induces a functorial morphism

\[
\mathbb{S}(\mu) : A^* \beta^* S \to B^* \alpha^* S
\]

(we again omit the base \( S \) from the notation for better readability — it is always determined by the argument) and therefore a functorial morphism

\[
\mathbb{D}(\mu) : A^* \beta^* \mathcal{E} \to B^* \alpha^* \mathcal{E}
\]

over \( \mathbb{S}(\mu) \), or — if we are in the (op)fibered situation — equivalently

\[
A^* \beta^* \mathcal{E} \to (\mathbb{S}(\mu))^* B^* \alpha^* \mathcal{E}
\]

respectively

\[
(\mathbb{S}(\mu))^* A^* \beta^* \mathcal{E} \to B^* \alpha^* \mathcal{E}
\]

in the fibre above \( A^* \beta^* S \), resp. \( B^* \alpha^* S \),

Let now \( \mathcal{F} \) be an object over \( \alpha^* S \). If relative right homotopy Kan extensions exist, we may form the following composition which will be called the (right) base-change morphism:

\[
\beta^* \alpha_* \mathcal{F} \to A_* A^* \beta^* \alpha_* \mathcal{F} \to A_* (\mathbb{S}(\mu))^* B^* \alpha^* \alpha_* \mathcal{F} \to A_* (\mathbb{S}(\mu))^* B^* \mathcal{F}
\]

(2)

Let now \( \mathcal{F} \) be an object over \( \beta^* S \). If relative left homotopy Kan extensions exist, we may form the composition, the (left) base-change morphism:

\[
B_!(\mathbb{S}(\mu))^* A_* \mathcal{F} \to B_!(\mathbb{S}(\mu))^* A^* \beta^* \beta_! \mathcal{F} \to B_! B^* \alpha^* \beta_! \mathcal{F} \to \alpha^* \beta_! \mathcal{F}
\]

(3)

We say that the square (1) is homotopy exact if (2) is an isomorphism for all right fibered derivators and (3) is an isomorphism for all left fibered derivators. It is obvious a priori that for a left and right fibered derivator (2) is an isomorphism if and only if (3) is, one being the adjoint of the other (see [7, §1.2] for analogous reasoning in the case of usual derivators).

**Definition 1.3.5.** \(^3\) We consider the following axioms on a pre-(multi-)derivator \( \mathbb{D} \):

(\text{Der1}) For \( I, J \) in \( \text{Dia} \), the natural functor \( \mathbb{D}(I || J) \to \mathbb{D}(I) \times \mathbb{D}(J) \) is an equivalence of (multi-)categories. Moreover \( \mathbb{D}(\varnothing) \) is not empty.

(\text{Der2}) For \( I \) in \( \text{Dia} \) the ‘underlying diagram’ functor

\[
dia : \mathbb{D}(I) \to \text{Hom}(I, \mathbb{D}(\cdot))
\]

is conservative.

\(^3\)The numbering is compatible with that of [7] in the case of non-fibered derivators.
In addition, we consider the following axioms for a morphism of pre-(multi-)derivators $p : \mathbb{D} \to \mathbb{S}$ (here we write down the left versions of the axioms; they all have corresponding dual right versions):

(FDer0 left) For each $I$ in $\text{Dia}$ the morphism $p$ specializes to an opfibered (multi-)category and the morphisms $\alpha : I \to J$ induce a diagram

$$
\begin{aligned}
\mathbb{D}(J) & \xrightarrow{\alpha^*} \mathbb{D}(I) \\
\downarrow & \downarrow \\
\mathbb{S}(J) & \xrightarrow{\alpha^*} \mathbb{S}(I)
\end{aligned}
$$

of opfibered (multi-)categories, i.e. the top horizontal functor maps coCartesian arrows to coCartesian arrows.

(FDer3 left) For each morphism $\alpha : I \to J$ in $\text{Dia}$ and $S \in \mathbb{S}(J)$ the functor $\alpha^*$ between fibers $\mathbb{S}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$ has a left-adjoint $\alpha_1^\#$.

(FDer4 left) For each morphism $\alpha : I \to J$ in and object $j \in J$ and the 2-cell

$$
\begin{array}{ccc}
I \times_J j & \xrightarrow{j} & I \\
\downarrow & & \downarrow \alpha \\
\{j\} & \xrightarrow{j} & J
\end{array}
$$

we get that the induced natural transformation of functors $\alpha_{j_1}(\mathbb{S}(\mu)) \circ \alpha^* \to j^* \alpha_1$ is an isomorphism.

(FDer5 left) (only for the multiderivator case needed). For any $I \in \text{Dia}$ and the morphism $\pi : I \to \cdot$, and a morphism $\xi \in \text{Hom}(S_1, \ldots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors

$$
\pi_j(\alpha^* \xi)(\pi^* - , \ldots , \pi^* - , - , \pi^* - , \ldots , \pi^* - ) \equiv \xi(-, \cdots , - , \pi_1 - , - , \cdots , - )
$$

are isomorphisms.

**Definition 1.3.6.** A morphism of pre-(multi-)derivators $p : \mathbb{D} \to \mathbb{S}$ with domain $\text{Dia}$ is called a left fibered (multi-)derivator with domain $\text{Dia}$, if axioms (Der1–2) hold for $\mathbb{D}$ and $\mathbb{S}$ and (FDer0–5 left) hold for $p$. Similarly it is called a right fibered (multi-)derivator with domain $\text{Dia}$, if instead the corresponding dual axioms (FDer0–5 right) hold. It is called just fibered if it is both left and right fibered.

The squares in axiom (FDer4) are in fact homotopy exact and it follows from (FDer4) that many more are (see [1.3.16]).

There is some redundcancy in the axioms, cf. [1.3.20]

**Question 1.3.7.** It seems natural to allow also (symmetric) multicategories, in particular operads, as domain for a fibered (symmetric) multiderivator. The author however did not succeed in writing down a neat generalization of (FDer3–4) which would encompass (FDer5).

---

4 This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$. 10
1.3.8. The pre-derivator associated with an ∞-category $\mathcal{S}$ is actually a left and right derivator (in the usual sense, i.e. fibered over ·) if $\mathcal{S}$ is complete and co-complete \[9\]. This includes the case of pre-derivators associated with categories, which is, of course, classical — axiom (FDer4) expressing nothing else than Kan’s formulas.

1.3.9. Let $S \in \mathcal{S}(\cdot)$ be an object and $p : \mathcal{D} \to \mathcal{S}$ be a (left, resp. right) fibered derivator. The association

$$I \mapsto \mathbb{D}(I)_{p^* S},$$

where $p : I \to \cdot$ is the projection, defines a (left, resp. right) derivator in the usual sense which we call its fibre $\mathbb{D}_{S}$ over $S$. The following axioms involve only these fibers.

More generally, if $S \in \mathcal{S}(J)$ we may consider the association

$$I \mapsto \mathbb{D}(I \times J)_{p^* S},$$

where $p : I \times J \to J$ is the projection. This defines again a (left, resp. right) derivator in the usual sense which we call its fibre $\mathbb{D}_{S}$ over $S$. Let $p : \mathcal{D} \to \mathcal{S}$ be a (left, resp. right) fibered derivator and $S : \cdot \to \mathcal{S}(\cdot)$ a functor of multicategories. This is equivalent to the choice of an object $S \in \mathcal{S}(\cdot)$ and a collection of morphisms $\alpha_n \in \operatorname{Hom}_{\mathcal{S}(\cdot)}(S, \ldots, S, S)$ for all $n \geq 2$, compatible with composition.

Then the fibre

$$I \mapsto \mathbb{D}(I)_{p^* S},$$

defines even a (left, resp. right) multiderivator (i.e. a fibered multiderivator over ·). The same holds analogously for a functor of multicategories $S : \cdot \to \mathcal{S}(I)$.

It follows from axiom (FDer5 left) and Lemma A.2.5 that a left fibered multiderivator $\mathbb{D} \to \cdot$ is the same as a left monoidal derivator in the sense of \[6\].

**Definition 1.3.10.** We call a pre-derivator $\mathbb{D}$ strong, if the following axiom holds:

(\text{Der8}) For any $K$ the ‘partial underlying diagram’ functor

$$\text{dia} : \mathbb{D}(K \times \Delta_1) \to \text{Hom}(\Delta_1, \mathbb{D}(K))$$

is full and essentially surjective.

**Definition 1.3.11.** Let $p : \mathcal{D} \to \mathcal{S}$ be a fibered (left and right) derivator. We call $\mathcal{D}$ pointed (rel. to $p$) if the following axiom holds:

(\text{FDer6}) For any $S \in \mathcal{S}(\cdot)$, the category $\mathbb{D}(\cdot)_{S}$ has a zero object.

**Definition 1.3.12.** Let $p : \mathcal{D} \to \mathcal{S}$ be a fibered (left and right) derivator. We call $\mathcal{D}$ stable (rel. to $p$) if its fibers are strong and the following axiom holds:

(\text{FDer7}) For any $S \in \mathcal{S}(\cdot)$, in the category $\mathbb{D}(\square)_{p^* S}$ an object is homotopy Cartesian if and only if it is homotopy coCartesian.

This condition can be weakened (cf. \[8\], Corollary 8.13)).

1.3.13. Recall from \[7\] that axiom (FDer7) imply that the fibers of a stable fibered derivator are triangulated categories in a natural way. Actually the proof shows that it suffices to have a derivator of domain Posf (finite posets). Since push-forward, resp. the (relative and absolute) tensor product commute with homotopy colimits (FDer5 left, cf. also \[1.3.19\] below) they induce, in particular, triangulated functors between the fibers.
1.3.14. (left) The following is a consequence of (FDer0): For a morphism \( \alpha : I \to J \) and a morphism in \( f : S \to T \in S(J) \), we get a natural isomorphism

\[
S(\alpha^*f) \cdot \alpha^* \to \alpha^*S(f) .
\]

W.r.t. this natural isomorphism we have the following:

**Lemma 1.3.15 (left).** Given a “pasting” diagram

\[
\begin{array}{ccc}
N & \xrightarrow{G} & L \\
\downarrow{A} & & \downarrow{\alpha} \\
M & \xrightarrow{\gamma} & K \\
\end{array}
\]

we get for the pasted natural transformation \( \nu \circ \mu = \beta \nu \circ \mu G \) that the following diagram is commutative:

\[
\begin{array}{ccc}
A \triangleright S(\beta \nu) \cdot G^*S(\mu) \cdot B^* & \xrightarrow{\sim} & \gamma^* a S(\mu S) \cdot B^* \\
& & \xrightarrow{\gamma^* \beta^* \alpha!} \\
A \triangleright S(\nu \circ \mu) \cdot G^*B^* & \xrightarrow{\sim} & \gamma^* \beta^* \alpha!
\end{array}
\]

Here the morphisms going to the right are (induced by) the various base-change morphisms. In particular, the pasted square is homotopy exact if the individual two squares are.

**Proof.** This is an analogue of [7, Lemma 1.17] and proven similarly. \( \square \)

**Proposition 1.3.16.**

1. Any square of the form

\[
\begin{array}{ccc}
I \times_J K & \xrightarrow{B} & I \\
\downarrow{A} & & \downarrow{\alpha} \\
K & \xrightarrow{\beta} & J
\end{array}
\]

(where \( I \times_J K \) is the slice category) is homotopy exact (in particular the ones from axiom FDer4 left and FDer4 right are).

2. A Cartesian square

\[
\begin{array}{ccc}
I \times_J K & \xrightarrow{B} & I \\
\downarrow{A} & & \downarrow{\alpha} \\
K & \xrightarrow{\beta} & J
\end{array}
\]

(where \( I \times_J K \) is the fibre product) is homotopy exact, if \( \alpha \) is a Grothendieck opfibration or if \( \beta \) is a Grothendieck fibration.

3. If \( \alpha : I \to J \) is a morphism of Grothendieck opfibrations over a diagram \( E \), then

\[
\begin{array}{ccc}
I_e & \xrightarrow{w_j} & I \\
\downarrow{\alpha_e} & & \downarrow{\alpha} \\
J_e & \xrightarrow{w_j} & J
\end{array}
\]

is homotopy exact for all objects \( e \in E \).
4. If a square

\[
\begin{array}{ccc}
L & \xrightarrow{B} & I \\
A & \downarrow & \phi^\alpha \\
K & \xrightarrow{\beta} & J
\end{array}
\]

is homotopy exact then so is

\[
\begin{array}{ccc}
L \times X & \xrightarrow{B} & I \times X \\
A & \downarrow & \phi^\alpha \\
K \times X & \xrightarrow{\beta} & J \times X
\end{array}
\]

for any diagram \(X\).

**Proof.** This proof is completely analogous to the non-fibered case. We sketch the arguments here (for the left-case only, the other case follows by logical duality):

3. Let \(j\) be an object in \(J_e\) and consider the cube:

\[
\begin{array}{ccc}
I_e \times_{/I_e,j} I_e & \xrightarrow{\iota_e} & I_e \\
I \times_{/I,j} I & \xrightarrow{\iota} & I \\
J_e & \xrightarrow{p_e} & I_e \\
J & \xrightarrow{p} & I
\end{array}
\]

By standard arguments on homotopy exact squares it suffices to show that the left square is homotopy exact on constant diagrams, i.e. that

\[p_{e,!}w^* \cong p!
\]

holds true for all usual derivators. By [7, Proposition 1.23] it suffices to show that \(w\) has a left adjoint.

Consider the map

\[
\begin{array}{ccc}
I_e \times_{/I_e,j} I_e & \xrightarrow{w} & I \times_{/I,j} I\end{array}
\]

where \(w\) is the map given by the inclusions \(\iota_{I,e}\) resp. \(\iota_{J,e}\), and where \(c\) is given by mapping \((i, \mu : \alpha(i) \rightarrow j)\) to \((i', \mu' : \alpha(i') \rightarrow j)\) where \(\alpha\) maps coCartesian arrows to coCartesian arrows by assumption, \(\alpha(\xi_{i,\mu}) : \alpha(i) \rightarrow \alpha(i')\) is coCartesian, and therefore there is a unique factorization

\[
\alpha(i) \xrightarrow{\alpha(\xi)} \alpha(i') \xrightarrow{\mu'} j
\]

of \(\mu\). A morphism \(\alpha : (i_1, \mu_1 : \alpha(i_1) \rightarrow j) \rightarrow (i_2, \mu_2 : \alpha(i_2) \rightarrow j)\), by definition of coCartesian, gives rise to a unique morphism \(\alpha' : i'_1 \rightarrow i'_2\) over \(p(i_1) \rightarrow p(i_2)\) such that \(\alpha' \xi_{i_1,\mu_1} = \xi_{i_2,\mu_2} \alpha'\), and we set \(c(\alpha) := \alpha'\). We have \(c \circ w = \text{id}\) and a morphism \(\text{id}_{I \times_{/I,j} I} \rightarrow w \circ c\) given by \((i, \mu) \mapsto \xi_{i,\mu}\). This makes \(w\) right adjoint to \(c\).
2. By axiom (Der2) it suffices to show that for any object \( k \) of \( K \), the induced morphism

\[ k^* A \overset{!}{\to} B \to k^* \beta^* \alpha! \]

is an isomorphism.

Consider the following pasting:

\[
\begin{array}{c}
I \times J \xrightarrow{j} I \times J K \xrightarrow{e} I \times J K \xrightarrow{B} I \\
\downarrow{\pi} \quad \downarrow{p} \quad \downarrow{\phi^\mu} \quad \downarrow{A} \quad \downarrow{\alpha} \\
K \xrightarrow{k} \beta \xrightarrow{J}
\end{array}
\]

Lemma 1.3.15 shows that the following composition

\[ \pi_! S(\beta \mu) \cdot j^* B^* \to \pi_! j^* S(\beta \mu) \cdot \nu^* B^* \to p_! S(\beta \mu) \cdot \nu^* B^* \to k^* A_1 B^* \to k^* \beta^* \alpha! \]

is the base-change associated with the pasting of the 3 squares. All morphisms in this sequence are isomorphisms except possibly for the rightmost one. The second from the left is an isomorphism because \( j \) is a right adjoint \([7, \text{Proposition 1.23}]\). The base-change morphism of the pasting is an isomorphism because of 3.

1. By axiom (Der2) it suffices to show that for any object \( k \) of \( K \), the induced morphism

\[ k^* A \overset{!}{\to} S(\mu) \cdot B^* \to k^* \beta^* \alpha! \]

is an isomorphism. Consider the following pasting:

\[
\begin{array}{c}
I \times J K \xrightarrow{e} I \times J K \xrightarrow{B} I \\
\downarrow{p} \quad \downarrow{A} \quad \downarrow{\alpha} \\
K \xrightarrow{k} \beta \xrightarrow{J}
\end{array}
\]

Lemma 1.3.15 shows that the following diagram is commutative

\[ p_! S(\mu) \cdot \nu^* B^* \xrightarrow{\sim} k^* \beta^* \alpha! \]

where the bottom horizontal morphism is an isomorphism by 2., and the top horizontal morphism is an isomorphism by (FDer4 left). Therefore also the right vertical morphism is an isomorphism.

4. (cf. also \([7, \text{Theorem 1.30}]\)). For any \( x \in X \) consider the cube

\[
\begin{array}{c}
L \xrightarrow{B} I \\
\downarrow{A} \quad \downarrow{\alpha} \\
I \times X \xrightarrow{\phi^\mu} I \times X \xrightarrow{\alpha} J \times X \\
\downarrow{A} \quad \downarrow{\beta} \quad \downarrow{\phi^\mu} \quad \downarrow{\beta} \\
K \xrightarrow{\beta} J \times X \xrightarrow{\alpha} K \times X \xrightarrow{\beta}
\end{array}
\]
The left and right hand side squares are homotopy exact because of 3., whereas the rear one is homotopy exact by assumption. Therefore the pasting

\[
\begin{array}{c}
L \rightarrow I \times X \\
\downarrow A \downarrow \alpha \\
K \rightarrow J \times X
\end{array}
\]

is homotopy exact. Therefore we have an isomorphism

\[
(id,x)^* A_! S(\mu)_* B^* \rightarrow (id,x)^* \beta^* \alpha_!
\]

where the morphism is induced by the base change of the given 2-commutative square. This suffices by axiom (Der2).

1.3.17. (LEFT) If \(S\) is strong the pull-backs and push-forwards along a morphism in \(S(\cdot)\), or more generally along a morphism in \(S(I)\), can be expressed using only the relative Kan-extension functors:

Let \(p : \mathbb{D} \rightarrow S\) be a left fibered derivator such that \(S\) is strong. Consider the 2-commutative square

\[
\begin{array}{c}
I \rightarrow I \\
\downarrow \phi^\mu \downarrow p \\
I \rightarrow I \times \Delta_1
\end{array}
\]

and consider a morphism \(f : S \rightarrow T\) in \(S(I)\). By strongness of \(S\), \(f\) may be lifted to an object \(F \in S(I \times \Delta_1)\), and this means that the morphism

\[
S(\mu)_* : p^* F \rightarrow \iota^* F,
\]

is isomorphic to \(f\). Since the square is homotopy exact by Proposition 1.3.16 1., we get that the natural transformation

\[
f_* \rightarrow \iota^* p_!
\]

is an isomorphism.

1.3.18. (LEFT) Let \(\alpha : I \rightarrow J\) a morphism in \(\text{Dia}\) and let \(f : S \rightarrow T\) be a morphism in \(S(J)\). Axiom (FDer0) of a left fibered derivator implies that we have a canonical isomorphism

\[
(\alpha^*(f))_* \alpha^* = \alpha^* f_*,
\]

determined by the choice of the push-forward functors. We get an associated exchange morphism

\[
\alpha_!(\alpha^*(f))_* \rightarrow f_* \alpha_!
\]  \(6\)

Proposition 1.3.19. If \(p : \mathbb{D} \rightarrow S\) is a left fibered derivator then the natural transformation \(6\) is an isomorphism. The corresponding dual statement holds for a right fibered derivator.

Proof. Consider the 2-commutative squares (the third and fourth are even commutative on the nose):

\[
\begin{array}{c}
\begin{array}{c}
I \rightarrow I \\
\downarrow \phi_1 \downarrow p_1 \\
I \rightarrow I \times \Delta_1
\end{array} & \begin{array}{c}
J \rightarrow J \\
\downarrow \phi^\mu \downarrow p_1 \\
J \rightarrow J \times \Delta_1
\end{array} & \begin{array}{c}
I \rightarrow J \\
\downarrow \iota_1 \downarrow \iota_1 \\
I \times \Delta_1 \rightarrow J \times \Delta_1
\end{array} & \begin{array}{c}
I \rightarrow J \\
\downarrow \alpha \downarrow \alpha \\
I \times \Delta_1 \rightarrow J \times \Delta_1
\end{array}
\end{array}
\]
They are all homotopy exact.
Consider the diagram
\[ \begin{array}{c}
\alpha_! (\alpha^*(f)) \\
\downarrow \\
\alpha_! \iota_!^p p_{I!} \\
\downarrow \\
\alpha_! \iota_!^p \alpha_! \iota_!^p p_{J!} \\
\downarrow \\
f_\bullet \alpha_!
\end{array} \]
where the vertical morphisms come from \([1.3.17]\) — these are the base change morphism for the first and second square above — and the lower horizontal morphisms are the base change for the third diagram above and the natural morphism associated with the commutativity of the fourth diagram above. Repeatedly applying Lemma \([1.3.15]\) shows that this diagram is commutative. Since all the morphisms except possibly the upper horizontal one are isomorphisms also the upper horizontal morphism is an isomorphism.

1.3.20. The last proposition does state nothing else than the fact that push-forward commutes with homotopy colimits (left case) and pull-back commutes with homotopy limits (right case). This also follows from \((\text{FDer}0 \text{ left})\) and \((\text{FDer}0 \text{ right})\) because, in that case, \(f_\bullet\) is a left adjoint. In particular (\(\text{FDer}5 \text{ left}\)) follows from the other axioms if we are considering plain fibered derivators (not multiderivators). Even in the multi-case, (\(\text{FDer}5 \text{ left}\)) follows from \((\text{FDer}0 \text{ left})\) and \((\text{FDer}0 \text{ right})\) together.

1.3.21. (left) Let \(\alpha : I \to J\) be a morphism in \(\text{Dia}\). Proposition \([1.3.19]\) allows us to extend the functor \(\alpha_!\) to a functor
\[ \alpha_! : \mathbb{D}(I) \times_{S(I)} S(J) \to \mathbb{D}(J), \]
which is still left adjoint to \(\alpha^*\) (more precisely: to \((\alpha^*, p(J))\)). Here the fibre product is formed w.r.t. \(p(I)\) and \(\alpha^*\), respectively. We sketch its construction: \(\alpha_!(\mathcal{E}, S)\) is given by \(\alpha^S_! \mathcal{E}\), where \(\alpha^S_!\) is the functor from axiom \((\text{FDer}3 \text{ left})\) with base \(S\). Let a pair of a morphism \(f : S \to T\) in \(S(J)\) and \(F : \mathcal{E} \to \mathcal{F}\) in \(\mathbb{D}(I)\) over \(\alpha^*(f)\) be given. We define \(\alpha_!(F,f)\) as follows: \(F\) corresponds to a morphism
\[ (\alpha^* f)_\bullet \mathcal{E} \to \mathcal{F}. \]
Applying \(\alpha^T_!\) we get a morphism
\[ \alpha^T_! (\alpha^* f)_\bullet \mathcal{E} \to \alpha^T_! \mathcal{F} \]
composition with the inverse of the morphism \([6]\) yields
\[ f_\bullet \alpha^S_! \mathcal{E} \to \alpha^T_! \mathcal{F} \]
or, equivalently, a morphism which we define to be \(\alpha_!(F,f)\)
\[ \alpha^S_! \mathcal{E} \to \alpha^T_! \mathcal{F} \]
over \(f\).
For the adjunction, we have to give a functorial isomorphism
\[ \text{Hom}_{\alpha^* f} (\mathcal{E}, \alpha^* \mathcal{F}) \cong \text{Hom}_f (\alpha_!(\mathcal{E}, S), \mathcal{F}), \]
where $\mathcal{E} \in \mathbb{D}(I)_{\alpha^*S}$ and $\mathcal{F} \in \mathbb{D}(J)_T$. We define it to be the succession

$$\text{Hom}_{\alpha^*f}(\mathcal{E}, \alpha^*\mathcal{F})$$

$$\cong \text{Hom}_{\text{id}_T}(\alpha(\alpha^*f)_\bullet\mathcal{E}, \mathcal{F})$$

$$\cong \text{Hom}_{\text{id}_T}(f_\bullet\alpha\mathcal{E}, \mathcal{F})$$

$$\cong \text{Hom}_{f}(\alpha\mathcal{E}, \mathcal{F}).$$

A dual statement holds for a right fibered derivator and the functor $\alpha_*$. From Proposition 1.3.19 we also get a vertical version of Lemma 1.3.15:

**Lemma 1.3.22** (left). Given a “pasting” diagram

```
N \arrow{r}{B} \downarrow{\Gamma} \downarrow{\gamma} \downarrow{b} & M \downarrow{\alpha} \downarrow{\beta} \downarrow{a} \\
\L \arrow{r}{\phi} \downarrow{\phi^\mu}\downarrow{\gamma} & \I \downarrow{\mu} \downarrow{\alpha} \downarrow{a} \\
\K \arrow{r}{\beta} \downarrow{\alpha} & \J \downarrow{\beta} \downarrow{\alpha} \downarrow{a}
```

we get for the pasted natural transformation $\mu \circ \nu := \mu \Gamma \circ \alpha \nu$ that the following diagram is commutative:

```
a_1S(\mu)_\bullet \Gamma S(\alpha \nu)_\bullet B^* \to a_1S(\mu)_\bullet b^* \gamma_\bullet \to \beta^* \alpha_\Gamma \gamma_\bullet
```

Here the morphisms going to the right are (induced by) the various base-change morphisms and the upper horizontal morphism is the isomorphism from Proposition 1.3.19. In particular, the pasted square is homotopy exact if the individual two are.

### 1.4 Transitivity

**Proposition 1.4.1.** Let

$$E \xrightarrow{p_1} D \xrightarrow{p_2} S$$

be two left (resp. right) fibered multiderivators. Then also the composition $p_3 = p_2 \circ p_1 : E \to S$ is a left (resp. right) fibered multiderivator.

**Proof.** We will show the statement for left fibered multiderivators. The other statement follows by logical duality.

Axiom (FDer0): For any $I \in \text{Dia}$, we have a sequence

$$E(I) \to D(I) \to S(I)$$
of fibered multicategories. It is well-known that then also the composition $\mathbb{E}(I) \to \mathbb{S}(I)$ is a fibered multicategory (see \ref{A.2}). The other statement of (FDer0) is immediate, too. Let $\alpha : I \to J$ be a functor like in axioms (FDer3 left) and (FDer4 left). We denote the relative homotopy Kan-extension functors w.r.t. the 2 fibered derivators by $\alpha_1^3$, and $\alpha_2^3$, respectively. As always, the base will be understood from the context or explicitly given as extra argument as in \ref{1.3.21}.

Axiom (FDer3 left): Let $S \in \mathbb{S}(J)$ be given. We define a functor

$$\alpha_3^3 : \mathbb{E}(I)_{\alpha^* S} \to \mathbb{E}(J)_S$$

in the fibre (under $p_2$) of $\mathcal{E} \in \mathbb{E}(I)_{\alpha^* S}$ as the composition

$$\mathbb{E}(I)_{\alpha^* S} \xrightarrow{(\nu^* \alpha_1^3 p_1)} \mathbb{E}(I)_{\alpha^* S} \times \mathbb{E}(J)_S \xrightarrow{\alpha_1^3} \mathbb{E}(J)_S$$

where $\nu$ is the unit

$$\nu : \mathcal{E} \to \alpha^* \alpha_1^3 \mathcal{E}$$

and $\alpha_1^3$ with two arguments is the extension given in \ref{1.3.21}.

Let $\mathcal{F}_1 \in \mathbb{E}(I)_{\alpha^* S}$ and $\mathcal{F}_2 \in \mathbb{E}(J)_S$ be given with images $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively under $p_1$. The adjunction is given by the succession of isomorphisms:

\begin{align*}
\text{Hom}_S(\alpha_3^3 \mathcal{F}_1, \mathcal{F}_2) &= \text{Hom}_S(\alpha_1^3 (\nu^* \mathcal{F}_1, \alpha_1^3 \mathcal{E}_1), \mathcal{F}_2) \quad \text{Definition} \\
&= \{ f \in \text{Hom}_S(\alpha_1^3 \mathcal{E}_1, \mathcal{E}_2) ; \xi \in \text{Hom}_F(\alpha_3^3 (\nu^* \mathcal{F}_1, \alpha_1^3 \mathcal{E}_1), \mathcal{F}_2) \} \quad \text{Definition} \\
&\cong \{ f \in \text{Hom}_S(\alpha_1^3 \mathcal{E}_1, \mathcal{E}_2) ; \xi \in \text{Hom}_{\alpha^* F}(\nu^* \mathcal{F}_1, \alpha^* \mathcal{F}_2) \} \quad \text{Adjunction} \ref{1.3.21} \\
&= \{ f \in \text{Hom}_{\alpha^* S}(\mathcal{E}_1, \alpha^* \mathcal{E}_2) ; \xi \in \text{Hom}_F(\mathcal{F}_1, \alpha^* \mathcal{F}_2) \} \quad \text{Note below} \\
&= \text{Hom}_{\alpha^* S}(\mathcal{F}_1, \alpha^* \mathcal{F}_2) \quad \text{Definition}
\end{align*}

Note that the composition

$$\tilde{f} : \mathcal{E}_1 \xrightarrow{\nu} \alpha^* \alpha_1^3 \mathcal{E}_1 \xrightarrow{\alpha^* f} \alpha^* \mathcal{E}_2$$

determined by $f$ via the adjunction of (FDer3) (base $S$) for $p_2 : \mathbb{D} \to \mathbb{S}$.

Axiom (FDer4 left): Let $\mathcal{E}$ be in $\mathbb{E}(I)_{\alpha^* S}$ and let $\mathcal{F}$ be its image under $p_1$. We have to show that the natural morphism

$$\alpha_3^3 \mathbb{S}(\mu)_{\alpha_2^3 \mathcal{E}} \to j^* \alpha_3^3$$

is an isomorphism. Inserting the definition of the push-forwards, resp. Kan extensions for $p_3$, we get

$$\alpha_1^3 (\nu_j)_1 \text{cart} \nu^* \mathcal{E} \to j^* \alpha_1^3 \nu^* \mathcal{E}.$$ 

Here $\nu_j : \mathbb{S}(\mu)_{\alpha_2^3 \mathcal{F}} \to \alpha_j^* \mathbb{S}(\mu)_{\alpha_2^3 \mathcal{F}}$ is the unit and $\nu : \mathcal{F} \to \alpha^* \alpha_1^3 \mathcal{F}$ is the unit. ‘cart’ is the Cartesian morphism $\iota^* \mathcal{F} \to \mathbb{S}(\mu)_{\alpha_2^3 \mathcal{F}}$. Consider the the base-change isomorphism (FDer4 for $p_2$

$$\text{bc} : \alpha_3^3 \mathbb{S}(\mu)_{\alpha_2^3 \mathcal{F}} \to j^* \alpha_3^3 \mathcal{F}$$

Furthermore, we have the morphism

$$\mathbb{D}(\mu) : \iota^* \alpha^* \alpha_1^3 \mathcal{F} \to \alpha_j^* j^* \alpha_1^3 \mathcal{F}$$

**Claim:** We have the equality

$$(\alpha_j^* \text{bc}) \circ \nu_j \circ \text{cart} = \mathbb{D}(\mu) \circ \iota^* (\nu)$$

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Proof: Consider the diagram (which affects only the fibered derivator $p_2 : \mathbb{D} \to \mathbb{S}$, hence we omit superscripts):

Clearly all squares, resp. triangles are commutative in this diagram. The two given morphisms are the compositions of the extremal paths hence they are equal.

We have a natural isomorphism induced by $bc$:

$$\alpha_{j!}^1(\ldots, \alpha_{j!}^2 \mathbb{S}(\mu)^i \mathbb{F}) \cong \alpha_{j!}^1((\alpha_{j!}^1 bc)(\ldots), j^* \alpha_{j!}^2 \mathbb{F})$$

(this is true for any isomorphism).

We therefore have

$$\alpha_{j!}^1(\nu_j)^1 \mathbb{F} \cong \alpha_{j!}^1((\alpha_{j!}^1 bc)(\nu_j)^1 \mathbb{F})$$

$$\cong \alpha_{j!}^1 \mathbb{D}(\mu)^1(\mathbb{F} \cong \alpha_{j!}^1 \mathbb{D}(\mu)^1 \nu_j \mathbb{E})$$

Thus we are left to show that

$$\alpha_{j!}^1 \mathbb{D}(\mu)^1 \mathbb{F} \cong j^* \alpha_{j!}^1 \mathbb{F}$$

is an isomorphism. A tedious check shows that this is the base change morphism associated with $p_1$. It is an isomorphism by (FDer4 left) for $p_1$.

\[\square\]

1.5 (Co)Local morphisms

1.5.1. Let $\operatorname{Dia}$ be a diagram category and let $\mathbb{S}$ be a strong right derivator with domain $\operatorname{Dia}$. Strongness implies that for each diagram

\[
\begin{array}{ccc}
U & \downarrow & \mathbb{S} \\
\downarrow & & \downarrow \\
S & \rightarrow & \mathbb{T}
\end{array}
\]

in $\mathbb{S}(\cdot)$ there exists a homotopy pull-back "$U \times_T S$" which is well-defined up to (non-unique!) isomorphism. A Grothendieck pre-topology on $\mathbb{S}$ is basically a Grothendieck pre-topology in the usual sense on $\mathbb{S}(\cdot)$ except that pull-backs are replaced by homotopy pull-backs. We state the precise definition:

Definition 1.5.2. A Grothendieck pre-topology on $\mathbb{S}$ is the datum for any $S \in \mathbb{S}(\cdot)$ a collection of families $\{U_i \to S\}_{i \in \mathbb{S}}$ of morphisms in $\mathbb{S}(\cdot)$ called covers, such that
1. Every family consisting of isomorphisms is a cover,

2. If \( \{ U_i \to S \}_{i \in I} \) is a cover and \( T \to S \) is any morphism then the family \( \{ U_i \times_S T \to T \}_{i \in I} \) is a cover for any choice of particular members of the family \( \{ U_i \times_S T \} \).

3. If \( \{ U_i \to S \}_{i \in I} \) is a cover and for each \( i \), \( \{ U_{i,j} \to U_i \}_{j \in J_i} \) is a cover then the family of compositions \( \{ U_{i,j} \to U_i \to S \}_{i \in I, j \in J_i} \) is a cover.

**Definition 1.5.3** (left). Let \( p : \mathbb{D} \to \mathbb{S} \) be a left fibered derivator satisfying also \( (\text{FDer}0 \text{ right}) \). Assume that pull-backs exist in \( \mathbb{S} \). We call a morphism \( f : U \to X \) in \( \mathbb{S}(\cdot) \) \( \mathbb{D} \)-local if

\( (\text{Dloc1 left}) \) The morphism \( f \) satisfies base change: for any diagram \( Q \in \mathbb{S}(\square) \) with underlying diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{G} & & \downarrow{\tilde{g}} \\
C & \xrightarrow{f} & D
\end{array}
\]

such that \( p(Q) \) in \( \mathbb{S}(\square) \) is a pullback-diagram, i.e. is (homotopy) Cartesian, if \( \tilde{F} \) and \( \tilde{f} \) are Cartesian, and \( \tilde{g} \) is coCartesian then also \( \tilde{G} \) is coCartesian.\(^5\)

\( (\text{Dloc2 left}) \) The morphism of derivators

\[ f^* : \mathbb{D}_X \to \mathbb{D}_U \]

commutes with homotopy colimits.

A morphism \( f : U \to X \) in \( \mathbb{S}(\cdot) \) is called universally \( \mathbb{D} \)-local if any homotopy pull-back of \( f \) is \( \mathbb{D} \)-local.

**Definition 1.5.4** (left). Assume that \( \mathbb{S} \) is equipped with a Grothendieck pre-topology \( \text{[1.5.2]} \). A left fibered derivator \( p : \mathbb{D} \to \mathbb{S} \) as in Definition \( \text{[1.5.3]} \) is called local w.r.t. the pre-topology on \( \mathbb{S} \), if

1. Every morphism \( U_i \to S \) which is part of a covering family is \( \mathbb{D} \)-local.

2. For a covering \( \{ f_i : U_i \to S \} \) the family

\[ f_i^* : \mathbb{D}(S) \to \mathbb{D}(U_i) \]

is jointly conservative.

**Definition 1.5.5** (right). Let \( p : \mathbb{D} \to \mathbb{S} \) be a right fibered derivator satisfying also \( (\text{FDer}0 \text{ left}) \). Assume that push-outs exist in \( \mathbb{S} \). We call a morphism \( f : X \to U \) in \( \mathbb{S}(\cdot) \) \( \mathbb{D} \)-colocal if

\( ^5 \)In other words, if

\[
\begin{array}{ccc}
\text{"}U \times_X Y\text{"} & \xrightarrow{F} & Y \\
\downarrow{G} & & \downarrow{\theta} \\
U & \xrightarrow{f} & X
\end{array}
\]

is the underlying diagram of \( p(Q) \) then the exchange morphism

\[ G_* F^* \to f^* \theta_* \]

is an isomorphism.
The morphism $f$ satisfies **base change**: for any diagram $Q \in \mathbb{D}(\square)$ with underlying diagram:

\[
\begin{array}{ccc}
A & \xleftarrow{F} & B \\
\downarrow{\bar{G}} & & \downarrow{\bar{g}} \\
C & \xleftarrow{\bar{f}} & D
\end{array}
\]

such that $p(Q)$ in $\mathbb{S}(\square)$ is a pushout-diagram, i.e. is (homotopy) coCartesian, if $\bar{F}$ and $\bar{f}$ are coCartesian, and $\bar{g}$ is Cartesian then also $\bar{G}$ is Cartesian.

The morphism of derivators

\[f_* : \mathbb{D}_X \to \mathbb{D}_U\]

commutes with homotopy limits.

A morphism $f : X \to U$ in $\mathbb{S}(\cdot)$ is called universally $\mathbb{D}$-colocal if any homotopy push-out of $f$ is $\mathbb{D}$-colocal.

**Definition 1.5.6** (right). Assume that $\mathbb{S}^{op}$ is equipped with a Grothendieck pre-topology [1.5.2]. A right fibered derivator $p : \mathbb{D} \to \mathbb{S}$ as in Definition [1.5.5] is called colocal w.r.t. the pre-cotopology on $\mathbb{S}$, if

1. Every morphism $S \to U_i$ which is part of a cocovering family is $\mathbb{D}$-colocal.
2. For a cocovering $\{f_i : S \to U_i\}$ the family

\[(f_i)_* : \mathbb{D}(\cdot)S \to \mathbb{D}(\cdot)U_i\]

is jointly conservative.

### 1.6 The associated pseudo-functor

Let $p : \mathbb{D} \to \mathbb{S}$ be a morphism of pre-derivators with domain Dia.

**1.6.1. (Left)** Let $\text{Dia}(\mathbb{S})$ the 2-category of diagrams over $\mathbb{S}$, where objects are pairs of $I \in \text{Dia}$ and $F \in \mathbb{S}(I)$, morphisms $(I,F) \to (J,G)$ are pairs of $\alpha : I \to J, f : F \to \alpha^*G$ and 2-morphisms $(\alpha,f) \to (\beta,g)$ are the natural transformations $\mu : \alpha \to \beta$ such that $\mathbb{S}(\mu)(G) \circ f = g$.

We call a morphism $(\alpha,f)$ of **fixed shape** if $\alpha = \text{id}$, and of **diagram type** if $f$ consists of identities. Every morphism is obviously a composition of one of diagram type by one of fixed shape.

**1.6.2. (Right)** There is a dual notion of a 2-category $\text{Dia}^{op}(\mathbb{S})$. Explicitly, objects are pairs of $I \in \text{Dia}$ and $F \in \mathbb{S}(I)$, morphisms $(I,F) \to (J,G)$ are pairs of $\alpha : I \to J, f : \alpha^*G \to F$ and 2-morphisms $(\alpha,f) \to (\beta,g)$ are the natural transformations $\mu : \alpha \to \beta$ such that $f \circ \mathbb{S}(\mu)(G) = g$.

The association $(I,F) \to (I^{op},F^{op})$ induces an isomorphism $\text{Dia}^{op}(\mathbb{S}) \to \text{Dia}(\mathbb{S}^{op})^{2-op}$.

We are interested in associating to a fibered derivator a pseudo-functor like for classical fibered categories.

**1.6.3. (Left)** We associate to a morphism of pre-derivators $p : \mathbb{D} \to \mathbb{S}$ which satisfies (FDer0 right) a (contravariant) 2-pseudo-functor

\[\mathbb{D} : \text{Dia}(\mathbb{S})^{1-op} \to \mathcal{CAT}\]
mapping \((I, F)\) to \(\mathbb{D}(I)_F\), a morphism \((\alpha, f) : (I, F) \to (J, G)\) to \(f^* \circ \alpha^* : \mathbb{D}(J)_G \to \mathbb{D}(I)_F\). A natural transformation given by \(\mu : \alpha \to \beta\) is mapped to the natural transformation pasted from the following two 2-commutative triangles:

\[
\begin{array}{ccc}
\mathbb{D}(I)_G & \mathbb{D}(I)_F \\
\downarrow f^* & \downarrow \alpha^* \\
\mathbb{D}(J)_G & \mathbb{D}(I)_G \beta^* & \mathbb{D}(I)_F \end{array}
\]

Proof of the pseudo-functor property. For a composition \((\beta, g) \circ (\alpha, f) = (\beta \circ \alpha, \alpha^* (g) \circ f)\) we have: \(f^* \circ \alpha^* \circ g^* \circ \beta^* = f^* \circ (\alpha^* g)^* \circ \alpha^* \circ \beta^*\). This follows from the isomorphism (axiom FDer0): \(\alpha^* \circ g^* \cong (\alpha^* g)^* \circ \alpha^*\). One checks that this indeed yields a pseudo-functor. \(\square\)

1.6.4. (Right) We associate to a morphism of pre-derivators \(p : \mathbb{D} \to \mathbb{S}\) which satisfies (FDer0 left) a (contravariant) 2-pseudo-functor

\[
\mathbb{D} : \text{Dia}^{op}(\mathbb{S})^{1-\text{op}} \to \text{CAT}
\]

mapping \((I, F)\) to \(\mathbb{D}(I)_{F(I)}\), a morphism \((\alpha, f) : (I, F) \to (J, G)\) to \(f^* \circ \alpha^*\) from \(\mathbb{D}(J)_G \to \mathbb{D}(I)_F\). This defines a functor by the same reason as before.

1.6.5. (Left) We assume that \(\mathbb{S}\) is a strong right derivator. There is a notion of “comma object” in Dia(\(\mathbb{S}\)) which we describe here for the case that \(\mathbb{S}\) is the pre-derivator associated with a category \(\mathbb{S}\) and leave it to the reader to formulate the derivator version. In that case the corresponding object will be determined up to non-unique isomorphism only.

Given three diagrams \(D_1 = (I_1, F_1), D_2 = (I_2, F_2), D_3 = (I_3, F_3)\) in Dia(\(\mathbb{S}\)) and morphisms \(\beta_1 : D_1 \to D_3, \beta_2 : D_2 \to D_3\), we form the comma diagram \(D_1 \times_{D_3} D_2\) as follows: The underlying diagram \(I_1 \times_{I_3} I_2\) consists of objects \(i_1 \in I_1, i_2 \in I_2, \alpha_1(i_1) \to \alpha_2(i_2)\) in \(I_3\). A morphism is a pair \(\beta_1 : i_i \to i_i'\) such that

\[
\begin{array}{ccc}
\alpha_1(i_1) & \alpha_1(i_1') \\
\downarrow \mu & \\
\alpha_2(i_2) & \alpha_2(i_2') \\
\end{array}
\]

commutes in \(I_3\).

The corresponding functor \(\mathcal{F} \in \mathbb{S}(I_1 \times_{I_3} I_2)\) maps a tripel like this to

\[
F_1(i_1) \times_{F_3(\alpha_2(i_2))} F_2(i_2).
\]

We define \(P_j\) to be \((i_j, p_j)\) for \(j = 1, 2\), where \(i_j\) maps a tripel \((i_1, i_2, \mu)\) to \(i_j\), and \(p_j\) is the corresponding projection of the fibre product.

We then get a 2-cell

\[
\begin{array}{ccc}
D_1 \times_{D_3} D_2 & \rightarrow & D_1 \\
\downarrow P_1 & \downarrow \phi & \downarrow \beta_1 \\
D_2 & \rightarrow & D_3
\end{array}
\]
If we are given $I_2, I_3$ only and two maps $I_1 \to I_3$ and $I_2 \to I_3$ we also form $D_1 \times_{I_3} I_2$ by the same underlying category, with functor $F_1 \circ \iota_1$.

1.6.6. (right) We assume that $S$ is a strong left derivator. There is a dual notion of “comma object” in $\text{Dia}^{\text{op}}(S)$ which we describe here again for the case that $S$ is the pre-derivator associated with a category $S$ and leave it to the reader to formulate the derivator version. In that case the corresponding object will be determined up to (non-unique!) isomorphism only.

Given three diagrams $D_0 = (I_1, F_1), D_2 = (I_2, F_2)$ in $\text{Dia}^{\text{op}}(S)$ mapping to $D_3 = (I_3, F_3)$, we form the comma diagram $D_0 \times_{D_3} D_2$ as follows: The underlying diagram is $I_1 \times_{I_3} I_2$ consisting of objects $i_1 \in I_1, i_2 \in I_2, \mu : \alpha_1(i_1) \to \alpha_2(i_2)$ in $I_3$. A morphism is a pair $\beta : i_i \to i'_i$ such that

$$\begin{array}{ccc}
\alpha_1(i_1) & \xrightarrow{\alpha_1(\beta_1)} & \alpha_1(i'_1) \\
\downarrow & & \downarrow \\
\alpha_2(i_2) & \xrightarrow{\alpha_2(\beta_2)} & \alpha_2(i'_2)
\end{array}$$

commutes in $I_3$.

The corresponding functor $\tilde{F}$ maps a triple like this to

$$F_1(i_1) \cup_{F_3(\alpha_1(i_1))} F_2(i_2)$$

We then get a 2-cell

$$D_0 \times_{D_3} D_2 \xrightarrow{\phi} D_0 \xrightarrow{\mu} D_2 \xrightarrow{\mu'} D_3$$

This language allows us to restate Lemma 1.3.15 and Lemma 1.3.22 in a more convenient way:

**Lemma 1.6.7 (left).** 1. Given a “pasting” diagram in $\text{Dia}(S)$:

$$\begin{array}{ccc}
D_1 & \xrightarrow{\Gamma} & D_3 \\
\downarrow & & \downarrow \\
D_2 & \xrightarrow{\gamma} & D_4 \\
\downarrow & & \downarrow \\
D_5 & \xrightarrow{\beta} & D_6
\end{array}$$

we get for the pasted natural transformation $\nu \otimes \mu := \beta \nu \circ \mu \Gamma$ that

$$(\nu \otimes \mu)_1,$$

2. Given a “pasting” diagram in $\text{Dia}(S)$:

$$\begin{array}{ccc}
D_1 & \xrightarrow{B} & D_2 \\
\downarrow & & \downarrow \\
D_3 & \xrightarrow{b} & D_4 \\
\downarrow & & \downarrow \\
D_5 & \xrightarrow{\beta} & D_6
\end{array}$$

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we get for the pasted natural transformation $\nu \odot \mu = \alpha \nu \circ \mu \Gamma$ that

$$\mu \circ \nu = (\nu \circ \mu) !.$$  

**Definition 1.6.8.** If $\mathbb{S}$ is equipped with a Grothendieck pre-topology \([1.5.2]\) then we call $(\alpha, f) : (I, F) \to (J, G)$ $\mathbb{D}$-**local**, if $f_i : F(i) \to G \circ \alpha(i)$ is $\mathbb{D}$-local \([1.5.3]\) for all $i \in I$. Likewise for the notions universally $\mathbb{D}$-local, $\mathbb{D}$-colocal, universally $\mathbb{D}$-colocal.

**Theorem 1.6.9** (left). Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator satisfying also (FDer0 right) and such that $\mathbb{S}$ is a strong right derivator. Then the associated pseudo-functor satisfies the following properties:

1. For a morphism of diagrams $(\alpha, f) : D_1 \to D_2$ the corresponding pullback

$$(\alpha, f)^* : \mathbb{D}(D_2) \to \mathbb{D}(D_1)$$

has a left-adjoint $(\alpha, f) !$.

2. For a diagram like in \([1.6.5]\)

$$D_1 \times_{/D_3} D_2 \xrightarrow{P_1} D_1 \\
D_2 \xrightarrow{\beta_2} D_3$$

the corresponding exchange morphism

$$P_2 ! P_1^* \to \beta_2 \beta_1 !$$

is an isomorphism in $\mathbb{D}(D_2)$ provided $\beta_2$ is $\mathbb{D}$-local.

**Proof.** 1. By (FDer0 left) and (FDer3 left) we can form $(\alpha, f) ! := \alpha_1 \circ f_\bullet$ which is clearly left adjoint to $(\alpha, f)^*$.  

2. We first reduce to the case where $I_2$ is the trivial category. For look at the diagram:

$$D_1 \times_{/D_3} (\{j_2\}, F_2(j_2)) \xrightarrow{\text{can}} D_1 \times_{/D_3} D_2 \times_{/D_3} (\{j_2\}, F_2(j_2)) \xrightarrow{\phi} D_1 \times_{/D_3} D_2 \xrightarrow{P_1} D_1 \\
(\{j_2\}, F_2(j_2)) \xrightarrow{\phi} (\{j_2\}, F_2(j_2)) \xrightarrow{P_2} D_2 \xrightarrow{\beta_2} D_3$$

The exchange morphism of the middle square and outmost rectangle are isomorphisms by the reduced case. The morphism $\text{can.}$ of the left hand square is of diagram type and its underlying diagram functor has an adjoint. The exchange morphism is therefore an isomorphism by \([7, 1.23]\). Using Lemma \([1.6.7]\) therefore, applying this for all $j_2 \in I_2$, also the exchange morphism of the right square has to be an isomorphism (this uses axiom Der2).

Now we may assume $D_2 = (\{j_2\}, F_2(j_2))$. Consider the following diagram, in which we denote $\beta_1 = (\alpha_1, f_1)$, $\beta_2 = (\alpha_2, f_2)$.

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\[
(I_1 \times_{I_3} \{i_2\}, \tilde{F}) \xrightarrow{p_1} (I_1 \times_{I_3} \{i_2\}, F_1 \circ i_1) \xrightarrow{i_1} (I_1, F_1) \\
\downarrow p_2 \quad \phi \downarrow \quad \phi \quad \downarrow f_1 \\
(I_1 \times_{I_3} \{i_2\}, \tilde{F}') \xrightarrow{p_1'} (I_1 \times_{I_3} \{i_2\}, F_3 \circ \alpha_1 \circ i_1) \xrightarrow{i_1} (I_1, F_3 \circ \alpha_1) \\
\downarrow p_2' \quad \phi \downarrow \quad \phi \quad \downarrow \alpha_1 \\
(I_1 \times_{I_3} \{i_2\}, F_2(i_2)) \xrightarrow{i_2} (I_1 \times_{I_3} \{i_2\}, F_3(\alpha_2(i_2))) \xrightarrow{\alpha_1} (I_3, F_3) \\
\downarrow \quad \phi \downarrow \quad \phi \quad \downarrow \\
(\{i_2\}, F_2(i_2)) \xrightarrow{f_2} (\{i_2\}, F_3(\alpha_2(i_2))) \xrightarrow{\alpha_2} (I_3, F_3)
\]

where \(\tilde{F}\) is the functor defined in page 1.6.5 mapping a tripl \((i_1, i_2, \mu: \alpha_1(i_1) \rightarrow \alpha_2(i_2))\) to

\[F_1(i_1) \times_{F_3(\alpha_2(i_2))} F_2(i_2)\]

and \(\tilde{F}'\) is the functor mapping a tripl \((i_1, i_2, \mu: \alpha_1(i_1) \rightarrow \alpha_2(i_2))\) to

\[F_3(\alpha_1(i_1)) \times_{F_3(\alpha_2(i_2))} F_2(i_2)\]

We have to show that the exchange morphism for the outer square is an isomorphism. Using Lemma 1.6.7 it suffices to show this for the squares 1–5. The exchange morphism for the squares 1 and 2, where the morphisms are of fixed shape, being an isomorphism can be checked point-wise by (Der2). Then it boils down to the base change condition (Dloc1 left). Note that the squares are pull-back squares in \(S\) by construction of \(\tilde{F}'\) resp \(\tilde{F}\). The exchange morphism for 4 is an isomorphism because of (FDer0). The exchange morphism for 3 is an isomorphism because of (Dloc2 left). The exchange morphism for 5 is an isomorphism because of (FDer4 left).

Dualizing, there is a corresponding right variant of the theorem, which uses \(\text{Dia}^{\text{op}}(\mathcal{S})\) instead. We leave its formulation to the reader.

## 2 (Co)homological descent

### 2.1 Fundamental (co)localizers

**Definition 2.1.1.** A class of morphisms \(\mathcal{W}\) in a category is called weakly saturated, if it satisfies the following properties:

\((WS1)\) Identities are in \(\mathcal{W}\).

\((WS2)\) \(\mathcal{W}\) has the 2-out-of-3 property.

\((WS3)\) If \(p: Y \rightarrow X\) and \(s: X \rightarrow Y\) are morphisms such that \(p \circ s = \text{id}_X\) and \(s \circ p \in \mathcal{W}\) then \(p \in \mathcal{W}\) (and hence \(s \in \mathcal{W}\) by (WS2)).

**Definition 2.1.2.** Let \(\mathcal{S}\) be a strong right derivator. Assume we are given a Grothendieck pre-topology on \(\mathcal{S}\) (1.5.2).

A class \(\mathcal{W}\) of morphisms in \(\text{Dia}(\mathcal{S})\) is called a fundamental localizer if the following properties are satisfied:
(L1) $\mathcal{W}$ is weakly saturated.

(L2) If $D = (I,F) \in \text{Dia}(\mathcal{S})$, and $I$ has a final object $e$, then the projection $D \to (\{e\}, F(e))$ is in $\mathcal{W}$.

(L3) If there is a commutative diagram in $\text{Dia}$

$$
\begin{array}{ccc}
I_1 & \xrightarrow{\alpha} & I_2 \\
\downarrow & & \downarrow \\
I_3 & & \\
\end{array}
$$

and $w : D_1 := (I_1, F_1) \to D_2 := (I_2, F_2)$ is an extension of the horizontal morphism, and

$w_i : D_1 \times_{I_3} i \to D_2 \times_{I_3} i$

is in $\mathcal{W}$ for all $i \in I_3$, then $w \in \mathcal{W}$.

(L4) If there is a commutative diagram of diagrams

$$
\begin{array}{ccc}
D_1 & \xrightarrow{w} & D_2 \\
\downarrow & & \downarrow \\
(\cdot, S) & & \\
\end{array}
$$

and $\{U_j \to S\}_j$ is a cover and the corresponding morphism

$w_j : D_1 \times_S U_j \to D_2 \times_S U_j$

is in $\mathcal{W}$ for all $j$, then $w \in \mathcal{W}$.

(L5) Let $D = (E, F) \in \text{Dia}(\mathcal{S})$ and let $p : I \to E$ be a Grothendieck fibration in $\text{Dia}$ such that $I_e \to \cdot$ is in $\mathcal{W}_{\text{Dia}}^\text{min}$ for all $e \in E$. Then the morphism $(I, p^* F) \to D$ is in $\mathcal{W}$. Here $\mathcal{W}_{\text{Dia}}^\text{min}$ is the smallest fundamental localizer in the classical sense of $\text{Dia}$-diagrams (cf. Remark 2.1.4 below).

2.1.3. There is variant of this notion, where $\text{Dia}(\mathcal{S})$ is replaced by a full subcategory $\text{Dia}'(\mathcal{S}) \subset \text{Dia}(\mathcal{S})$ with the properties:

1. $(I, p^*_\cdot S) \in \text{Dia}'(\mathcal{S})$ for all $S \in \mathcal{S}(\cdot)$ and $I \in \text{Dia}$.
2. If $D \in \text{Dia}'(\mathcal{S})$ then also all $D \times_{/E} e$ occurring in (L3 left) are in $\text{Dia}'(\mathcal{S})$.
3. All $D \times_S U$ occurring in (L4 left) are in $\text{Dia}'(\mathcal{S})$.

Remark 2.1.4. If $\mathcal{S}$ is the trivial derivator, (L1–L3 left) are precisely the definition of fundamental localizer of Grothendieck. There is a dual notion of fundamental colocalizer in $\text{Dia'}(\mathcal{S})$, which assumes that on $\mathcal{S}$ a Grothendieck pre-cotopology \([1.5.2]\) has been chosen. If $\mathcal{S} = \cdot$ and $\text{Dia} = \text{Cat}$, Grothendieck has shown (cf. \([16]\)) that (L1–L3 left) are equivalent to (L1–L3 right). In that case (L5 left) is basically a special case of (L3 right). In particular there is no difference between localizer and colocalizer.

The class of localizers is obviously closed under intersection, hence there is a smallest localizer $\mathcal{W}_{\text{Dia}(\mathcal{S})}^\text{min}$. If $\mathcal{S}$ is the trivial derivator and $\text{Dia} = \text{Cat}$, Cisinski \([4]\) has shown that $\mathcal{W}_{\text{Cat}}^\text{min}$ is precisely the set $\mathcal{W}_\infty$ of functors $\alpha : I \to J$ such that $N(\alpha)$ is a weak equivalence in the classical sense (of simplicial sets, resp. topological spaces). For a fundamental localizer in the sense of Definition 2.1.2 this implies the following:
Theorem 2.1.5. If $\text{Dia} = \text{Cat}$, for a functor $\alpha \in \mathcal{W}_\infty$, i.e. $\alpha : I \to J$ such that $N(\alpha)$ is a weak equivalence of topological spaces, the morphism $(\alpha, \text{id}) : (I, p_i^*S) \to (J, p_j^*S)$ is in $\mathcal{W}$ for all $S \in \mathcal{S}(\cdot)$.

Proof. The class of functors $\alpha : I \to J$ in $\text{Cat}$ such that $(\alpha, \text{id}) : (I, p_i^*S) \to (J, p_j^*S)$ is in $\mathcal{W}$ obviously form a fundamental localizer in the classical sense. \hfill $\square$

Remark 2.1.6. Let $\text{Dia}$ be a self-dual diagram category. Since the 2-morphisms play no role in the definition of fundamental localizer we get that for a fundamental localizer $\mathcal{W}$ in $\text{Dia}(\mathcal{S})$ the corresponding class $\mathcal{W}^{\text{op}}$ is a fundamental colocalizer in $\text{Dia}^{\text{op}}(\mathcal{S}^{\text{op}})$ and vice-versa. In particular the smallest fundamental localizer in $\text{Dia}(\mathcal{S})$ and smallest fundamental colocalizer in $\text{Dia}^{\text{op}}(\mathcal{S}^{\text{op}})$ arise from each other in this way.

2.1.7. We will for (notational) simplicity\footnote{Otherwise coproducts have always to be taken in the small-category argument.} assume that

1. $\mathcal{S}$ has all relative finite coproducts.

2. For all finite families $(S_i)_{i \in I}$ of object in $\mathcal{S}(\cdot)$ the collection $\{S_i \to \coprod_{j \in I} S_j\}_{i \in I}$ is a cover.

Let $\emptyset$ be the initial object of $\mathcal{S}$ (which exists by 1.). Then the map

$$\emptyset \to (\cdot, \emptyset),$$

where $\emptyset$ on the left denotes the empty diagram, is in $\mathcal{W}$ by (L4 left) applied to the empty cover. From this and (L4 left) again follows that for a finite collection $(S_i)_{i \in I}$ of objects of $\mathcal{S}(\cdot)$ the map

$$(I, (S_i)_{i \in I}) \to (\cdot, \coprod_{i \in I} S_i)$$

is in $\mathcal{W}$. More generally, if we have a Grothendieck opfibration with finite discrete fibers $p : O \to I$ and a diagram $F \in \mathcal{S}(O)$, the morphism

$$(O, F) \to (I, p_!F)$$

is in $\mathcal{W}$, because $\mathcal{S}$ has relative finite coproducts (i.e. if the above functors $p_!$ exist and can be computed fibre-wise).

Example 2.1.8 (Mayer-Vietoris). For the simplest non-trivial example of a non-constant map in $\mathcal{W}$ consider a cover $\{U_1 \to S, U_2 \to S\}$ in $\mathcal{S}(\cdot)$ consisting of two monomorphisms. Then the projection

$$p : \left( \begin{array}{c} \text{"}U_1 \times_S U_2\text{"} \to U_1 \\ \downarrow \ \\ U_2 \end{array} \right) \to S$$

is in $\mathcal{W}$ as is easily derived from the axioms (L1–L4). See\footnote{2.4.14} for how the Mayer-Vietoris long exact sequence is related to this.

2.1.9. Let $\alpha, \beta : I \to J$ be two functors in $\text{Dia}$. Recall that it is the same to give a natural transformation of functors $\alpha \Rightarrow \beta$ or a morphism $I \times \Delta_1 \to J$ such that the compositions $I \xrightarrow{e_i} I \times \Delta_1 \to J$, for $i = 1, 2$ are $\alpha$ and $\beta$ respectively. We call $\alpha$ and $\beta$ homotopic, if they are equivalent in the smallest equivalence relation generated by the following relation: $\alpha \sim \beta$, if there exists a natural transformation $\alpha \Rightarrow \beta$. In other words $\alpha$ and $\beta$ are homotopic if there is a finite set of functors $\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_n = \beta : I \to J$ and natural transformations

$$\gamma_0 \Leftarrow \gamma_1 \Rightarrow \gamma_2 \Leftarrow \cdots \Rightarrow \gamma_n.$$
Proposition 2.1.10. Let \( W \) be a fundamental localizer in \( \text{Dia}(S) \) (or in one of the variants of 2.1.3). \( W \) satisfies the following properties:

1. \( W \) is closed under coproducts.

2. Let \( I_1, I_2 \) be in \( \text{Dia} \) and \( p: I_1 \to I_2 \) and \( s: I_2 \to I_1 \) such that \( p \) is left adjoint to \( s \), and \( F \in S(I_1) \), then the obvious morphisms: \( \overline{p}: (I_1, F) \to (I_2, F \circ s) \) and \( \overline{s}: (I_2, F \circ s) \to (I_1, F) \) are in \( W \).

3. Given a commutative diagram in \( \text{Dia} \)

\[
\begin{array}{ccc}
I_1 & \xrightarrow{\alpha} & I_2 \\
\downarrow & & \downarrow \\
I_3 & & \\
\end{array}
\]

where the vertical morphisms are Grothendieck opfibrations and \( \alpha \) is a morphism of opfibrations, and given an extension \( w: (I_1, F_1) \to (I_2, F_2) \) of the horizontal morphism, if \( w_i: D_{1,i} \to D_{2,i} \)

is in \( W \) for all \( i \in I \), then \( w \in W \).

4. If \( f: D_1 \to D_2 \) is in \( W \) then also \( f \times E: D_1 \times E \to D_2 \times E \) is in \( W \) for any \( E \in \text{Dia} \).

5. Any morphism which is homotopic (in the sense of 2.1.3) to a morphism in \( W \) is in \( W \).

Proof. 1. follows immediately from (L3 left) applied to a diagram

\[
\begin{array}{ccc}
\bigsqcup_{i \in I} I_{1,i} & \xrightarrow{\alpha} & \bigsqcup_{i \in I} I_{2,i} \\
\downarrow & & \downarrow \\
I & & \\
\end{array}
\]

where \( I \) is considered as a discrete category.

2. We first show that \( \overline{p} \in W \). Using (L3 left), it suffices to show that \( \overline{p}_i: D_1 \times_{I_2} i \to D_2 \times_{I_2} i \) is in \( W \) for all \( i \in I_2 \), however by the adjunction we have \( I_1 \times_{I_2} i = I_1 \times_{I_1} s(i) \) and therefore \( I_1 \times_{I_2} i \) has a final object. The two diagrams are therefore both equivalent to \((\cdot, F(s(i)))\). That \( \overline{s} \) is in \( W \) will follow from 4. because this implies that \( \overline{s} \circ \overline{p} \) and \( \overline{p} \circ \overline{s} \) are in \( W \) therefore by L1 also \( \overline{s} \) is in \( W \). For note that the unit and the counit extend to \((I_1, F)\), and \((I_2, F \circ s)\) respectively.

3. Using (L3 left), we have to show that \( D_1 \times_{/E} e \to D_2 \times_{/E} e \) is in \( W \). We have the diagram

\[
\begin{array}{ccc}
D_1 \times_{/E} e & \xrightarrow{s_e} & D_2 \times_{/E} e \\
\downarrow i_e & & \downarrow i_e \\
D_1 \times_{/E} e & \xrightarrow{s_e} & D_2 \times_{/E} e \\
\end{array}
\]

where \( i_e \) is of diagram type and is right adjoint to \( s_e \). Therefore \( s_e \) is in \( W \) by 1. and hence also \( i_e \) because \( s_e i_e = \text{id} \) (using L1). Note: we are not using the not yet proven part of 1. Therefore the top arrow being in \( W \) implies that the bottom arrow is in \( W \).

4. This is but a special case of 2.
5. A natural transformation \( \mu : f \Rightarrow g \) for \( f, g : D_1 \to D_2 \) can be seen as a morphism of diagrams \( \mu : \Delta_1 \times D_1 \to D_2 \). Since the projection \( p : \Delta_1 \times D_1 \to D_1 \) is in \( \mathcal{W} \) by 3. also the morphisms \( e_{0,1} : D_1 \to \Delta_1 \times D_1 \) are in \( \mathcal{W} \). Since \( \mu \circ e_0 = f \) and \( \mu \circ e_1 = g \), we have that \( f \) is in \( \mathcal{W} \) if and only if \( g \in \mathcal{W} \).

**Proposition 2.1.11.** Axiom (L5 left) is in the presence of (L1–L3 left) equivalent to the following, apparently stronger axiom:

(L5’ left) Let \( D = (E, F) \in \text{Dia}(\mathbb{S}) \) and let \( p : I \to E \) be any functor in \( \text{Dia} \) such that \( e \times_{IF} I \to \cdot \) is in \( \mathcal{W}_{\text{Dia}}^{\text{min}} \) for all \( e \in E \). Then the morphism \( (I, p^*F) \to D \) is in \( \mathcal{W} \).

**Proof.** (L5 left) implies (L5’ left): Consider the following 2-commutative diagram

\[
(E, F) \times_{(E, F)} (I, p^*F) \quad \xrightarrow{(\text{id}, p^*F)} \quad (I, p^*F) \\
\downarrow \quad \downarrow \quad \downarrow \\
(E, F) \quad \xrightarrow{i} \quad (E, F)
\]

The underlying diagram functor of the horizontal map (which is not purely of diagram type) is a Grothendieck opfibration and hence by Proposition 2.1.10 3. is in \( \mathcal{W} \), provided that the morphisms of the fibers \( (E \times_{IF} e, \text{pr}_1^*F) \to (\cdot, F(e)) \) are. However \( E \times_{IF} e \) has the final object id, whose value under \( \text{pr}_1^*F \) is \( F(e) \). The morphisms of the fibers are therefore in \( \mathcal{W} \) by (L2 left). The underlying diagram functor of the left vertical map is a Grothendieck fibration and \( \text{pr}_1^*F \) is constant along the fibers. Therefore the fact that all \( e \times_{IF} I \to \cdot \) are in \( \mathcal{W}_{\text{Dia}}^{\text{min}} \) implies that the left vertical map is in \( \mathcal{W} \) by (L5 left). Therefore also the right vertical map is in \( \mathcal{W} \). (This uses Proposition 2.1.10 5. and the fact that the 2 compositions in the diagram are homotopic).

(L5’ left) implies (L5 left): If \( I \to E \) is a Grothendieck fibration as in Axiom (L5 left), the morphism \( e \times_{IF} I \to e \times_{IF} I \) is in \( \mathcal{W}_{\text{Dia}}^{\text{min}} \) (being part of an adjunction), therefore (L5’ left) applies. \( \square \)

### 2.2 Simplicial objects in a fundamental localizer

2.2.1. In this section, we fix a diagram category \( \text{Dia} \) and a stable right derivator \( \mathbb{S} \) equipped with a Grothendieck pre-topology and with the assumptions of 2.1.7. Assume that either \( \Delta^{\text{op}} \) and also all truncations \( (\Delta^n)^{\text{op}} \) are in \( \text{Dia} \) or that \( (\Delta^\circ)^{\text{op}} \) (injective simplex diagram) and all truncations \( (\Delta^{\leq n})^{\text{op}} \) are in \( \text{Dia} \). The reasoning in the section does use little of the explicit definition of \( \Delta^{\text{op}} \), resp. \( (\Delta^\circ)^{\text{op}} \). For comparison with classical texts on cohomological descent we stick to the particular diagram \( \Delta^{\text{op}} \), resp. \( (\Delta^\circ)^{\text{op}} \).

Consider the category \( \mathbb{S}(\Delta^{\text{op}}) \). Since \( \mathbb{S} \) has all (relative) finite coproducts, \( \mathbb{S}(\cdot) \) is actually tensored over \( \text{SET} \mathcal{F} \) and so \( \mathbb{S}(\Delta^{\text{op}}) \) will be tensored over \( \text{SET} \mathcal{F}^{\Delta^{\text{op}}} \). We sketch this construction. A finite simplicial set, i.e. a functor \( \xi : \Delta^{\text{op}} \to \text{SET} \mathcal{F} \), can be seen as a functor with values in finite discrete categories. The corresponding Grothendieck construction yields a Grothendieck opfibration \( \pi_\xi : \int \xi \to \Delta^{\text{op}} \). We define for \( X_\bullet \in \mathbb{S}(\Delta^{\text{op}}) \):

\[
\xi \otimes X_\bullet := (\pi_\xi)^*(\pi_\xi)^*X_\bullet.
\]

Recall that ‘\( \mathbb{S} \) has relative finite coproducts’ means that all functors \( (\pi_\xi) \); arising this way exist and can be computed fibre-wise. We consider a simplicial object \( X_\bullet \in \mathbb{S}(\Delta^{\text{op}}) \) as the diagram \( (\Delta^{\text{op}}, X_\bullet) \in \text{Dia}(\mathbb{S}) \) and likewise a semi-simplicial object \( X_\bullet \in \mathbb{S}((\Delta^\circ)^{\text{op}}) \) as the diagram \( ((\Delta^\circ)^{\text{op}}, X_\bullet) \in \text{Dia}(\mathbb{S}) \).
2.2.2. Consider the full subcategory $\Delta^\leq_n$ consisting of $\Delta_0, \ldots, \Delta_n$. Since $S$ is assumed to be a right derivator, the restriction functor

$$t^* : S(\Delta^{\mathrm{op}}) \to S((\Delta^\leq_n)^{\mathrm{op}})$$

has a right adjoint $t_*$ which is usually called the **coskelet** and denoted $\cosk^n$.

Let some simplicial object $Y_* \in S(\Delta^{\mathrm{op}})$ and a morphism $\alpha : X_{\leq n} \to t^* Y_*$ be given. Consider the full subcategory $(\Delta^{\mathrm{op}} \times \Delta_1)^{\leq-\leq_n}$ of all objects $\Delta_i \times [1]$ for all $i$, and $\Delta_i \times [0]$ for $i \leq n$. The restriction

$$t^* : S(\Delta^{\mathrm{op}} \times \Delta_1) \to S((\Delta^{\mathrm{op}} \times \Delta_1)^{\leq-\leq_n})$$

has again an adjoint $t_*$ Since $S$ is assumed to be strong we can consider $\alpha$ as an object over $(\Delta \times \Delta_1)^{\leq-\leq_n}$. The first row of $t_* \alpha$ is called the **relative coskelet** $\cosk^n(X_{\leq n} Y_*)$ of $X_{\leq n}$. For $n = -1$ we understand $\cosk^{-1}(-|Y_*|) = Y_*$.

These constructions work the same way with $\Delta$ replaced by $\Delta^\circ$. The functor ‘coskelet’ and ‘relative coskelet’ is in both cases even the *same functor*, i.e. these functors commute with the restriction of a simplicial to a semi-simplicial object.\footnote{E.g., for the case of the ‘coskelet’, observe that there is an adjunction:

$$\Delta_m \times (\Delta^{\circ})^{\mathrm{op}} (\Delta^\leq_n)^{\mathrm{op}} \leftrightarrow \Delta_m \times (\Delta^{\circ})^{\mathrm{op}} (\Delta^\leq_n)^{\mathrm{op}}.$$}

(\text{This would not at all be true for the corresponding left adjoint, the functor ‘skelet’}).

**Lemma 2.2.3.** Let $\mathcal{W}$ be a fundamental localizer in $\text{Dia}(S)$ (or in one of the variants of 2.1.3). Let $((\Delta^{\mathrm{op}})^2, F_{\bullet, \bullet}) \in \text{Dia}(\mathcal{W})$ (or in one of the variants) be a bisimplicial diagram and let $\delta : \Delta^{\mathrm{op}} \to (\Delta^{\mathrm{op}})^2$ be the diagonal. Then

$$(\Delta^{\mathrm{op}}, \delta^* F_{\bullet, \bullet}) \to ((\Delta^{\mathrm{op}})^2, F_{\bullet, \bullet})$$

is in $\mathcal{W}$. The same holds true if $\Delta$ is replaced by $\Delta^\circ$.

**Proof.** Consider the forgetful functor $f : \Delta^{\mathrm{op}} \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}} \to (\Delta^{\mathrm{op}})^2$. We first show that the obvious lift of the diagonal

$$s : (\Delta^{\mathrm{op}}, \delta^* F_{\bullet, \bullet}) \to (\Delta^{\mathrm{op}} \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}, f^* F_{\bullet, \bullet})$$

is in $\mathcal{W}$. There is also a projection

$$p : (\Delta^{\mathrm{op}} \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}, f^* F_{\bullet, \bullet}) \to (\Delta^{\mathrm{op}}, \delta^* F_{\bullet, \bullet})$$

given on diagrams by mapping an $\alpha : \Delta_n \to \Delta_m$ to $\Delta_n$ and the morphism $F((\mathrm{id}, \alpha)) : F_{n,m} \to F_{n,n}$. We have $p \circ s = \text{id}$ and there is a 2-morphism $\text{id} \Rightarrow s \circ p$. Therefore by Proposition 2.1.10 4., and by (L1), the morphisms $s$ and $p$ are in $\mathcal{W}$. We will now show that the morphism induced by $f$

$$f : (\Delta^{\mathrm{op}} \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}, f^* F_{\bullet, \bullet}) \to (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, F_{\bullet, \bullet})$$

is in $\mathcal{W}$. By (L5' left), cf. Proposition 2.1.11, we have to show that

$$\Delta_m \times (\Delta^{\mathrm{op}})^2 (\Delta^{\mathrm{op}} \times_{/\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}})$$

is contractible w.r.t. every fundamental localizer on $\text{Dia}$. Equivalently we have to show this for the dual category. Objects of that category are diagrams of the form:

\[
\begin{array}{ccc}
\Delta_{m'} & \xrightarrow{\alpha} & \Delta_{n'} \\
\downarrow & & \downarrow \\
\Delta_m & & \Delta_n
\end{array}
\]
Obviously we have an adjunction with the category of diagrams of the form:

\[
\begin{array}{ccc}
\Delta_{m'} & \xrightarrow{\cdot} & \Delta_m \\
\downarrow & & \downarrow \\
\Delta_n & \xleftarrow{\cdot} & \Delta_{m} \\
\end{array}
\]

This is the category \(\Delta/(\Delta_m \times \Delta_n)\) which is contractible by \([1, \text{Proposition 2.2.3}]\). Note that this is the only feature of \(\Delta^\text{op}\) used in the proof of this Lemma. Note that likewise \(\Delta^\circ/(\Delta_m \times \Delta_n)\) is contractible.

**Remark 2.2.4.** The previous lemma should be seen in the following context: The Grothendieck construction is a way of embedding the category of simplicial sets into the category of small categories. This construction maps weak equivalences to weak equivalences and induces an equivalence of the corresponding homotopy categories. A bisimplicial set can be seen as a simplicial object in the category of simplicial sets. Its homotopy colimit is given by the diagonal simplicial set. On the other hand the homotopy colimit in the category of small categories is again just given by the Grothendieck construction. From this perspective, the Lemma is clear if \(S\) is the derivator associated with the category of sets (equipped with the discrete topology).

**Lemma 2.2.5.** Let \(W\) be a fundamental localizer in \(\text{Dia}(\mathbf{S})\) (or in one of the variants of \([2.1.3]\)). Consider a simplicial diagram \((\Delta^\text{op}, F_* ) \in \text{Dia}(\mathbf{S})\) (or in one of the variants). The morphism

\[
(\Delta^\text{op}, F_* \otimes \Delta_n ) \to (\Delta^\text{op}, F_* )
\]

is in \(W\). The same holds true if \(\Delta\) is replaced by \(\Delta^\circ\).

**Proof.** Note that \((\Delta^\text{op}, F_* \otimes \Delta_n )\) is the diagonal of the bisimplicial diagram

\[
((\Delta^\text{op})^2, \text{pr}_1^* F_* \otimes \text{pr}_2^* \Delta_n )
\]

By the previous Lemma it suffices to show that the obvious map to

\[
((\Delta^\text{op})^2, \text{pr}_1^* F_*)
\]

is in \(W\). This can be checked row-wise by Proposition \([2.1.10]\). The rows are equal to

\[
(\Delta^\text{op}, F_m \otimes \Delta_n ) \to (\Delta^\text{op}, F_m )
\]

and \((\Delta^\text{op}, F_m \otimes \Delta)\) is however equivalent to \((f \Delta_n, \pi^* F_m )\) (cf. \([2.1.7]\)). Since the resulting map

\[
(\int \Delta_n, \pi^* F_m ) \to (\Delta^\text{op}, F_m )
\]

is a map of constant diagrams, the statement follows from the fact that \((f \Delta_n )\) and \(\Delta^\text{op}\) are contractible w.r.t. any fundamental localizer on \(\text{Dia}\) (cf. \([1, \text{Proposition 2.2.3}]\)).

**Corollary 2.2.6.** Let \(W\) be a fundamental localizer in \(\text{Dia}(\mathbf{S})\) (or in one of the variants of \([2.1.3]\)). Let \(f, g : (\Delta^\text{op}, F_*) \to (\Delta^\text{op}, G_* )\) be two homotopic maps of simplicial objects. Then \(f \in W\) if and only if \(g \in W\). The same holds true if \(\Delta\) is replaced by \(\Delta^\circ\).

**Proof.** The statement follows by the standard argument because the projection \((\Delta^\text{op}, F_* \otimes \Delta_1 ) \to (\Delta^\text{op}, F_*)\) is in \(W\) by Lemma \([2.2.5]\).
**Proposition 2.2.7** (Čech resolutions are in \(\mathcal{W}\)). Let \(\mathcal{W}\) be a fundamental localizer in \(\text{Dia}(\mathcal{S})\) (or in one of the variants of \([2.1.3]\)).

Let \(U \to S\) be a local epimorphism. Then

\[ p : (\Delta^{\text{op}}, \cosk_0(U|S)) \to (\cdot, S) \]

is in \(\mathcal{W}\). The same holds true if \(\Delta\) is replaced by \(\Delta^o\).

**Proof.** The assumption means that there is a cover \(\mathcal{U} = \{U_i \to S\}\) in the given pre-topology, such that for all \(i\), the induced map

\[ U \times_S U_i \to U_i \]

has a section \(s_i\). By axiom (L4 left) it suffices to show that for all \(i\) the map

\[ p_i : (\Delta^{\text{op}}, \cosk_0(U \times_S U_i|U_i)) \to (\cdot, U_i) \]

is in \(\mathcal{W}\). Explicitly the simplicial object \(\cosk_0(U \times_S U_i|U_i)\) is given by

\[
\ldots \xrightarrow{\sim} U \times_S U \times_S U \times_S U_i \xrightarrow{\sim} U \times_S U \times_S U \times_S U_i \xrightarrow{\sim} U \times_S U_i
\]

Since \(\Delta\) is acyclic, it suffices to show that the map

\[ p_i : (\Delta^{\text{op}}, \cosk_0(U \times_S U_i|U_i)) \to (\Delta^{\text{op}}, p^*U_i) \]

is in \(\mathcal{W}\). There is a section

\[ s_i : (\Delta^{\text{op}}, p^*U_i) \to (\Delta^{\text{op}}, \cosk_0(U \times_S U_i|U_i)) \]

such that \(p_i \circ s_i = \text{id}\). By (L1) it then suffices to see that \(s_i \circ p_i \in \mathcal{W}\). We will construct a homotopy in the sense of simplicial objects between \(\text{id}\) and \(s_i \circ p_i\).

\[
(\Delta^{\text{op}}, \Delta \times \cosk_0(U \times_S U_i|U_i)) \to (\Delta^{\text{op}}, \cosk_0(U \times_S U_i|U_i))
\]

Since \(\Delta_1\) is 1-coskeletal, and the \(\cosk_0\)'s anyway, it suffices to construct the homotopy in degrees 0 and 1:

\[
\text{Hom}(\Delta_1, \Delta_1) \times U \times_S U \times_S U_i \xrightarrow{\sim} \text{Hom}(\Delta_0, \Delta_1) \times U \times_S U_i
\]

\[
U \times_S U \times_S U_i \xrightarrow{\sim} U \times_S U_i
\]

This can be achieved by mapping \(\text{id}_{\Delta_1} \times (U \times_S U_i) \times_{U_i} (U \times_S U_i)\) to \((U \times_S U_i) \times_{U_i} (U \times_S U_i)\) via \((s_i \circ \text{pr}_2) \times \text{id}\).

\[ \square \]

**Definition 2.2.8.** Recall that a morphism \(X_\bullet \to Y_\bullet\) of simplicial objects (resp. semi-simplicial objects) is called a **hypercover** if the following two equivalent conditions hold:

1. In any diagram of simplicial objects

\[
\partial \Delta_n \otimes U \longrightarrow X_\bullet
\]

\[
\Delta_n \otimes U \longrightarrow Y_\bullet
\]
there is a cover $U = \{ U_i \to U \}$ and for all $i$ there is a lift in the diagram

$$\begin{array}{ccc}
\partial \Delta_n \otimes U_i & \to & \partial \Delta_n \otimes U \\
\downarrow & & \downarrow \\
\Delta_n \otimes U_i & \to & \Delta_n \otimes U
\end{array}$$

2. For any $n \geq 0$ the morphism

$$X_n \to \cosk_{n-1}(i_{\leq n-1}^*X_\bullet | Y_\bullet)_n$$

admits local sections in the pre-topology on $S$ (i.e. is a local epimorphism).

Remark 2.2.9. 1. In particular the notion of hypercover depends only on the Grothendieck topology generated by the pre-topology because a morphism is a local epimorphism precisely if the sieve generated by it is a covering sieve.

2. Condition 1. shows that, if $S$ is the derivator associated with $\mathcal{SET}$ equipped with the discrete topology, then a hypercover is precisely a trivial Kan fibration.

3. Note that according to whether we consider simplicial or semi-simplicial objects, $\Delta_n$ and $\partial \Delta_n$ mean different objects.

Definition 2.2.10. If in [2.2.8] 2. the morphism is even an isomorphism for all sufficiently big $n$, then $\alpha$ is called a finite (or bounded) hypercover. Equivalently we have $X_\bullet \simeq \cosk_n(i_{\leq n}^*X_\bullet | Y_\bullet)$ for some $n$.

Lemma 2.2.11. Let $W$ be a fundamental localizer in Dia$(S)$ (or in one of the variants of 2.1.3). For a finite hypercover $X_\bullet \to Y_\bullet$ such that $X_\bullet \simeq \cosk_{i+1}(X_\bullet | Y_\bullet)$ and $i_{\leq n}^*X_n \simeq i_{\leq n}^*Y_n$ the morphism $(\Delta^\text{op}, X_\bullet) \to (\Delta^\text{op}, Y_\bullet)$ is in $W$. The same holds true if $\Delta$ is replaced by $\Delta^\circ$.

Proof. We may assume $i \geq 1$ because otherwise we are in the situation of Lemma 2.2.7. The conditions imply that the map $X_i \to Y_i$ is a local epimorphism. Indeed, this is the map $X_i \to Y_i = \cosk_{i-1}(i_{\leq i-1}^*X_\bullet | Y_\bullet)_i$ in this case. Therefore the morphism $X_j \to Y_j$ is actually a local epimorphism for all $j$.

Consider the following diagram in Dia$(S)$:

$$\begin{array}{ccc}
(\Delta^\text{op} \times \Delta^\text{op}, (X_\bullet \times Y_\bullet, X_\bullet | X_\bullet)) & \to & (\Delta^\text{op} \times \Delta^\text{op}, (X_\bullet | Y_\bullet)) \\
\downarrow & & \downarrow \\
(\Delta^\text{op}, X_\bullet) & \to & (\Delta^\text{op}, Y_\bullet)
\end{array}$$

where

$$(X_\bullet | Y_\bullet)_{m,n} := \cosk_0(X_n | Y_n)_m = \underbrace{X_n \times_{Y_n} \cdots \times_{Y_n} X_n}_{m+1 \text{ factors}}.$$  

$$(X_\bullet \times Y_\bullet, X_\bullet | X_\bullet)_{m,n} := \cosk_0(X_n \times_{Y_n} X_n | X_n)_m = \underbrace{X_n \times_{Y_n} \cdots \times_{Y_n} X_n}_{m+2 \text{ factors}}.$$  

The vertical morphisms are in $W$ by Proposition 2.1.10 3. because the columns are in $W$, which is the case by Lemma 2.2.7. Again by 2.1.10 3. it suffices therefore to show that the rows

$$p: (\Delta^\text{op} \times \Delta^\text{op}, (X_\bullet \times Y_\bullet | X_\bullet))_{m,\bullet} \to (\Delta^\text{op} \times \Delta^\text{op}, (X_\bullet | Y_\bullet))_{m,\bullet}.$$
of the top horizontal morphism are in \( W \). These are again hypercovers of the form considered in this Lemma, in particular \( i \)-coskeletal, where the \( i \)-truncation is given by

\[
\begin{array}{c}
X_i \times Y_i \cdots \times Y_i X_i \rightarrow X_{i-1} = Y_{i-1} \cdots \rightarrow X_0 = Y_0 \\
m+2 \\
\downarrow \\
\end{array}
\]

where the left-most vertical arrow is induced by the map \( \Delta_{m+1} \rightarrow \Delta_{m+2}, i \mapsto i \). There is a section \( s \), with \( s_i \) induced by the map

\[
\Delta_{m+2} \rightarrow \Delta_{m+1}, \quad i \mapsto \begin{cases} i & i < m + 2, \\ m + 1 & i = m + 2. \end{cases}
\]

We will construct a homotopy \( \mu : \text{id} \Rightarrow s \circ p \) of truncated simplicial objects:

\[
\begin{array}{c}
\text{Hom}(\Delta_i, \Delta_1) \times X_i \times Y_i \cdots \times Y_i X_i \rightarrow \text{Hom}(\Delta_{i-1}, \Delta_1) \times Y_{i-1} \cdots \rightarrow \text{Hom}(\Delta_0, \Delta_1) \times Y_0 \\
\downarrow \mu_i \\
X_i \times Y_i \cdots \times Y_i X_i \rightarrow Y_{i-1} \cdots \rightarrow Y_0 \\
\end{array}
\]

The morphism \( \mu_i \) at the constant morphism \( 0 : \Delta_1 \rightarrow \Delta_1 \) is given by the identity, at the constant morphism \( 1 : \Delta_1 \rightarrow \Delta_1 \) given by \( s_i \circ p_i \), and at the other morphisms \( \Delta_i \rightarrow \Delta_1 \) arbitrarily. The existence of this homotopy allows by Lemma 2.2.6 and (L1) to conclude. \( \square \)

**Theorem 2.2.12.** Let \( W \) be a fundamental localizer in \( \text{Dia}(S) \) (or in one of the variants of \( \text{2.1.3} \)). Any finite hypercover considered as a morphism of diagrams

\[
(\Delta^{\text{op}}, X_*) \rightarrow (\Delta^{\text{op}}, Y_*)
\]

is in \( W \). The same holds true if \( \Delta \) is replaced by \( \Delta^\circ \).

**Proof.** Any finite hypercover is a finite succession of hypercovers of the form considered in Lemma 2.2.11 \( \square \)

### 2.3 Cartesian and coCartesian objects

**Definition 2.3.1.** Let \( D \rightarrow S \) be a fibered derivator of domain \( \text{Dia} \). Let \( I \in \text{Dia} \) be a diagram and let \( \alpha : I \rightarrow E \) be a functor in \( \text{Dia} \). We say that an object

\[
X \in D(I)
\]

is \( E \)-(co-)Cartesian, if for any \( \mu : i \rightarrow j \) mapping to an identity in \( E \), the corresponding morphism \( D(\mu) : i^* X \rightarrow j^* X \) is (co-)Cartesian.

If \( E \) is the trivial category, we omit it from the notation, and talk about (co-)Cartesian objects.
These notions define full subcategories $D(I)^{E\text{-cart}}$ (resp. $D(I)^{E\text{-cocart}}$) of $D(I)$, and $D(I)^{E\text{-cart}}_F$ (resp. $D(I)^{E\text{-cocart}}_F$) of $D(I)_F$ for any $F \in S(I)$.

**Lemma 2.3.2.** The functor $\alpha^*$ w.r.t. a morphism $\alpha : D_1 \to D_2$ in $\text{Dia}(S)$ maps Cartesian objects to Cartesian objects. The functor $\alpha^*$ for a morphism $\alpha : D_1 \to D_2$ in $\text{Dia}^{\text{op}}(S)$ maps coCartesian objects to coCartesian objects.

**Definition 2.3.3.** Let $D \to S$ be a fibered derivator of domain $\text{Dia}$. We say that in $D \to S$ left Cartesian projections exist if for all $I, E, S$ as above, the fully-faithful inclusion

$$D(I)^{E\text{-cart}} \to D(I)_F$$

has a left adjoint $\square^E_I$.

More generally we have four notions with the following notations:

- $\square^E_I$ left adjoint left Cartesian projection
- $\square^E_I$ right adjoint right Cartesian projection
- $\square^E_I$ left adjoint left coCartesian projection
- $\square^E_I$ right adjoint right coCartesian projection

We will, in general, only use left Cartesian and right coCartesian projection, the others being somewhat unnatural. In 3.3.1 we will show (using Brown representability) that for an infinite fibered derivator whose fibers are stable and well-generated a right coCartesian projection exists. Similarly if, in addition, Brown representability for the dual holds, e.g. if the fibers are compactly generated, then a left Cartesian projection exists in many cases. Note that for a usual (non fibered) derivator, Cartesian and coCartesian are equivalent notions. If in a fibered derivator with stable fibers both left and right Cartesian projections exist, then there is actually a recollement [14, Proposition 4.13.1]:

Example 2.3.4. The projections are difficult to describe explicitly, except in very special situations. Here a rather trivial example where this is possible. Let $D$ be a stable derivator and consider $I = \Delta_1$, the projection $p : \Delta_1 \to \cdot$ and the inclusions $e_0, e_1 : \cdot \to \Delta_1$. Then a left and a right Cartesian projection exists and the recollement above is explicitly given by:

$$D(\Delta_1)^{\text{cart}} \cong D(\cdot) \xleftarrow{e_0^*} D(\Delta_1) \xrightarrow{p^*} D(\cdot) \xrightarrow{e_1^*} C \xrightarrow{[-1]e_0} D(\cdot)$$

Note that the functor $C$ (Cone) may be described as either $[1] \circ e_0^*$ or $e_1^*$ (cf. [7, §3]) and that the essential image of $p^*$ is precisely the kernel of $C$ which also coincides with the full subcategory of Cartesian=coCartesian objects.

### 2.4 Weak and strong $D$-equivalences

**Definition 2.4.1 (left).** Let $\text{Dia}$ be a diagram category and let $S$ be a strong right derivator with domain $\text{Dia}$ equipped with Grothendieck pre-topology.
Let $\mathbb{D} \rightarrow S$ be a left fibered derivator satisfying $(\text{FDer}0 \ \text{right})$ and $S \in S(\cdot)$. A morphism $f : D_1 \rightarrow D_2$ in $\text{Dia}(S)/(\cdot, S)$ is called a weak $\mathbb{D}$-equivalence relative to $S$ if the natural transformation

$$p_1!p_1^* \to p_2!p_2^*$$

is an isomorphism of functors.

A morphism $f \in \text{Dia}(S)$ is called a strong $\mathbb{D}$-equivalence if $f^*$ induces an equivalence of categories

$$f^* : \mathbb{D}(D_2)^{\text{cart}} \rightarrow \mathbb{D}(D_1)^{\text{cart}}$$

Note that weak is a relative notion whereas strong is absolute.

**Definition 2.4.2 (right).** Let $\text{Dia}$ be a diagram category and $S$ again be a strong left derivator with domain $\text{Dia}$ equipped with Grothendieck pre-cotopology.

Let $\mathbb{D} \rightarrow S$ be a right fibered derivator satisfying $(\text{FDer}0 \ \text{left})$ and $S \in S(\cdot)$. A morphism $f : D_1 \rightarrow D_2$ in $\text{Dia}^\text{op}(S)/(\cdot, S)$ is called a weak $\mathbb{D}$-equivalence relative to $S$, if the natural transformation

$$p_2*p_2^* \rightarrow p_1*p_1^*$$

is an isomorphism of functors.

A morphism $f \in \text{Dia}^\text{op}(S)$ is called a strong $\mathbb{D}$-equivalence if $f^*$ induces an equivalence of categories

$$f^* : \mathbb{D}(D_2)^{\text{cocart}} \rightarrow \mathbb{D}(D_1)^{\text{cocart}}$$

For a derivator (i.e. $S = \cdot$) there is no difference between $\text{Dia}(S)$ and $\text{Dia}^\text{op}(S)$ and then obviously also the two different definitions of weak, resp. strong $\mathbb{D}$-equivalence coincide (for the case of weak $\mathbb{D}$-equivalences, note that the two conditions become adjoint to each other). These notions of $\mathbb{D}$-equivalence (right version) should be compared to the classical notions of cohomological descent, see [1, Exposé Vbis].

**Lemma 2.4.3 (left).** Let $f : D_1 \rightarrow D_2$ in $\text{Dia}(S)/(\cdot, S)$. Then the following implication holds:

$$f \text{ strong } \mathbb{D} \text{-equivalence } \Rightarrow f \text{ weak } \mathbb{D} \text{-equivalence relative to } S.$$ 

**Proof.** The morphism in the definition of weak $\mathbb{D}$-equivalence is induced by the counit w.r.t. the adjunction $f^*, f!$:

$$p_1!p_1^* \cong p_2!f_!f^*p_2^* \rightarrow p_2!p_2^*$$

Now let $f_{\Box}$ be an inverse to $f^*$, as is required by the definition of strong $\mathbb{D}$-equivalence. From $f^*p_2^* \cong p_1^*$ follows $p_2!f_{\Box} \cong p_1!$ and the diagram

$$\begin{array}{ccc}
p_2!f_{\Box}f^*p_2^* & \rightarrow & p_2!p_2^* \\
& \downarrow & \\
p_2!f_{\Box}f^*p_2^* & \rightarrow & \\
\end{array}$$

is commutative. Since the diagonal morphism is a natural isomorphism the statement follows. □

Of course there is an analogous right version of this Lemma.

The goal of this section is to prove the following Theorems:
Main Theorem 2.4.4 (right). Let Dia be a diagram category and S be a strong left derivator with domain Dia equipped with Grothendieck pre-cotopology.

1. Let $\mathcal{D} \to S$ be a fibered derivator with domain Dia which is colocal for the Grothendieck pre-cotopology on S. Then the set $W^w_{\mathcal{D}}$ consisting of those morphisms $f : D_1 \to D_2$ in $\text{Dia}^{\text{op}}(S)$ which are weak $\mathcal{D}$-equivalences relative to any $S \in S(\cdot)$ form a fundamental colocalizer (cf. Remark 2.1.3).

2. Let $\mathcal{D} \to S$ be an infinite fibered derivator with domain Dia which is colocal for the Grothendieck pre-cotopology on S, with stable, compactly generated fibers. The set $W^s_{\mathcal{D}}$ consisting of those morphisms $f : D_1 \to D_2$ in $\text{Dia}^{\text{op}}(S)$ which are strong $\mathcal{D}$-equivalences form a fundamental colocalizer.

Main Theorem 2.4.5 (left). Let Dia be a diagram category and let S be a strong right derivator with domain Dia equipped with Grothendieck pre-topology.

1. Let $\mathcal{D} \to S$ be a fibered derivator with domain Dia, which is local for the Grothendieck pre-topology on S. Then the set $W^w_{\mathcal{D}}$ consisting of those morphisms $f : D_1 \to D_2$ in $\text{Dia}(S)$ which are weak $\mathcal{D}$-equivalences relative to any $S \in S(\cdot)$ form a fundamental localizer (cf. Remark 2.1.3).

2. Let $\mathcal{D} \to S$ be an infinite fibered derivator with domain Dia, which is local for the Grothendieck pre-topology on S, with stable, compactly generated fibers. The set $W^s_{\mathcal{D}}$ consisting of those morphisms $f : D_1 \to D_2$ in $\text{Dia}^{\text{op}}(S)$ which are strong $\mathcal{D}$-equivalences form a fundamental localizer, where $\text{Dia}'(S)$ is the full subcategory of $\text{Dia}(S)$ of the diagrams which consist of universally $\mathcal{D}$-local morphisms (cf. Remark 2.1.3).

Remark 2.4.6. The restriction onto $\text{Dia}'(S)$ in the left-variant of the theorem is needed because otherwise we do not know whether left Cartesian projections exist (cf. Theorem 3.3.2).

The weak-$\mathcal{D}$-equivalences for the case of usual derivators (i.e. $S = \cdot$) were called just ‘$\mathcal{D}$-equivalences’ by Cisinski [5] and it is rather straight-forward to see from the definition of derivator that they form a fundamental localizer.

We will only prove the left-variant of the Theorem. The other follows by logical duality and the restriction to $\text{Dia}'(S)$ is not necessary because Lemma 2.4.11 is used instead of Lemma 2.4.10.

Before proving the Theorem we need a couple of Lemmas. We assume for the rest of this section that Dia is a diagram category and that S is a strong right derivator with domain Dia equipped with a Grothendieck pre-topology.

Definition 2.4.7. Two morphisms (in $\text{Dia}(S)$ or $\text{Dia}^{\text{op}}(S)$)

$$ D_1 \xrightarrow{p} D_2 $$

such that chains of 2-morphisms

$$ p \circ s \Rightarrow \cdots \Leftarrow \cdots \Rightarrow \text{id}_{D_1}, \quad s \circ p \Rightarrow \cdots \Leftarrow \cdots \Rightarrow \text{id}_{D_2} $$

exist are called a homotopy equivalence (or p is called if an s with this property exists).
Lemma 2.4.8 (left). Let \( \mathcal{D} \) be a left fibered derivator satisfying (FDer0 right) and \( D_1, D_2 \in \text{Dia}(S) \). Given any homotopy equivalence \((p,s)\), then \( p^* \) and \( s^* \) induce an equivalence

\[
\mathcal{D}(D_2)^{\text{cart}} \xrightarrow{p^*} \mathcal{D}(D_1)^{\text{cart}} \xleftarrow{s^*} \mathcal{D}(D_2)^{\text{cart}}
\]

Proof. The 2-morphisms in the chains induce morphisms between the pull-back functors, e.g.

\[(\alpha,f)^* \mathcal{E} \rightarrow (\beta,g)^* \mathcal{E}\]

which are isomorphisms on Cartesian objects. \(\square\)

Example 2.4.9 (cf. also Proposition 2.1.10 2.). If

\[
I_1 \xrightarrow{p} I_2 \xleftarrow{s} I_2
\]

is an adjunction with \( p \) left adjoint to \( s \), and if \( F \in S(I_1) \) then we get an equivalence

\[
\mathcal{D}(D_2)^{\text{cart}} \xrightarrow{p^*} \mathcal{D}(D_1)^{\text{cart}} \xleftarrow{s^*} \mathcal{D}(D_2)^{\text{cart}}
\]

for \( D_1 = (I_1, F) \) and \( D_2 = (I_2, F \circ s) \).

Lemma 2.4.10 (left). Let \( \text{Dia} \) be an infinite diagram category and let \( \mathcal{D} \rightarrow S \) be an infinite fibered derivator with domain \( \text{Dia} \) with stable, compactly generated fibers. Consider a morphism \( D = (I, F) \rightarrow (\cdot, S) \) such that \( F \) is a diagram of universally \( \mathcal{D} \)-local morphisms. Let \( U \rightarrow S \) be a universally \( \mathcal{D} \)-local morphism. Denote \( D_U := D \times (\cdot, S) (\cdot, U) \) in \( \text{Dia}(S) \). Then the following diagram is commutative (via the exchange natural isomorphism):

\[
\begin{array}{ccc}
\mathcal{D}(D) & \xrightarrow{\text{pr}} & \mathcal{D}(D)^{\text{cart}} \\
\downarrow{\text{pr}^*_1} & & \downarrow{\text{pr}^*_1} \\
\mathcal{D}(D_U) & \xrightarrow{\text{pr}} & \mathcal{D}(D_U)^{\text{cart}}
\end{array}
\]

Note that by Theorem 3.3.2 left Cartesian projectors exist for \( D \) and \( D_U \).

Proof. The functor \( \text{pr}^*_1 \) has a right adjoint \( \text{pr}_{1*} \) by (Dloc2 left) and Brown representability. (Dloc1 left) says that \( \text{pr}^*_1 \) preserves coCartesian morphisms, hence \( \text{pr}_{1*} \) preserves Cartesian morphisms. Therefore the right adjoint of the given diagram is the following diagram:

\[
\begin{array}{ccc}
\mathcal{D}(D) & \xleftarrow{\text{pr}_{1*}} & \mathcal{D}(D)^{\text{cart}} \\
\uparrow{\text{pr}^*_1} & & \uparrow{\text{pr}^*_1} \\
\mathcal{D}(D_U) & \xleftarrow{\text{pr}_1} & \mathcal{D}(D_U)^{\text{cart}}
\end{array}
\]

whose exchange morphism is an isomorphism. Consequently the exchange morphism of our diagram is also a natural isomorphism. \(\square\)
Lemma 2.4.11 (right). Let $\text{Dia}$ be an infinite diagram category and let $\mathbb{D} \to \mathbb{S}$ be an infinite fibered derivator with domain $\text{Dia}$ with stable, compactly generated fibers. Consider a morphism $D = (I, F) \to (\cdot, S)$. Let $S \to U$ be a universally $\mathbb{D}$-colocal morphism. Denote $D_U := D \times_{(\cdot, S)} (\cdot, U)$ in $\text{Dia}^{\text{op}}(\mathbb{S})$. Then the following diagram is commutative (via the exchange natural isomorphism):

$$
\begin{array}{ccc}
\mathbb{D}(D) & \xrightarrow{\square^*} & \mathbb{D}(D)^{\text{cocart}} \\
\pr^*_1 \downarrow & & \downarrow \pr^*_1 \\
\mathbb{D}(D_U) & \xrightarrow{\square^*} & \mathbb{D}(D_U)^{\text{cocart}}
\end{array}
$$

Note that by Theorem 3.3.1 right Cartesian projectors exist for $D$ and $D_U$.

Proof. The functor $\pr^*_1$ has a left adjoint $\pr_{1!}$ by (Dloc2 right) and Brown representability for the dual. (Dloc1 right) says that $\pr^*_1$ preserves Cartesian morphisms, hence $\pr_{1!}$ preserves coCartesian morphisms. Therefore the right adjoint of the given diagram is the following diagram:

$$
\begin{array}{ccc}
\mathbb{D}(D) & \xleftarrow{\pr_{1!}} & \mathbb{D}(D)^{\text{cocart}} \\
\pr_{1!} \uparrow & & \uparrow \pr_{1!} \\
\mathbb{D}(D_U) & \xleftarrow{\pr_{1!}} & \mathbb{D}(D_U)^{\text{cocart}}
\end{array}
$$

whose exchange morphism is an isomorphism. Consequently the exchange morphism of our diagram is also a natural isomorphism. \hfill \Box

Lemma 2.4.12 (left). Let $\mathbb{D} \to \mathbb{S}$ be a left fibered derivator with domain $\text{Dia}$ admitting a left Cartesian projection. For any Grothendieck opfibration

$$
\begin{array}{ccc}
I & \xrightarrow{\pi} & E \\
\downarrow & & \\
E
\end{array}
$$

in $\text{Dia}$, for any diagram in $F \in \mathbb{S}(I)$, and for each element $e \in E$, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{D}(I)_F & \xrightarrow{\square^E} & \mathbb{D}(I)^{E-\text{cart}}_F \\
\iota^* \downarrow & & \downarrow \iota^* \\
\mathbb{D}(I_e)_{F_e} & \xrightarrow{\square^0} & \mathbb{D}(I_e)^{\text{cart}}_{F_e}
\end{array}
$$

where $\iota : I_e \to I$ is the inclusion of the fiber.

Lemma 2.4.13 (right). Let $\mathbb{D} \to \mathbb{S}$ be a fibered derivator with domain $\text{Dia}$ admitting a right coCartesian projection. For a Grothendieck fibration

$$
\begin{array}{ccc}
I & \xrightarrow{\pi} & E \\
\downarrow & & \\
E
\end{array}
$$
in Dia, for any diagram in \( F \in S(I) \), and for each element \( e \in E \), the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{D}(I)_F & \xrightarrow{\xi^F} & \mathbb{D}(I)^{E,cart}_F \\
\downarrow \xi^e & & \downarrow \xi^e \\
\mathbb{D}(I_e)_{F_e} & \xrightarrow{\xi^e_*} & \mathbb{D}(I_e)^{cocart}_{F_e}
\end{array}
\]

where \( \iota : I_e \to I \) is the inclusion of the fiber.

Proof. We restrict to the right-variant, the other being dual. We will show that \( \iota! \) maps coCartesian objects to \( E \)-coCartesian ones. Then the left adjoint of the given diagram is the following diagram:

\[
\begin{array}{ccc}
\mathbb{D}(I)_F & \xleftarrow{\xi^F} & \mathbb{D}(I)^{E,cart}_F \\
\uparrow \iota_L & & \uparrow \iota_L \\
\mathbb{D}(I_e)_{F_e} & \xleftarrow{\xi^e_*} & \mathbb{D}(I_e)^{cocart}_{F_e}
\end{array}
\]

which is commutative and consequently also our diagram is commutative.

Let \( \alpha_k \) be the inclusions of \( \cdot \) into \( I \) with image \( i_k \). Let \( \nu : i_1 \to i_2 \) be a map of \( I \) mapping to an identity \( \text{id}_f \) of \( E \). It yields a natural transformation \( \nu : \alpha_1 \to \alpha_2 \).

Consider the diagram

\[
e \times / I E f \xrightarrow{\pi} I_e \times / I i_k \xrightarrow{A_k} I_e \\
\downarrow \rho' \quad \downarrow p_k \quad \downarrow \varphi^k \quad \downarrow \iota \\
i' \xrightarrow{\alpha_k} I
\]

where \( c_k \) is given on a morphism \( \beta : e \to f \) in \( E \) by the choice of a Cartesian arrow \( i'_k \to i_k \). It is right adjoint to \( \pi \) by the definition of Cartesian arrow.

There is a functor (composition with \( \nu \))

\[
\tilde{\nu} : I_e \times / I i_1 \to I_e \times / I i_2
\]

such that \( A_2 \tilde{\nu} = A_1 \) and \( p_2 \tilde{\nu} = p_1 \). We have therefore a natural (point-wise) coCartesian morphism \( S(\mu_1)_* \tilde{\nu}^* \to \tilde{\nu}^* S(\mu_2)_* \) of functors \( \mathbb{D}(I_e \times / I i_2)_{A^*_2 F_e} \to \mathbb{D}(I_e \times / I i_1) \).

We have also a natural transformation \( \rho : \tilde{\nu}c_1 \to c_2 \) given for a morphism \( \beta : e \to f \) in \( E \) by the unique arrow \( \rho(\beta) \) over \( \text{id}_e \) making the following diagram commutative:

\[
i'_1 \xrightarrow{c_1(\beta)} i_1 \\
\rho(\beta) \downarrow \quad \nu \downarrow \\
i'_2 \xrightarrow{c_2(\beta)} i_2
\]

The resulting morphism \( \mathbb{D}(\rho) : c_1^* \tilde{\nu}^* \to c_2^* \) is point-wise coCartesian on coCartesian objects.
We get a commutative diagram of natural transformations

\[
\begin{array}{c}
\xymatrix{ S(\mu_1) \cdot A_1^* \ar[r]^{\nabla(\mu_1)'} \ar[d]_{\sim} & p_1^* \alpha_1^* t_1 \ar[d]_{\sim} \\
S(\mu_1) \cdot \nabla^* A_2^* \ar[r]_{\nabla^* p_2^* \alpha_2^* t_1} \ar[d]_{\sim} & \nabla^* S(\mu_2) \cdot A_2^* \ar[r]_{\nabla^* (\nabla(\mu_2)')} \ar[d]_{\sim} & p_2^* \alpha_2^* t_1 \\
\nabla^* S(\mu_2) \cdot A_2^* \ar[r]_{\nabla^* (\nabla(\mu_2)')} \ar[d]_{\sim} & \nabla^* \nabla^* S(\mu_2) \cdot A_2^* \ar[r]_{\nabla^* (\nabla(\mu_2)')} \ar[d]_{\sim} & \nabla^* p_2^* \alpha_2^* t_1 \\
\end{array}
\]

where the first two top vertical morphisms are the natural isomorphisms induced by \(A_2 \nabla = A_1\), the third top vertical morphism is the natural isomorphism induced by \(p_2 \nabla = p_1\), and the first two lower vertical morphisms are point-wise coCartesian. Here we use the notation \(\nabla(\mu_1)\) for the morphism \(S(\mu_1) \cdot \nabla^* \rightarrow \nabla^* \cdot S(\mu_1)\) induced by \(D(\mu_1) : \nabla^* \rightarrow \nabla^*\).

Now we apply \(p_1\) to the outer square:

\[
\begin{array}{c}
p_1 S(\mu_1) \cdot A_1^* \ar[r]^{p_1 \nabla^* \alpha_1^* t_1} \ar[d]_{\sim} & p_1 \nabla^* \nabla^* p_2^* \alpha_2^* t_1 \ar[d]_{\sim} \\
p_1 \nabla^* S(\mu_2) \cdot A_2^* \ar[r]_{p_1 \nabla^* \nabla^* p_2^* \alpha_2^* t_1} \ar[d]_{\sim} & p_1 \nabla^* \nabla^* \nabla^* p_2^* \alpha_2^* t_1 \\
\end{array}
\]

The left vertical map is still coCartesian (Homotopy colimits preserve coCartesian morphisms). There is a canonical isomorphism \(p_1^! c_1^* \nabla^* \rightarrow p_2^!\) \[7, Prop. 1.23\] and \(\nabla(\rho) : c_1^* \nabla^* \rightarrow c_2^*\) is an isomorphism on coCartesian objects over constant diagrams. Consider the commutative diagram:

\[
\begin{array}{c}
p_1^! c_1^* \nabla^* \ar[r]^{\sim} \ar[d]_{\nabla(\rho)^{ad}} & p_2^! c_1^* \nabla^* \ar[r] \ar[d]_{\nabla(\rho)} & p_2! \ar[d]_{\nabla(\rho)} \\
p_2^! c_2^* \ar[r]^{\sim} & p_2^! c_2^* \nabla^* \ar[r] & p_2! \\
\end{array}
\]

where the rightmost horizontal morphisms are the respective counits. Since \(\nabla(\rho)\) is an isomorphism on coCartesian objects over constant diagrams, so is the morphism \(p_1^! c_1^* \nabla^* \rightarrow p_2!\). Now we have the commutative diagram

\[
\begin{array}{c}
p_1^! c_1^* \nabla^* \ar[r] \ar[d]_{\sim} & p_2! \ar[d]_{\sim} \\
p_{11} \nabla^* \ar[r] & p_{21} \nabla^* \\
\end{array}
\]

which shows that also the natural map \(p_{11} \nabla^* \rightarrow p_{21}\) is an isomorphism on coCartesian objects over constant diagrams.
We get a commutative diagram

\[
\begin{array}{ccc}
\pi_1! S(\mu_1) \bullet A_1^* & \overset{p_1! p_1^* \alpha_1^*}{\longrightarrow} & \alpha_1^* \\
\pi_1! S(\mu_2) \bullet A_2^* & \overset{p_1! p_2^* \alpha_2^*}{\longrightarrow} & \alpha_2^* \\
p_2! S(\mu_2) \bullet A_2^* & \overset{p_2! p_2^* \alpha_2^*}{\longrightarrow} & \alpha_2^* \\
\end{array}
\]

where the composition of the left vertical arrows is coCartesian on coCartesian objects because the functor \( S(\mu_2) \bullet A_2^* \) maps coCartesian objects to coCartesian objects over constant diagrams. The composition of the horizontal morphisms are isomorphisms by (FDer4 left). Hence the rightmost vertical map is coCartesian, too. This had to be shown.

\[\square\]

Proof of Main Theorem 2.4.5. (L1) is clear for weak and strong \( \mathbb{D} \)-equivalences.

For (L2 left), let \( D_1 = (I, F) \) and \( D_2 = (\{e\}, F(e)) \). The projection \( p \) and the inclusion \( i \) of the final object induce morphisms:

\[
\begin{array}{ccc}
D_1 & \overset{p}{\longrightarrow} & D_2 \\
\downarrow & & \downarrow \\
(\cdot, S) & \overset{p}{\longrightarrow} & (\cdot, S)
\end{array}
\]

We have \( p \circ i = \text{id} \) and a 2-morphism \( \beta : \text{id} \Rightarrow i \circ p \). Therefore the statement is clear for weak \( \mathbb{D} \)-equivalences and for strong \( \mathbb{D} \)-equivalences it follows from Lemma 2.4.8. (Actually \( (i \circ p)^* \) is left adjoint to the inclusion \( \mathbb{D}(D_1)_{\text{cart}} \to \mathbb{D}(D_1) \).)

(L3 left): Here the case of weak \( \mathbb{D} \)-equivalences is easier and requires less assumptions: Let \( w : D_1 \to D_2 \) be a morphism as in (L3 left) and assume we are given a base \( S \in S(\cdot) \) for \( w \). Then we automatically have a factorization

\[
\begin{array}{ccc}
D_1 & \overset{w}{\longrightarrow} & D_2 \\
\downarrow & & \downarrow \\
(\cdot, S) & \overset{w}{\longrightarrow} & (\cdot, S)
\end{array}
\]

We have to show that

\[ p_1! p_1^* \to p_2! p_2^* \]

is an isomorphism and it suffices to show that the morphism

\[ p_1^! (p_1^*)^* \to p_2^! (p_2^*)^* \]

is an isomorphism. This may be checked point-wise, so fix \( e \in E \) and let \( \iota_i : D_i \times/(e, p_E^* S) (e, S) \to D_i \) be the projection. Applying \( e^* \), we get

\[ e^* p_1^! (p_1^*)^* \to e^* p_2^! (p_2^*)^* \]

which is the same as

\[ (p_1^! (p_1^*)^*) e^* \to (p_2^! (p_2^*)^*) e^*. \]
By Lemma 1.6.7 this is induced by the canonical natural transformation which is an isomorphism by assumption.

Now we proceed to show (L3 left) for strong $\mathbb{D}$-equivalences under the additional assumptions. Consider the following diagram over $E$

\[
\begin{array}{ccc}
D_1 & \xrightarrow{w} & D_2 \\
\downarrow & & \downarrow \\
D_1 \times_{/\mathbb{D} E} D_2 & \xrightarrow{w'} & D_2 \times_{/\mathbb{D} E} D_2
\end{array}
\]

where the vertical morphisms are of pure diagram type. We have an adjunction

\[
I_i \xleftarrow{\kappa_i} I_i \times_{/\mathbb{D} E} E
\]

where $\kappa_i$ maps an object $i$ to $(i, \text{id}_{p(i)})$. We have a natural transformation $\kappa_i \circ \iota_i \Rightarrow \text{id}_{I_i \times_{/\mathbb{D} E} E}$ and $\iota_i \circ \kappa_i = \text{id}_{E}$. Actually this defines an adjunction with $\kappa_i$ left-adjoint to $\iota_i$. Furthermore, we get lifts to diagrams:

\[
D_i \xleftarrow{\overline{\kappa}_i} (I_i \times_{/\mathbb{D} E} E, \iota_i \circ F) = D_1 \times_{/\mathbb{D} E} E,
\]

a 2-morphism $\overline{\kappa}_i \circ \overline{\iota}_i \Rightarrow \text{id}_{D_1 \times_{/\mathbb{D} E} E}$, and we have $\overline{\iota}_i \circ \overline{\kappa}_i = \text{id}_{D_1}$.

Hence, by Lemma 2.4.8 the pullbacks along $\overline{\kappa}_i$ and $\overline{\iota}_i$ induce equivalences on Cartesian objects, so we are reduced to showing that $w'$ is an equivalence on Cartesian objects. Furthermore the underlying diagrams $I_k \times_{/\mathbb{D} E} E$ are Grothendieck opfibrations over $E$ and the functor underlying $w'$ is a map of Grothendieck opfibrations (the pushforward along a map $\mu : e \to f$ being given by mapping $(i, \nu : p(i) \to e)$ to $(i, \nu \circ \mu)$). Hence w.l.o.g. we may assume that $I_1 \to E$ is a Grothendieck opfibration and the morphism $I_1 \to I_2$ underlying $f$ is a morphism of Grothendieck opfibrations.

We keep the notation $w : D_1 \to D_2$, however, and the assumption translates to the statement that the composition

\[
\mathbb{D}(D_{2,e})^\text{cart} \xrightarrow{w_2^*} \mathbb{D}(D_{1,e})^\text{cart}
\]

for the fibers is an equivalence with inverse $\Box_{1,w_e}^*$. Consider the diagram

\[
\mathbb{D}(D_2)^E_{-\text{cart}} \xrightarrow{\text{incl}} \mathbb{D}(D_2) \xrightarrow{w_2^*} \mathbb{D}(D_1).
\]

We first show that the counit

\[
\Box_1^F w_1 w^* \mathcal{E} \to \mathcal{E}
\]

is an isomorphism for every $E$-Cartesian $\mathcal{E}$.

This can be checked after pulling back to the fibers. Let $\iota_k : I_{k,e} \to I_k$ be the inclusion of the fibre over some $e \in E$.

We have the isomorphisms:

\[
\iota_2^* \Box_1^F w_1 w^* \mathcal{E} \cong \Box_{1,w_e} \iota_1^* w^* \mathcal{E} \cong \Box_{1,w_e} w_e^* \iota_2^* \mathcal{E} \cong \iota_2^* \mathcal{E}.
\]

where we used the isomorphism $\iota_2^* \Box_1^F \cong \Box \iota_2^*$ (Lemma 2.4.12) and the isomorphism $\iota_2^* w_1 \cong w_e \iota_1^*$ (exists for pure diagram type, because we have a morphism of Grothendieck cofibrations, see Proposition 1.3.16 3. and for fixed shape by axiom (FDer0 left)). The morphism $\Box_{1,w_e} w_e^* \mathcal{E} \to \mathcal{E}$ is an isomorphism for Cartesian $\mathcal{E}$ by assumption.
We now show that the unit
\[ \mathcal{E} \to w^* \square_1^E w! \mathcal{E} \]
is an isomorphism for \( \mathcal{E} \) \( E \)-Cartesian. This can be checked again on the fibers:
\[ i_1^* w^* \square_1^E w! \mathcal{E} \cong w_e^* i_2^* \square_1^E w! \mathcal{E} \cong w_e^* \square_1 w_e^* i_1^* \mathcal{E} \cong i_1^* \mathcal{E} . \]
Therefore we have already proven that the functors
\[ \mathbb{D}(D_2)^{E-cart} \xrightarrow{w^*} \mathbb{D}(D_1)^{E-cart} \]
are an equivalence. We conclude by showing that \( \square_1^E w! \) maps Cartesian objects to Cartesian objects:
Let \( \nu : e \to f \) be a morphisms of \( E \). It induces a morphism (choice of push-forward for \( I_k \to E \))
\[ \overline{\nu}_k : D_{k,e} \to D_{k,f} \]
(not of diagram type!) and a 2-morphism: \( i_{k,e} \to i_{k,f} \circ \overline{\nu}_k \).

\textbf{Claim:} It suffices to show that for all \( \nu : e \to f \) the induced morphism
\[ i_{2,e}^* \square_1^E w! \mathcal{E} \to \overline{\nu}_2^* i_{2,f}^* \square_1^E w! \mathcal{E} \]
is an isomorphism for every Cartesian \( \mathcal{E} \).

\textbf{Proof of the claim:} Every morphism \( \mu : i \to i'' \) in \( I \) with \( p(\mu) = \nu \), say, is the composition of a coCartesian \( \mu' \) and some morphism \( \mu'' \) with \( p(\mu'') = \text{id}_f \). Since \( \mathcal{E} \) is \( E \)-Cartesian, the morphism \( \mathcal{E}(\mu'') \) is Cartesian. Hence to show that \( \mathcal{E}(\mu) \) is Cartesian it suffices to see that \( \mathcal{E}(\mu') \) is Cartesian. A reformulation is, however, that the morphism of the claim be an isomorphism.

Using the same argument as in the first part of the proof, we have to show that
\[ \square_1 w_e^* i_{1,e}^* \mathcal{E} \to \overline{\nu}_2^* \square_1 w_f^* i_{1,f}^* \mathcal{E} \]
is an isomorphism for every Cartesian \( \mathcal{E} \).
Since both sides are Cartesian objects, this can be checked after applying \( w_e^* \) which is an equivalence on Cartesian objects.

\[ w_e^* \square_1 w_e^* i_{1,e}^* \mathcal{E} \to w_e^* \overline{\nu}_2^* \square_1 w_f^* i_{1,f}^* \mathcal{E} \]
We have \( w_e^* \overline{\nu}_2^* = \overline{\nu}_1^* w_e^* \) because the map of diagrams underlying \( w \) is a morphism of Grothendieck op fibrations. Hence, after applying \( w_e^* \), we get
\[ w_e^* \square_1 w_e^* i_{1,e}^* \mathcal{E} \to \overline{\nu}_1^* w_f^* \square_1 w_f^* i_{1,f}^* \mathcal{E} \]
Since \( w_e^* \square_1 w_e^* \) and \( w_f^* \square_1 w_f^* \) are equivalences on Cartesian objects, we get
\[ i_{1,e}^* \mathcal{E} \to \overline{\nu}_1^* i_{1,f}^* \mathcal{E} \]
A slightly tedious check shows that this is again the morphism induced by the 2-morphism \( i_{1,e} \to i_{1,f} \circ \overline{\nu}_1 \). It is an isomorphism because \( \mathcal{E} \) is Cartesian.

\textbf{(L4 left)} Let \( w \) be a morphism as in (L4 left) and let \( T \) be a base for \( w \), i.e. we have again a factorization
\[ D_1 \xrightarrow{w} D_2 \]
\[ \xymatrix{ D_1 \ar[rr]^w & & D_2 \\
S \times T \ar[rru]^{p'_1} \ar[rru]^{p'_2} \ar[u]_{p_1} \ar[d]_p \ar[rrl] & & D_2 \ar[lll]_{p_2} \ar[u]_{p_2} } \]
\[ 44 \]
For any $i$ (index of the cover in $L4$ left) we have the following commutative diagrams of objects in $\text{Dia}(\mathcal{S})$:

\[
\begin{array}{ccc}
D_1 \times_S U_i & \xrightarrow{w_i} & D_2 \times_S U_i \\
pr_1^{(i)} \downarrow & & \downarrow pr_1^{(i)} \\
D_1 & \xrightarrow{w} & D_2
\end{array}
\]

The morphisms $pr_1^{(i)}$ are of fixed shape. We also have the diagram:

\[
\begin{array}{ccc}
D_1 \times_S U_i & \xrightarrow{w} & D_2 \times_S U_i \\
pr'_1 \downarrow & & \downarrow pr'_1 \\
U_i \times T & \xrightarrow{p'_2} & U_i \times T
\end{array}
\]

Again the case of weak $\mathbb{D}$-equivalences is easier and requires less assumptions. It suffices to show that the morphism induced by $w$

\[p'_1(p'_1)^* \to p_2(p'_2)^*\]

is an isomorphism.

Since $\mathbb{D}$ is local, this may be checked after pulling back along the cover $\iota_i: U_i \times T \to S \times T$, i.e. the morphism

\[\iota_i^*p'_1(p'_1)^* \to \iota_i^*p'_2(p'_2)^*\]

has to be an isomorphism. Applying Proposition 1.6.9, 2. we get the morphism induced by $w_i$:

\[P'_1(P'_1)^* \iota_i^* \to P'_2(P'_2)^* \iota_i^*\]

This morphism is an isomorphism by assumption.

We now proceed to the case of strong $\mathbb{D}$-equivalences. We first show that the unit is an isomorphism:

\[\mathcal{E} \to w^* \Box_1 w_! \mathcal{E}\]

for any Cartesian $\mathcal{E}$. Note that by the stability axiom of a Grothendieck pre-topology also the collections $(D_1 \times_S U_i)_j \to D_{1,j}$ are covers for any $j \in I_1$, where $I_1$ is the underlying diagram of $D_1$.

Since $\mathbb{D}$ is local w.r.t. the Grothendieck pre-topology (and by axiom Der2), the family $(pr_1^{(i)})^*$ is jointly conservative. Therefore it suffices to show that the unit is an isomorphism after applying $(pr_1^{(i)})^*$. We get

\[(pr_1^{(i)})^* \mathcal{E} \to (pr_1^{(i)})^*w^* \Box_1 w_! \mathcal{E}\]

which is

\[(pr_1^{(i)})^* \mathcal{E} \to (pr_1^{(i)})^*w_!(pr_1^{(i)})^* \Box_1 w_! \mathcal{E}\]

and since $(pr_1^{(i)})^*$ commutes with $\Box_1$ (Lemma 2.4.10) and with $w_!$ (Theorem 1.6.9, 2.), we get:

\[(pr_1^{(i)})^* \mathcal{E} \to w_!^* \Box_1 w_!((pr_1^{(i)})^* \mathcal{E}\]

This morphism is an isomorphism by assumption.

In the same way the counit is an isomorphism.

(L5 left): For the case of weak $\mathbb{D}$-equivalences we will show that the morphism

\[p_!p^* \to \text{id}\]
where \( p : (I, p^* F) \to (E, F) = D \), is an isomorphism. This is the same as showing that

\[
\text{id} \to p_* p^*
\]

be an isomorphism. Note that \( p_* \) exists because this is a morphism of diagram type and \( \mathcal{D} \to S \) is assumed to be a right fibered derivator, too (this is the only place, where this assumption is used for the case of weak \( \mathbb{D} \)-equivalences). Now, since \( p \) is a Grothendieck fibration, \( p_* \) can be computed fiber-wise. So we have to show that

\[
\text{id} \to p_{e,*} p^*_e
\]

is an isomorphism, or equivalently, that

\[
p_{e,*} p^*_e \to \text{id}
\]

is an isomorphism. Since by assumption the map of fibers \( I_e \to e \) is in \( \mathcal{W}^\text{min}_{\text{Dia}} \), we get is an isomorphism for all right derivators. For the case of strong \( \mathbb{D} \)-equivalences: We have shown that

\[
p_* p^* \to \text{id}
\]

is an isomorphism, hence on Cartesian objects the same holds for

\[
\Box \ p_* p^* \to \text{id}
\]

We have to show that also the counit

\[
\text{id} \to p^* \Box \ p^* \tag{7}
\]

is an isomorphism on Cartesian objects. First note that \( p_* \) also is a right adjoint of \( p^* \) when restricted to the full subcategories of Cartesian objects because \( p_* \) preserves Cartesian objects. Indeed, \( p_* \) can be computed fiber-wise because \( p \) is a Grothendieck fibration. The fibers being contractible in the sense of any localizer on \( \text{Dia} \) implies that \( p^*_e, p_{e,*} \) induce an equivalence \( \mathbb{D}(D_e)_{\text{cart}} \cong \mathbb{D}(-)_{F(e)} \). Note: This uses that (L1–L3 left) hold for the class of strong \( \mathbb{D} \)-equivalences on the fibre \( \mathbb{D}_{F(e)} \), a fact which has been proven already. Therefore we pass to the right adjoints of the functors in (7) and have to show that the counit

\[
p^* p_* \to \text{id}
\]

is an isomorphism on Cartesian objects. Again this can be checked fiber-wise, i.e. we have to show that the counit

\[
p^*_e p_{e,*} \to \text{id}
\]

is an isomorphism on Cartesian objects. But the pair of functors is an equivalence as we have seen, and the claim follows.

We proceed to state some consequences of the fact that weak \( \mathbb{D} \)-equivalences form a fundamental localizer.

**Example 2.4.14** (Mayer-Vietoris). Let \( S \) be a strong right derivator (e.g. associated with a category with limits) with Grothendieck pre-topology. We saw in Example 2.1.8 that for a cover \( \{U_1 \to S, U_2 \to S\} \) consisting of 2 monomorphisms, the projection

\[
p : \begin{pmatrix}
\text{“}U_1 \times_S U_2\text{”} & \to & U_1 \\
\downarrow & & \downarrow \\
U_2 & & \\
& & \to S
\end{pmatrix}
\]
belongs to any fundamental localizer. If $D \to S$ is a fibered derivator which is local w.r.t. the pre-topology on $S$, Theorem 2.4.5 implies therefore that $p$ is a weak $D$-equivalence in $\text{Dia}(S)/(\cdot, S)$, i.e. for $A \in D(\cdot)_S$ we have

$$ p\pi^*A \cong A, $$

i.e. the homotopy colimit of

$$ i_{1,2,*}i_{1,2}^!A \rightarrow i_{1,*}i_{1}^!A $$

is isomorphic to $A$. If $D$ has stable fibers, this translates into the usual distinguished triangle

$$ i_{1,2,*}i_{1,2}^!A \rightarrow i_{1,*}i_{1}^!A \oplus i_{2,*}i_{2}^!A \rightarrow A \rightarrow i_{1,2,*}i_{1,2}^!A[1] $$
in the language of triangulated categories.

Dually, if $D \to S^{\text{op}}$ is a fibered derivator colocal w.r.t. the pre-cotopology on $S^{\text{op}}$, Theorem 2.4.4 implies therefore that $p^{\text{op}}$ is a weak $D$-equivalence in $\text{Dia}^{\text{op}}(S^{\text{op}})/(\cdot, S)$, i.e. for $A \in D(\cdot)_S$ we have

$$ A \cong p_*\pi^*A, $$

i.e. the homotopy limit of

$$ i_{1,*}i_{1}^*A \rightarrow i_{2,*}i_{2}^*A \rightarrow i_{2,*}i_{1,2}^*A $$
is isomorphic to $A$. If $D$ has stable fibers, this translates into the usual distinguished triangle

$$ A \rightarrow i_{1,*}i_{1}^*A \oplus i_{2,*}i_{2}^*A \rightarrow i_{1,2,*}i_{1,2}^*A \rightarrow A[1] $$
in the language of triangulated categories. Note that $i_*$ denotes a left adjoint push-forward along a morphism in $S^{\text{op}}$, i.e. a left adjoint pull-back along a morphism in $S$.

**Example 2.4.15** ((Co)homological descent). Let $S$ be a strong right derivator and let $X_\bullet \in S(\Delta^{\text{op}})$

$$ \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 $$

be such that $p:\left(\Delta^{\text{op}}, X_\bullet\right) \rightarrow (\Delta^{\text{op}}, \pi^*S)$ is a finite hypercover. If $D \to S$ is a fibered derivator local w.r.t. the pre-topology on $S$, Theorem 2.4.5 implies therefore that $\pi \circ p$ is a weak $D$-equivalence in $S(\cdot)/(\cdot, S)$, i.e. for $A \in D(\cdot)_S$ we have

$$ A \cong \pi_0\pi p^*\pi^*A, $$

i.e. the homotopy colimit of $p\pi^*\pi^*A$ is equal to $A$. If the fibers of $D \to S$ are in fact derived categories, this yields a spectral sequence of homological descent because the homotopy colimit over a simplicial complex is the total complex of the associated double complex (a well-known fact). This double complex looks like

$$ \cdots \rightarrow p_2,*p_2^!A \rightarrow p_1,*p_1^!A \rightarrow p_0,*p_0^!A. $$

The point is that we get a coherent double complex. Knowing the individual morphisms $p_i,*p_i^!A \rightarrow p_{i-1,*}p_{i-1}^!A$ as morphisms in the derived category $D(\cdot)_A$ would not be sufficient!

Dually (applying everything to a fibered derivator $D \to S^{\text{op}}$, and working in $\text{Dia}^{\text{op}}(S^{\text{op}})$), one obtains the more classical spectral sequence of cohomological descent.
We proceed to state some consequences of the fact that strong $\mathbb{D}$-equivalences form a fundamental localizer.

**Corollary 2.4.16** (left). Let $\mathcal{S}$ be a strong right derivator. If $\mathbb{D} \to \mathcal{S}$ is an infinite fibered derivator which is local w.r.t. the pre-topology on $\mathcal{S}$ (1.3.2) with stable, compactly generated fibers then for any finite hypercover $f : X_\bullet \to Y_\bullet$ considered in $\text{Dia}(\mathcal{S})'$, the functor $f^*$ induces an equivalence

$$
\mathbb{D}(Y_\bullet)^\text{cart} \to \mathbb{D}(X_\bullet)^\text{cart}.
$$

Here $\text{Dia}(\mathcal{S})'$ is the full subcategory of diagrams with universally $\mathbb{D}$-local morphisms (cf. 2.1.3).

**Corollary 2.4.17** (right). Let $\mathcal{S}$ be a strong right derivator. If $\mathbb{D} \to \mathcal{S}^{\text{op}}$ is an infinite fibered derivator which is colocal w.r.t. the pre-cotopology on $\mathcal{S}^{\text{op}}$ (1.5.2) with stable, compactly generated fibers then for any finite hypercover $f : X_\bullet \to Y_\bullet$ considered in $\text{Dia}^{\text{op}}(\mathcal{S}^{\text{op}})$, the functor $f^*$ induces an equivalence

$$
\mathbb{D}(Y_\bullet)^\text{cocart} \to \mathbb{D}(X_\bullet)^\text{cocart}.
$$

**Corollary 2.4.18.** If $\mathbb{D}$ is an infinite derivator (not fibered) with domain $\text{Cat}$ with stable, well-generated fibers, then for each homotopy type $I$, we get a category $\mathbb{D}(I)^\text{cart}$ well-defined up to equivalence of categories. Moreover each morphism $I \to J$ of homotopy types gives rise to a corresponding functor $\alpha^*: \mathbb{D}(J)^\text{cart} \to \mathbb{D}(I)^\text{cart}$. It is, however, not possible to arrange those as a pseudo functor $\text{HOT} \to \text{CAT}$.

3 Representability

In this section we exploit the consequences that Brown representability type results have for fibered derivators. In particular these results are useful to see that under certain circumstances a left fibered (multi-)derivator is already a right fibered (multi-)derivator, provided that its fibers are nice (i.e. stable and well-generated derivators). Furthermore they provide us with (co)Cartesian projectors that are needed for the strong form of (co)homological descent. In contrast to the rest of the article the results are stated in a rather unsymmetric form. This is due to the fact that in applications the stable derivators will rather be well-generated or compactly generated whereas their duals will rather not satisfy this condition. All the results used are taken from [14] and [18].

3.1 Well-generated triangulated categories and Brown representability

**Definition 3.1.1** ([14, 5.1, 6.3]). Let $\mathcal{D}$ be a category with zero object and small coproducts. We call $\mathcal{D}$ perfectly generated if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that the following conditions hold

1. An object $X \in \mathbb{D}(\cdot)$ is zero if and only if $\text{Hom}(T, X) = 0$ for all $T \in \mathcal{T}$;

2. If $\{X_i \to Y_i\}$ is any set of maps, and $\text{Hom}(T, X_i) \to \text{Hom}(T, Y_i)$ is surjective for all $i$, then $\text{Hom}(T, \bigsqcup_i X_i) \to \text{Hom}(T, \bigsqcup_i Y_i)$ is also surjective.

$\mathcal{D}$ is called well-generated, if there is a set of objects $\mathcal{T}$ in $\mathcal{D}$ such that in addition to 1., 2. there is a regular cardinal $\alpha$ such that the following condition holds:

3. All $T \in \mathcal{T}$ are $\alpha$-small [14, 6.3].
It is called **compactly generated**, if there is a set of objects \( T \) in \( D \) such that in addition to 1., 2. the following equivalent conditions hold:

4. All \( T \in T \) are \( \aleph_0 \)-small.

4'. All \( T \in T \) are compact, i.e. for each morphism \( \gamma : T \to \bigsqcup_{i \in I} X_i \) there is a finite subset \( J \subseteq I \) such that \( \gamma \) factors through \( \bigsqcup_{i \in J} X_i \).

**Definition 3.1.2.** A pre-derivator \( D \) whose domain \( \Dia \) is infinite (i.e. closed under infinite disjoint unions) is called **infinite** if the restriction-to-\( I_j \) functors induce an equivalence:

\[
D(\bigsqcup_{j \in J} I_j) \cong \prod_{j \in J} D(I_j)
\]

for all sets \( J \).

Recall [14, 4.4] that a functor from a triangulated category \( D \) to an abelian category is called **cohomological** if it sends distinguished triangles to exact sequences.

We recall the following theorem

**Theorem 3.1.3** (right Brown representability). *Let \( D \) be a perfectly generated triangulated category with small coproducts. Then a functor \( F : D^{op} \to AB \) is cohomological and sends coproducts to products if and only if \( F \) is representable. An exact functor \( D \to E \) between triangulated categories commutes with coproducts if and only if it has a right adjoint.*

*Proof.* [14 Theorem 5.1.1]

It can be shown that for a compactly generated triangulated category \( D \) with small coproducts, \( D^{op} \) is perfectly generated and has small coproducts. Therefore the dual version of the previous theorem holds in this case:

**Theorem 3.1.4** (left Brown representability). *Let \( D \) be a compactly generated triangulated category with has small coproducts. Then a functor \( F : D \to AB \) is homological and sends products to products if and only if \( F \) is representable. An exact functor \( D \to E \) between triangulated categories commutes with products if and only if it has a left adjoint.*

**Theorem 3.1.5.** *Let \( D \) be a well-generated triangulated category with small coproducts. Consider a functor \( F : D \to AB \) which is cohomological and commutes with coproducts. Then there exists a right adjoint to the inclusion of the full subcategory of objects \( X \) such that \( F(X[n]) = 0 \) for all \( n \in \mathbb{Z} \) (i.e. this subcategory is reflective).*

*Proof.* [14 Theorem 7.1.1]

**Lemma 3.1.6.** *Let \( \Dia \) be an infinite diagram category \([1.1.1]\). Let \( D \to S \) be an infinite left fibered derivator with domain \( \Dia \). If \( D(\cdot) \) for all \( S \in S(\cdot) \) is perfectly generated (resp. well-generated, resp. compactly generated), then the same holds for \( D(I)_{S'} \) for all \( I \in \Dia \) and \( S' \in S(I) \). Furthermore the \( D(I)_{S'} \) all have small coproducts.*

*Proof.* A set of generators, as requested, is given by the set \( \mathcal{T}_I := \{ iT \}_{i \in I, T \in T} \). Indeed, an object \( X \in D(I) \) is zero if \( i^* X \) is zero for all \( i \in X \) by (Der2). Therefore \( X \) is zero, if \( \Hom(iT, X) = \Hom(T, i^* X) = 0 \) for all \( i \in I \) and for every \( T \in T \). We have to show that \( \Hom(iT, \bigsqcup_i X_i) \to \Hom(iT, \bigsqcup_i Y_i) \) is an isomorphism for a family \( \{ X_i \to Y_i \}_{i \in O} \) of morphisms as in 2. We have
Hom\((i_T, \bigsqcup_I X_i)\) = Hom\((T, i^* \bigsqcup_I X_i)\) = Hom\((T, \bigsqcup_I i^* X_i)\), where we used that \(i^*\) commutes with coproducts. This follows because the Cartesian diagram

\[
\begin{array}{ccc}
O & \longrightarrow & O \times I \\
\downarrow & & \downarrow \\
I & \longrightarrow & I
\end{array}
\]

is homotopy exact. Note that, since \(\mathbb{D}\) is infinite, coproducts exist and are equal to the corresponding homotopy coproducts. The map Hom\((T, \bigsqcup_I i^* X_i)\) → Hom\((T, \bigsqcup_I i^* Y_i)\) is surjective by assumption. We have to show that a morphism

\[
i_T \to \bigsqcup_{i \in I} Y_i
\]

in \(\mathbb{D}(I)_{S'}\) factors trough \(\bigsqcup_{i \in J} Y_i\) for some subset \(J \subset I\) of cardinality less than \(\alpha\). By the same reasoning as above, we get a morphism

\[
T \to \bigsqcup_{i \in I} i^* Y_i
\]

Hence, there is some subset \(J \subset I\), as required, such that this morphism factors through it. The same then holds for the original morphism. Since there is no need to enlarge \(J\), the same statement holds for finite subsets.

The categories \(\mathbb{D}(I)_{S'}\) have small coproducts because \(\mathbb{D} \to \mathbb{S}\) is infinite and left fibered. □

**Definition 3.1.7.** Let \(\mathbb{D} \to \mathbb{S}\) be an infinite left fibered derivator with domain \(\mathbb{D}\). We will say that \(\mathbb{D} \to \mathbb{S}\) has **perfectly-generated** (resp. **well-generated**, resp. **compactly-generated**) fibers, if all categories \(\mathbb{D}(\cdot)_{S}\) are perfectly-generated (resp. well-generated, resp. compactly-generated) for all \(S \in \mathbb{S}(\cdot)\). It follows from the previous Lemma that, in this case, for all \(I \in \mathbb{D}\) and \(S' \in \mathbb{S}(I)\) the category \(\mathbb{D}(I)_{S'}\) is also perfectly-generated (resp. well-generated, resp. compactly-generated).

### 3.2 Left and Right

**Theorem 3.2.1** (left). Let \(\mathbb{D}\) be an infinite diagram category \(\mathbb{D}(I)\). Let \(\mathbb{D}\) and \(\mathbb{E}\) be infinite left derivators with domain \(\mathbb{D}\) such that for all \(I \in \mathbb{D}\), the pre-derivators \(\mathbb{D}_I\) and \(\mathbb{E}_I\) are stable (left and right) derivators with domain \(\mathbb{Posf}\). Assume that \(\mathbb{D}\) is perfectly generated. Then a morphism of derivators \(F: \mathbb{D} \to \mathbb{E}\) commutes with all homotopy colimits w.r.t. \(\mathbb{D}\) if and only if it has a right adjoint.

**Proof.** Let \(I\) be in \(\mathbb{D}\). Since \(\mathbb{D}_I\) and \(\mathbb{E}_I\) are stable, \(\mathbb{D}(I)\) is canonically triangulated, and we may use Theorem 3.1.3 of right Brown representability. It follows that the functor \(F(I): \mathbb{D}(I) \to \mathbb{E}(I)\) has a right adjoint \(G(I)\), because it is triangulated, commutes with small coproducts and \(\mathbb{D}(I)\) is perfectly generated. To construct a morphism of derivators out of this collection, for any \(\alpha: I \to J\), we have to give an isomorphism: \(G(J)\alpha^* \to \alpha^* G(I)\). We take the adjoint of the isomorphism \(\alpha_! F(J) \to F(I)\alpha_!\) expressing that \(F\) commutes with all homotopy colimits (see [7 Lemma 2.1] for details).

Analogously, using Theorem 3.1.4 of left Brown representability, we obtain:

**Theorem 3.2.2** (right). Let \(\mathbb{D}\) be an infinite diagram category \(\mathbb{D}(I)\). Let \(\mathbb{D}\) and \(\mathbb{E}\) be infinite right derivators with domain \(\mathbb{D}\) such that for all \(I \in \mathbb{D}\), the pre-derivators \(\mathbb{D}_I\) and \(\mathbb{E}_I\) are stable (left and right) derivators with domain \(\mathbb{Posf}\). Assume that \(\mathbb{D}\) is compactly generated. Then a morphism of derivators \(F: \mathbb{D} \to \mathbb{E}\) commutes with all homotopy limits w.r.t. \(\mathbb{D}\) if and only if it has a left adjoint.
Theorem 3.2.3 (left). Let $\text{Dia}$ be an infinite diagram category. Let $D \to S$ be an infinite left fibered (multi-)derivator with domain $\text{Dia}$ whose fibers $D_S$ for all $S \in S(I)$ and every $I \in \text{Dia}$ are stable (left and right) derivators with domain $\text{Posf}$. Assume that $D$ has perfectly generated fibers. Then $D$ is a right fibered (multi-)derivator, too.

Proof. Let $I \in \text{Dia}$ and let $f \in \text{Hom}_{S(I)}(S_1, \ldots, S_n; T)$ be a multimorphism. By (FDer0 left) $f$ induces a push-forward $f_*: D_{S_1} \times \cdots \times D_{S_n} \to D_T$ between the corresponding fibers. These fibers satisfy the assumptions of Theorem 3.2.1 because by Proposition 1.3.19 (resp. FDer5) the “push-forward” $f_*$ commutes with homotopy colimits. Therefore it has a right adjoint $f^*$ w.r.t. any slot. That $f^* J$ is a morphism of derivators implies the axiom (FDer0 right). Similarly $\alpha : J \to I$ in $\text{Dia}$ induces a morphism of derivators $\alpha^*: D_S \to D_{\alpha^* S}$. It commutes with homotopy colimits by 1.3.16. Therefore $\alpha^*$ has a right adjoint $\alpha^*$ by the previous theorem, i.e. (FDer3 right) holds. Finally property (FDer5 right) is then a consequence of Lemma 1.3.16 1. Finally property (FDer5 right) is the adjoint of property (FDer5 left).

Analogously, using Theorem 3.1.4 of left Brown representability, we obtain:

Theorem 3.2.4 (right). Let $\text{Dia}$ be an infinite diagram category. Let $D \to S$ be an infinite right fibered (multi-)derivator with domain $\text{Dia}$, whose fibers $D_S$ for all $S \in S(I)$ and every $I \in \text{Dia}$ are stable (left and right) derivators with domain $\text{Posf}$. Assume that $D$ has compactly generated fibers. Then $D$ is a left fibered (multi-)derivator, too.

3.3 (Co)Cartesian projectors

Theorem 3.3.1 (right). Let $D \to S$ be an infinite fibered left derivator (w.r.t. $\text{Dia}$) whose fibers are stable derivators w.r.t. $\text{Posf}$. Assume that $D(\cdot)_S$ is well-generated for every $S \in S(\cdot)$. Then the fully-faithful inclusion

\[
D(I)_F^{\text{E-cocart}} \to D(I)_F
\]

has a right adjoint $\square^F$ for all functors $I \to E$ in $\text{Dia}$ and every $F \in S(I)$.

If $D \to S$ also satisfies (FDer0 right) and if $F$ is such that $F(\mu)$ satisfies (Dloc2 left) for every $\mu$ that maps to an identity in $E$, then the fully-faithful inclusion

\[
D(I)_F^{\text{E-cart}} \to D(I)_F
\]

has a right adjoint $\square^F$.

Proof. Consider the set $O$ of morphisms $\mu : i \to j$ which maps to an identity in $E$. For each morphism $\mu \in O$, we consider the composition $D_\mu$:

\[
\begin{array}{c}
D(I)_F \\
\xrightarrow{\mu^*} D(\rightarrow)_{\mu^* F} \\
\xrightarrow{F(\mu)*} D(\rightarrow)_{F^* F} \\
\xrightarrow{\text{Cone}} D(\cdot)_{i^* F}
\end{array}
\]

and the functor $D$ defined as

\[
\prod_{\mu \in O} D_\mu : D(I)_F \to \prod_{\mu \in O} D(\cdot)_{i^* F} = D(O)_{i^* F},
\]

where $i : O \to I$ is the map “source”. $D$ commutes with coproducts, as all functors in the succession do, and it is exact. Therefore, by Theorem 7.4.1, the triangulated subcategory $D(I)_F^{\text{E-cocart}} = \ker D$ is well-generated and hence the inclusion in the statement of the Theorem has a right adjoint.

In the Cartesian case, $F(\mu)^*$ commutes with coproducts only if $F(\mu)$ satisfies (Dloc2 left).

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Theorem 3.3.2 (left). Let $\mathbb{D} \to \mathbb{S}$ is an infinite fibered derivator (with domain $\text{Dia}$) whose fibers are stable. Assume that all $\mathbb{D}(\cdot)_S$ for $S \in \mathbb{S}(\cdot)$ are compactly generated.

Let $I \to E$ be a functor in $\text{Dia}$ and let $F \in \mathbb{S}(I)$, such that $F(\mu)$ satisfies (Dloc2 left) for every morphism $\mu$ in $I$ that maps to an identity in $E$. Then the fully-faithful inclusion

$$\mathbb{D}(I)_E^{\text{cart}} \to \mathbb{D}(I)_F$$

has a left adjoint $\Box_I^E$.

Proof. As in the proof of the previous theorem we have an exact functor which commutes with coproducts

$$F^{\text{cart}} : \mathbb{D}(I)_F \to \mathcal{T}$$

into another triangulated category such that the subcategory of $E$-Cartesian objects is precisely its kernel. Lemma 3.1.6 implies that $\mathbb{D}(I)_F$ is compactly generated, and hence $\mathbb{D}(I)_F^{\text{op}}$ is perfectly generated. Furthermore, Theorem 3.3.1 implies that $\mathbb{D}(I)_F^{\text{op}}/\mathbb{D}(I)_F^{\text{cart}}$ are locally small. Note that

$$\mathbb{D}(I)_F^{\text{op}}/(\mathbb{D}(I)_F^{\text{cart}})^{\text{op}} = (\mathbb{D}(I)_F^{\text{cart}}/\mathbb{D}(I)_F^{\text{cart}})^{\text{op}}$$

Therefore [14, Proposition 5.2.1] implies that a right adjoint to the inclusion $(\mathbb{D}(I)_F^{\text{cart}})^{\text{op}} \to (\mathbb{D}(I)_F)^{\text{op}}$ exists. So a left adjoint to the inclusion:

$$\mathbb{D}(I)_F^{\text{cart}} \to \mathbb{D}(I)_F$$

exists. For the coCartesian case one argues analogously.

Remark 3.3.3. We have the feeling that, in the compactly generated case, $\Box_I^E$ should exist unconditionally but were not able to prove this.

4 Constructions

4.1 The fibered multiderivator associated with a fibered multicategory

4.1.1. The most basic situation in which a (non-trivial) fibered (multi-)derivator can be constructed arises from a bifibration of (locally small) multicategories

$$p : \mathcal{D} \to \mathcal{S}$$

where we are given a set of weak equivalences $\mathcal{W}_S \subset \text{Mor}(\mathcal{D}_S)$ for each object $S$ of $\mathcal{S}$. In the examples we have in mind, the objects of $\mathcal{S}$ are spaces (or schemes), the objects of $\mathcal{D}$ are chain complexes of sheaves (coherent, etale Abelian, etc.) on them, and the morphisms in $\mathcal{W}_S$ are the quasi-isomorphisms. In these examples the multicategory-structure arises from the tensor product and is even, in most cases, the more natural structure because no particular tensor-product is chosen a priori.

Definition 4.1.2. In the situation above, let $\mathcal{S}$ be the pre-multiderivator associated with the multicategory $\mathcal{S}$. We define a pre-multiderivator $\mathbb{D}$ as follows (cf. A.2.1 for localizations of multicategories):

$$\mathbb{D}(I) = \text{Hom}(I, \mathcal{D})[\mathcal{W}_I^{-1}]$$

where $\mathcal{W}_I$ is the set of natural transformations which are element-wise in the union $\bigcup_S \mathcal{W}_S$. The functor $p$ obviously induces a morphism of pre-multiderivators

$$\bar{p} : \mathbb{D} \to \mathbb{S}$$

Observe that morphisms in $\mathcal{W}_I$, by definition, necessarily map to identities in $\text{Hom}(I, \mathcal{S})$. 

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In this section we prove that the above morphism of pre-(multi-)derivators is a left (resp. right) fibered (multi-)derivator on directed (resp. inverse) diagrams, provided that the fibers are model categories whose structure is compatible with the structure of bifibration. We use the definition of a model category from [13]. We also denote the cofibrant replacement functor by $Q$ and the fibrant replacement functor by $R$.

**Definition 4.1.3.** A bifibration of (multi-)model-categories is a bifibration of (multi-)categories $p: D \to S$ together with the collection for any object $S$ in $S$ a closed model structure $(D_S, \text{Cof}_S, \text{Fib}_S, W_S)$ on the fibre such that

1. for any $n \geq 1$ and for every multimorphism

$S_1 \xrightarrow{f} T \xleftarrow{f} S_n$

the push-forward $f_\bullet$ and the various pull-backs $f^\bullet_j$ define a Quillen adjunction in $n$-variables

$$\prod_i (D_{S_i}, \text{Cof}_{S_i}, \text{Fib}_{S_i}, W_{S_i}) \xrightarrow{f_\bullet} (D_T, \text{Cof}_T, \text{Fib}_T, W_T)$$

$$(D_T, \text{Cof}_T, \text{Fib}_T, W_T) \times \prod_{i \neq j} (D_{S_i}, \text{Cof}_{S_i}, \text{Fib}_{S_i}, W_{S_i}) \xrightarrow{f^\bullet_j} (D_{S_j}, \text{Cof}_{S_j}, \text{Fib}_{S_j}, W_{S_j})$$

2. For any 0-ary morphism $f$ in $S$, let $f_\bullet()$ be the corresponding unit object (i.e. the object representing the 0-ary morphism functor $\text{Hom}_f(\cdot, \cdot)$) and the cofibrant replacement $Qf_\bullet() \to f_\bullet()$. Then the natural morphism

$$F_\bullet(X_1, \ldots, X_{i-1}, Qf_\bullet(), X_i, \ldots, X_n) \to F_\bullet(X_1, \ldots, X_{i-1}, f_\bullet(), X_i, \ldots, X_n) \cong (F \circ f)_\bullet(X_1, \ldots, X_n)$$

is a weak equivalence for cofibrant $X_i$. Here $F$ is any morphism which is composable with $f$.

**Remark 4.1.4.** If $S = \cdot$, the above notion coincides with the notion of monoidal model-category in the sense of [13, Definition 4.2.6]. In this case it is enough to claim property 1. for $n = 1, 2$.

**Theorem 4.1.5.** Under the conditions of 4.1.3 the morphism of pre-derivators

$$\tilde{\rho}: D \to S,$$

defined in 4.1.2, is a left fibered multiderivator (satisfying also $\text{FDer}0$ right) with domain $\text{Dir}$ and a right fibered multiderivator (satisfying also $\text{FDer}0$ left) with domain $\text{Inv}$. Furthermore for all $S \in S(\cdot)$ its fibre $D_S$ (1.3.9) is just the pre-derivator associated with the pair $(D_S, W_S)$.

There are techniques invented by Cisinski [3] which allow to extend a derivator to more general diagram categories. We will explain in a forthcoming article how these can be applied to fibered (multi-)derivators.
4.1.6. The proof of the theorem will occupy the rest of this section. First note that also for any diagram category $I$, the functors

$$ p_I : \text{Hom}(I, \mathcal{D}) \rightarrow \text{Hom}(I, \mathcal{S}) = \mathcal{S}(I) $$

are bifibrations of multicategories. Morphisms in $\text{Hom}(I, \mathcal{D})$ are (co)Cartesian, if and only if they are point-wise (co)Cartesian.

If $I$ is directed or inverse we will show that also $p_I$ is a bifibration of multi-model-categories in the sense of Definition 4.1.3.

Afterwards we will apply the following variant and generalization to multicategories of the results in [2, Exposé XVII, §2.4].

**Proposition 4.1.7.** Let $p : \mathcal{D} \rightarrow \mathcal{S}$ be a bifibration of (multi-)model-categories in the sense of Definition 4.1.3. Let $W$ be the union of the $W_S$ over all objects $S \in \mathcal{S}$. Then the fibers of $\tilde{p} : \mathcal{D}[W^{-1}] \rightarrow \mathcal{S}$ (as ordinary categories) are the homotopy categories $\mathcal{D}_S[W^{-1}]$ and $\tilde{p}$ is again a bifibration of multicategories such that the push-forward $F_\ast$ along any $F \in \text{Hom}_S(X_1, \ldots, X_n; Y)$ (for $n \geq 1$) is the left derived functor of the corresponding push-forward w.r.t. $p$. Similarly the pull-back w.r.t. a slot $i$ is the right derived functor of the corresponding pull-back w.r.t. $p$.

4.1.8. The proposition and its proof have several well-known consequences which we mention, despite being all elementary, because the proof below gives a uniform treatment of all the cases.

1. The homotopy category of a model category is locally small and can be described as the category of cofibrant/fibrant objects modulo homotopy of morphisms. Apply the Proposition to the (trivial) bifibration of ordinary categories $\mathcal{D} \rightarrow \cdot$.

2. Quillen adjunctions lead to an adjunction of the derived functors on the homotopy categories. Apply the Proposition to a bifibration of ordinary categories $\mathcal{D} \rightarrow \Delta_1$.

3. The homotopy category of a closed monoidal model category is a closed monoidal category. Apply the Proposition to a bifibration of multicategories $\mathcal{D} \rightarrow \cdot$.

4. Quillen adjunctions in $n$ variables lead to an adjunction in $n$ variables on the homotopy categories. Apply the Proposition to a bifibration of multicategories $\mathcal{D} \rightarrow \Delta_{1,n}$, where the multicategory $\Delta_{1,n}$ consists of $n + 1$ objects and one $n$-ary morphism connecting them.

Before proving Proposition 4.1.7 we define homotopy relations on $\text{Hom}_F(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F})$ where $F \in \text{Hom}(X_1, \ldots, X_n; Y)$ is a multimorphism in $\mathcal{S}$.

**Definition 4.1.9.** 1. Two morphisms $f$ and $g$ are called right homotopic if there is a path object of $\mathcal{F}$

$$ \mathcal{F} \xrightarrow{\text{pr}_1} \mathcal{F}' \xrightarrow{\text{pr}_2} \mathcal{F} $$

and a morphism $\text{Hom}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F}')$ over the same multimorphism $F$ such that the compositions with $\text{pr}_1$ and $\text{pr}_2$ are $f$ and $g$, respectively.

2. For $n \geq 1$, two morphisms $f$ and $g$ are called $i$-left homotopic if there is a cylinder object $\mathcal{E}'_i$ of $\mathcal{E}_i$

$$ \mathcal{E}_i \xrightarrow{\iota_1} \mathcal{E}'_i \xrightarrow{\iota_2} \mathcal{E}_i $$

and a morphism $\text{Hom}(\mathcal{E}_1, \ldots, \mathcal{E}'_i, \ldots, \mathcal{E}_n; \mathcal{F})$ over $F$ such that the compositions with $\iota_1$ and $\iota_2$ are $f$ and $g$, respectively.
Lemma 4.1.10. 1. ‘Right homotopic’ is preserved under pre-composition, while ‘i-left homotopic’ is preserved under post-composition.

2. Let \( n \geq 1 \). If \( f, g \in \text{Hom}(E_1, \ldots, E_n; F) \) are i-left homotopic and all \( E_i \) are cofibrant then \( f \) and \( g \) are right homotopic. If \( f, g \in \text{Hom}(E_1, \ldots, E_n; F) \) are right homotopic, \( F \) is fibrant, and all \( E_j, j \neq i \) are cofibrant then \( f \) and \( g \) are i-left homotopic.

3. Let \( n \geq 1 \). In \( \text{Hom}(E_1, \ldots, E_n; F) \) if all \( E_i \) are cofibrant, right homotopy is an equivalence relation. In \( \text{Hom}(E_1, \ldots, E_n; F) \) if \( F \) is fibrant, and all \( E_j, j \neq i \) are cofibrant, i-left homotopy is an equivalence relation.

In particular on the category \( D^{\text{Cof,Fib}} \) of fibrant/cofibrant objects, i-left homotopy=right homotopy is an equivalence relation, which is compatible with composition.

Proof. 1. is obvious.

2. If all \( E_i \) are cofibrant then also \( F_\bullet(\xi_1, \ldots, \xi_n) \) is cofibrant and \( f \) and \( g \) correspond uniquely to morphisms \( f', g': F_\bullet(\xi_1, \ldots, \xi_n) \to F \). Since \( f \) and \( g \) are i-left homotopic, there is a cylinder object

\[
E_i \xrightarrow{\xi} E'_i \xrightarrow{\xi} E_i
\]

realizing the i-left homotopy. Since \( E_i \) is cofibrant so is \( E'_i \). Hence also

\[
F_\bullet(\xi_1, \ldots, \xi_n) \xrightarrow{\xi} F_\bullet(\xi_1, \ldots, \xi'_1, \ldots, \xi_n) \xrightarrow{\xi} F_\bullet(\xi_1, \ldots, \xi_n)
\]

is a cylinder object because all \( E_j \) are cofibrant, and hence also \( f' \) and \( g' \) are left homotopic. These are therefore also right homotopic and hence so are \( f \) and \( g \). Dually we obtain the second statement. 4. follows from [13, Proposition 1.2.5, (iii)].

Lemma 4.1.11. i-left homotopic maps become equal in \( D^{\text{Cof}}[(W^{\text{Cof}})^{-1}] \).

Proof. This follows from the fact, that a cylinder object

\[
E_i \xrightarrow{\xi_1} E'_i \xrightarrow{\pi} E_i
\]

automatically lies in \( D^{\text{Cof}} \) if \( E_i \) does, and the two maps \( \xi_1 \) and \( \xi_2 \) become equal because \( p \) becomes invertible.

We have to distinguish the easier case, in which all objects \( F_\bullet() \) for 0-ary morphisms \( F \) are cofibrant. Otherwise we define a category \( D^{\text{Cof}}[(W^{\text{Cof}})^{-1}] \) in which we set \( \text{Hom}_F(; F) = \text{Hom}_{D_S[W_S^{-1}]}(QF_\bullet(; F)) \) for all \( Y \), where \( F \) is a 0-ary morphism with domain \( S \). Composition given as follows: For a morphism \( f \in \text{Hom}_S(E_1, \ldots, E_n; F) \) with cofibrant \( E_i \) and \( F \) and \( \xi : QF\bullet() \to E_i \), we define the composition \( \xi \circ f \) as the following composition

\[
E_1 \xrightarrow{\xi} (F \circ G)\bullet(E_2, \ldots, E_n) \xrightarrow{\sim} G\bullet(E_1, \ldots, F\bullet(), \ldots, E_n) \xrightarrow{\sim} E_n.
\]
Proof. This follows from the fact, that there exists a path object $G_*(\mathcal{E}_1, \ldots, QF_*(\mathcal{E}_n)) \rightarrow G_*(\mathcal{E}_1, \ldots, \mathcal{E}_n) \rightarrow \mathcal{F}$.

One checks that the so-defined composition is associative and independent of the choices of the push-forwards made.

**Lemma 4.1.12.** If all $F_*(\cdot)$ for 0-ary morphisms $F$ are cofibrant then $\mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}] \rightarrow \mathcal{D}[\mathcal{W}^{-1}]$ is an equivalence of categories.

Otherwise it is, if we replace $\mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}]$ by $\mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}]$.

**Proof.** The inclusion $\mathcal{D}^\text{Cof} \rightarrow \mathcal{D}$ induces a functor $F : \mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}] \rightarrow \mathcal{D}[\mathcal{W}^{-1}]$. If the $F_*(\cdot)$ are not cofibrant then it may be modified to a functor

$$\mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}] \rightarrow \mathcal{D}[\mathcal{W}^{-1}]$$

as follows: A 0-ary morphism $QF_*(\cdot) \rightarrow \mathcal{F}$ is mapped to the composition

$$\xymatrix{ \text{c} \ar[r] & F_*(\cdot) \ar[r] & QF_*(\cdot) \ar[r] & \mathcal{F} }$$

in $\mathcal{D}[\mathcal{W}^{-1}]$.

We now specify a functor $G$ in the other direction. On objects, $G$ maps $X$ to a cofibrant replacement $QX$. For $n \geq 1$, a morphism $f \in \text{Hom}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F})$ over $F$, is mapped to the following morphism.

Composing with the morphisms $Q\mathcal{E}_i \rightarrow \mathcal{E}_i$, we get a morphism $f' \in \text{Hom}(Q\mathcal{E}_1, \ldots, Q\mathcal{E}_n; \mathcal{F})$ or equivalently a morphism $X_i \rightarrow F^{*i}(Q\mathcal{E}_1, \ldots, Q\mathcal{E}_n; \mathcal{F})$. Now choose a lift in

$$\xymatrix{ F^{*i}(Q\mathcal{E}_1, \ldots, Q\mathcal{E}_n; Q\mathcal{F}) \ar[d] \ar[r] & Q\mathcal{E}_i \ar@{-->}[r] & F^{*i}(Q\mathcal{E}_1, \ldots, Q\mathcal{E}_n; \mathcal{F}) }$$

which exists because the vertical map is again a trivial fibration because all the $Q\mathcal{E}_i$ are cofibrant. The resulting map in $\text{Hom}(Q\mathcal{E}_1, \ldots, Q\mathcal{E}_n; P\mathcal{F})$ is actually unique in $\mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}]$. Indeed, two lifts in the triangle above become left homotopic, because $Q\mathcal{E}_i$ is cofibrant [13, Proposition 1.2.5. (iv)], and therefore also the two maps in in $\text{Hom}(Q\mathcal{E}_1, \ldots, Q\mathcal{E}_n; Q\mathcal{F})$ become equal in $\mathcal{D}^\text{Cof}[(\mathcal{W}^\text{Cof})^{-1}]$ (Lemma 4.1.11). From this it follows that $G$ is indeed a functor on $n$-ary morphisms for $n \geq 1$.

For $n = 0$ a morphism $f \in \text{Hom}(; \mathcal{F})$ over $F$ corresponds to a morphism $F_*(\cdot) \rightarrow \mathcal{F}$. If $F_*(\cdot)$ is cofibrant, this morphism lifts (again uniquely up to right homotopy) to a morphism $F_*(\cdot) \rightarrow Q\mathcal{F}$, i.e. to a morphism in $\text{Hom}_F(; Q\mathcal{F})$.

If $F_*(\cdot)$ is not cofibrant then the composition lifts to morphism: $QF_*(\cdot) \rightarrow P\mathcal{F}$ which is defined to be the image of $G$.

Furthermore $G$ is inverse to the inclusion up to isomorphism.

**Lemma 4.1.13.** Right homotopic maps become equal in $\mathcal{D}^\text{Cof,Fib}[(\mathcal{W}^\text{Cof,Fib})^{-1}]$.

**Proof.** This follows from the fact, that there exists a path object

$$\mathcal{F} \xleftarrow{\text{pr}_1} \mathcal{F}' \xrightarrow{i} \mathcal{F}$$

with $\mathcal{F}'$ cofibrant and fibrant which realizes the right homotopy [13, Proposition 1.2.6.]. This uses that the all sources are cofibrant and the domain is fibrant. The two maps $\text{pr}_1$ and $\text{pr}_2$ become equal because $i$ becomes invertible.
Lemma 4.1.14. The functor \( D^{\text{Fib, Cof}}((W^{\text{Fib, Cof}})^{-1}) \to D^{\text{Cof}}((W^{\text{Cof}})^{-1}) \) (resp. \( D^{\text{Fib, Cof}}((W^{\text{Fib, Cof}})^{-1}) \to D^{\text{Cof}}((W^{\text{Cof}})^{-1}) \)) is an equivalence of multicategories.

Proof. The proof is completely analogous to the Lemma 4.1.12 but with some minor changes which require, in particular, the chosen order of restriction to cofibrant and fibrant objects. We specify again a functor \( G \) in the other direction. On objects, \( G \) maps \( \mathcal{E} \) to a fibrant replacement \( R\mathcal{E} \). Note that \( R\mathcal{E} \) is still cofibrant. A morphism \( f \in \text{Hom}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F}) \) over \( F \) corresponds to a morphism \( F_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n) \to \mathcal{F} \). Now choose a lift in

\[
\begin{array}{ccc}
F_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n) & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \rho \\
F_\bullet(R\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n) & & R\mathcal{F} \\
\end{array}
\]

It exists because the vertical maps are again trivial cofibrations because all the \( \mathcal{E}_i \) and \( R\mathcal{E}_i \) are cofibrant. The lift is actually unique in \( D^{\text{Cof, Fib}}((W^{\text{Cof, Fib}})^{-1}) \), because two lifts in the triangle above become right homotopic, because \( R\mathcal{F} \) is fibrant [13, Proposition 1.2.5. (iv)]. Therefore also the corresponding morphisms in \( \text{Hom}(\mathcal{E}_1, \ldots, \mathcal{E}_n; R\mathcal{F}) \) become equal in \( D^{\text{Cof, Fib}}((W^{\text{Cof, Fib}})^{-1}) \) by the previous Lemma. It follows that \( G \) is indeed a functor which is inverse to the inclusion up to isomorphism. \(\square\)

Lemma 4.1.15. If all objects \( F_\bullet() \) are cofibrant then the natural functor

\( D^{\text{Fib, Cof}}((W^{\text{Fib, Cof}})^{-1}) \to D^{\text{Fib, Cof}}/\sim \)

is an isomorphism of categories. Otherwise it is, if we modify the 0-ary morphisms as before.

Proof. The natural functor \( D^{\text{Fib, Cof}} \to D^{\text{Fib, Cof}}/\sim \) takes weak equivalences to isomorphisms [13, Proposition 1.2.8] and has the universal property of \( D^{\text{Fib, Cof}}((W^{\text{Fib, Cof}})^{-1}) \) by the same argument as in [13, Proposition 1.2.9]. \(\square\)

Proof of Proposition 4.1.4. The previous Lemmas showed that \( D[\mathcal{W}^{-1}] \) is equivalent to \( D^{\text{Fib, Cof}}/\sim \) if all objects \( F_\bullet() \) are cofibrant, or if we replace the right hand side by \( D^{\text{Fib, Cof}}/\sim \), which has \( \text{Hom}_{D^{\text{Fib, Cof}}/\sim}(F_\bullet(), \mathcal{F}) \) for all 0-ary morphism \( F \) in \( \mathcal{S} \) with domain \( S \) and for every \( \mathcal{F} \in D_S \).

It remains to show that

\[
p/\sim : D^{\text{Fib, Cof}}/\sim \to \mathcal{S}
\]

is bifibered if all \( F_\bullet() \) are cofibrant or otherwise bifibered for \( n \geq 1 \) (i.e. (co)Cartesian morphisms exist for \( n \geq 1 \)). (The modification \( D^{\text{Fib, Cof}}/\sim \) has been constructed such that it has coCartesian morphisms for \( n = 0 \)).

We show that \( p/\sim \) is op-fibered, the other case being similar. Let \( F \) be a multimorphism in \( \mathcal{S} \) with domain \( S \). The set \( \text{Hom}_F(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{F}) \) modulo right homotopy is in bijection with the set \( \text{Hom}_{D_S}(F_\bullet(\mathcal{E}_1, \ldots, \mathcal{E}_n), \mathcal{F}) \) modulo right homotopy. Since \( \mathcal{F} \) is fibrant the latter set is the same as
Hom_{D_S}(R(F_*(E_1,\ldots,E_n)),\mathcal{F})$ modulo right homotopy. Hence morphisms in $\text{Hom}_F(E_1,\ldots,E_n;\mathcal{F})$ uniquely decompose as the composition

$\begin{align*}
E_1 & \xrightarrow{\text{cocart}} F_*(E_1,\ldots,E_n) \\
& \xrightarrow{R(F_*(E_1,\ldots,E_n))}
\end{align*}$

followed by a morphism in $\text{Hom}_{D_S}(R(F_*(E_1,\ldots,E_n)),\mathcal{F})$ modulo right homotopy. More generally, by the same argument, a morphism in $\text{Hom}_{GF}(E_1,\ldots,E_n,\mathcal{F}_1,\ldots,\mathcal{F}_m;\mathcal{G})$ modulo right homotopy factorizes uniquely into the above composition followed by a morphism in

$\text{Hom}_G(\mathcal{F}_1,\ldots,R(F_*(E_1,\ldots,E_n)),...,\mathcal{F}_m;\mathcal{G})$

modulo right homotopy.

It remains to see that the push-forward in $D[\mathcal{W}^{-1}]$ corresponds to the left derived functor of $F_*$. For any objects $E_1,\ldots,E_n$ the morphism

$\begin{align*}
RQE_1 & \xrightarrow{\text{cocart}} F_*(RQE_1,\ldots,RQE_n) \\
& \xrightarrow{R(F_*(RQE_1,\ldots,RQE_n))}
\end{align*}$

is a coCartesian morphism lying over $F$, with codomains isomorphic to the $E_i$. However, $R(F_*(RQE_1,\ldots,RQE_n))$ is isomorphic to the value of the left derived functor of $F_*$ at $E_1,\ldots,E_n$. \hfill \Box

4.1.16. We now concentrate on the left case. For $I$ directed, we proceed to construct a model structure on the fibers of the bifibration of multicategories (cf. 4.1.6):

$\text{Hom}(I,D) \to \text{Hom}(I,S) = S(I)$

This model structure is an analogue of the classical Reedy model structure and has the property that pull-back w.r.t. diagrams and relative left Kan extension functors form a Quillen adjunction. Let $I \in \text{Dir}$ and let $F : I \to S$ be a functor. We will define a model-category structure

$(\mathcal{D}_F, \text{Cof}_F, \text{Fib}_F, \mathcal{W}_F)$

where $\mathcal{D}_F$ is the fibre of $\text{Hom}(I,D)$ over $F$ and where $\mathcal{W}_F$ is the class of morphisms which are element-wise in the corresponding $\mathcal{W}_{F(i)}$.

For any $G \in \mathcal{D}_F$, and any $i \in I$, we define a latching object

$L_i G := \text{colim}_{I_i} \{ F(\alpha) \ast_i G(j) \}_{\alpha,j-i}$

Here $I_i$ is the full subcategory of $I \times_i I$ consisting of all objects except $\text{id}_i$. 58
We have a canonical morphism

\[ L_iG \rightarrow G(i) \]

in \( \mathcal{D}_{F(i)} \).

We define \( \text{Fib}_F \) to be the class of morphisms which are element-wise in the corresponding \( \text{Fib}_{F(i)} \).

We define \( \text{Cof}_F \) to be the class of morphisms \( G \rightarrow H \) such that for any \( i \in I \) the induced morphism \( \delta \) in

\[
\begin{array}{ccc}
L_iG & \rightarrow & L_iH \\
\downarrow & & \downarrow \\
G(i) & \rightarrow & \text{push-out} \\
& & \delta \\
& & H(i)
\end{array}
\]

is in \( \text{Cof}_{F(i)} \). We call a morphism \( G \rightarrow H \) in \( \text{Cof}_F \) temporarily an \textbf{acyclic cofibration} if \( \delta \) is, in addition, a weak equivalence.

The proof that this yields a model-category structure is completely analogous to the classical case \[13, \S 5.1\] (if \( S \) is trivial). We need a couple of Lemmas:

**Lemma 4.1.17.** The class of cofibrations (resp. acyclic cofibrations) in \( \mathcal{D}_F \) consists precisely of the morphisms which have the left lifting property w.r.t. trivial fibrations (resp. fibrations). They are stable under retracts.

**Proof.** This is shown as in the classical case: We first show that acyclic cofibration have the lifting property w.r.t. fibrations. Let a diagram

\[
\begin{array}{ccc}
G_1 & \rightarrow & H_1 \\
\downarrow^{\alpha} & & \downarrow^{\beta} \\
G_2 & \rightarrow & H_2
\end{array}
\]

be given where \( \alpha \) is a acyclic cofibration and \( \beta \) is a fibration. We proceed by induction on \( n \) and assume that for all \( i \in I \) with \( \nu(i) < n \) a map \( G_2(i) \rightarrow H_1(i) \) has been constructed which is a lift in the above diagram, evaluated at \( i \). For each \( i \) of degree \( n \) consider the following diagram (where the map \( L_iG_2 \rightarrow L_iH_1 \rightarrow H_1(i) \) is constructed using the already constructed lifts):

\[
\begin{array}{ccc}
G_1 \amalg_{L_iG_1} L_iG_2 & \rightarrow & H_1(i) \\
\downarrow^{\alpha'(i)} & & \downarrow^{\beta(i)} \\
G_1(i) & \rightarrow & H_2(i)
\end{array}
\]

Here \( \alpha'(i) \) is a trivial \( \text{Cof}_{F(i)} \)-cofibration by definition, and \( \beta(i) \) a \( \text{Fib}_{F(i)} \)-fibration by definition. Hence a lift exists. In the same way the statement for cofibrations and trivial fibrations is shown. Closure under retracts is left as an exercise for the reader.

The statement that the class of acyclic cofibrations (resp. cofibrations) is precisely the class of morphisms that has the left lifting property w.r.t. fibrations (resp. trivial fibrations) follows from the retract argument as for model categories.

**Lemma 4.1.18.** There exists a functorial factorization of morphisms in \( \mathcal{D}_F \) into a fibration followed by an acyclic cofibration and into a trivial fibration followed by a cofibration.
Proof. We show this again by induction on \( n \). We do the case: factorization into an acyclic cofibration followed by a fibration, the other case being similar. Let \( G \to K \) a morphism in \( D_F \). We have the following diagram:

\[
\begin{array}{ccc}
L_iG & \longrightarrow & L_iH \\
\downarrow & & \downarrow \\
G(i) & \longrightarrow & H(i)
\end{array}
\]

Here the top row is constructed using the already defined factorizations and \( H(i) \) and the dotted maps are constructed as the factorization in the model category \( D_F(i) \) into trivial \( \text{Cof}_{F(i)} \)-cofibration followed by \( \text{Fib}_{F(i)} \)-fibration.

Lemma 4.1.19. The classes of cofibrations, acyclic cofibrations, fibrations and weak equivalences are stable under composition.

Proof. This follows from the characterization by a lifting property (resp. by definition for the case of the weak equivalences).

Lemma 4.1.20. Acyclic cofibrations are precisely the trivial cofibrations.

Proof. We begin by showing that an acyclic cofibration is a weak equivalence. It suffices to show that in the diagram

\[
\begin{array}{ccc}
L_iG & \longrightarrow & L_iH \\
\downarrow & & \downarrow \\
G(i) & \longrightarrow & H(i)
\end{array}
\]

the top horizontal morphism is a trivial cofibration because then the lower horizontal morphism is a composition of two trivial cofibrations and hence is a weak equivalence. The top morphism is indeed a trivial cofibration because the morphism of \( I_i \)-diagrams

\[
\{F(\alpha) \bullet G(j)\}_{\alpha:j\to i} \to \{F(\alpha) \bullet G(j)\}_{\alpha:j\to i}
\]

is a trivial cofibration in the classical sense (i.e. over the constant diagram over \( I_i \) with value \( F(i) \)) because of Lemma [4.1.21] and [4.1.22].

In the other direction let \( f \) be a trivial cofibration and factor it as \( f = pg \), where \( g \) is an acyclic cofibration and \( p \) is a fibration. It follows that \( p \) is a weak-equivalence. Now construct a lift in the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{g} & H \\
\downarrow & & \downarrow \\
G & \xrightarrow{p} & G
\end{array}
\]

This shows that \( f \) is a retract of \( g \), and hence is an acyclic cofibration, too.

Lemma 4.1.21. For each (1-ary) morphism of diagrams \( f \in \text{Hom}_S(X_1; Y) \) there is an associated push-forward and an associated pull-back, defined by taking the point-wise push-forward \( f_* \), and point-wise pull-back \( f^* \) (cf. [4.1.6]), respectively. \( f_* \) respects the classes of cofibrations and acyclic cofibrations. \( f^* \) respects the classes of fibrations and trivial fibrations.
Proof. It suffices (by the lifting property) to show that $f^\bullet$ respects fibrations and trivial fibrations. This is clear because they are defined point-wise.

A posteriori this will say that $f^*, f_\bullet$ define a Quillen adjunction between the corresponding model categories [4.1.26].

**Lemma 4.1.22.** Let $i \in I$ be an object, let $\iota: I_i \to I$ be the corresponding latching category with its natural functor to $I$, and let $F_i := \iota^* F: I_i \to \mathcal{S}$ be the restriction of $F$. The pullback $\iota^*: \mathcal{D}_F \to \mathcal{D}_{F_i}$ respects cofibrations and acyclic cofibrations.

**Proof.** It is easy to see that the pullback induces an isomorphism of the corresponding latching objects as in the classical case. 

**Corollary 4.1.23.** The structure constructed in [4.1.16] defines a model category.

**Proof.** This follows from the previous Lemmas.

**Proposition 4.1.24.** For any morphism of directed diagrams $\alpha: I \to J$, and for any functor $F: J \to \mathcal{S}$, the functor $\alpha^*: \mathcal{D}_F \to \mathcal{D}_{F^\alpha}$ has a left adjoint $\alpha_!$. They define a Quillen adjunction.

**Proof.** That the two functors define a Quillen-adjunction follows once we have shown that $\alpha_!$ exists because $\alpha^*$ preserves obviously fibrations and weak equivalences. Let $G$ be an object of $\mathcal{D}_F$. We define $(\alpha_!G)(j) := \text{colim}_{I \times_j j} S(\mu) \bullet \iota_j^* G$.

For each morphism $\mu: j \to j'$ we get a functor $\overline{\mu}: I \times_j j \to I \times_j j'$ and hence an induced morphism $F(\mu)_\bullet S(\mu) \bullet \iota_j^* G \to \overline{\mu}_* S(\mu') \bullet \iota_{j'}^*$. Taking the colimit and noticing that $F(\mu)_\bullet$ commutes with colimits we get a morphism $F(\mu)_\bullet \text{colim}_{I \times_j j} S(\mu) \bullet \iota_j^* G \to \text{colim}_{I \times_j j'} S(\mu') \bullet \iota_{j'}^*$ which we define to be $(\alpha_!G)(\mu)$.

We now proceed to show the functor we have constructed is indeed adjoint to $\alpha^*$. A morphism $\mu: G \to \alpha^* H$ is given by a collection of maps $a(i): G(i) \to H(\alpha(i))$ for all objects $i \in I$, subject to the condition that $F(\alpha(\mu))_\bullet G(i) \xrightarrow{F(\alpha(\mu))_\bullet a(i)} F(\alpha(\mu))_\bullet H(\alpha(i))$ commutes for each morphism $\mu: i \to i'$ in $I$. For each $j \in J$ and morphism $\mu: \alpha(i) \to j$ we get a morphism $\overline{H(\mu)} \circ (F(\mu)_\bullet a(i)) : F(\mu)_\bullet G(i) \to H(j)$.
and therefore for fixed \( j \) a morphism

\[
\text{colim}_{I \times j} S(\mu) \cdot \iota_j^* G \to H(j).
\]

One checks that this yields a morphism \( \alpha_1 G \to H \). On the other hand, let \( b : \alpha_1 G \to H \) be a morphism given by

\[
b(j) : \text{colim}_{I \times j} S(\mu) \cdot \iota_j^* G \to H(j)
\]
or equivalently for all \( \mu : \alpha(i) \to j \) by morphisms

\[
F(\mu)_* G(i) \to H(j).
\]

In particular, for \( \mu \) being the identity of \( \alpha(i) \), we get morphisms

\[
G(i) \to H(\alpha(i))
\]

which constitute a morphism of diagrams \( G \to \alpha^* H \). One checks that these associations are inverse to each other.

Lemma 4.1.25. The functor \( \iota_j^* : D_I \to D_{I \times j} \) respects cofibrations and trivial cofibrations.

Proof. This follows easily from the fact that \( \iota_j \) induces a canonical identification

\[
I_i = (I \times j)_\mu
\]

for any \( \mu = (i, \alpha(i) \to j) \). For this implies that we have a canonical isomorphism \( L_i G \cong L_\mu \iota_j^* G \).

Lemma 4.1.26. The bifibration of multicategories of \( \underline{4.1.6} \):

\[
\text{Hom}(I, D) \to \text{Hom}(I, S) = S(I)
\]

equipped with the model-category structures constructed in \( \underline{4.1.16} \) is a bifibration of multi-model-categories in the sense of \( \underline{4.1.3} \).

Proof. First for each multi-morphism of diagrams \( f \in \text{Hom}_S(X_1, \ldots, X_n; Y) \) we have to see that push-forward and the various pull-backs form a Quillen adjunction in \( n \) variables. The case \( n = 1 \) has been treated above. We do the case \( n = 2 \), the proof for higher \( n \) being similar. It suffices to see e.g. the following: For any cofibration \( E_1 \to E'_1 \) and for any fibration \( F \to F' \) the dotted induced map in the following diagram

\[
\begin{array}{ccc}
f^*#2(E'_1; F) & \dashrightarrow & \text{pull-back} \rightarrow f^*#2(E'_1; F') \\
\downarrow & & \downarrow \\
f^*#2(E_1; F) & \rightarrow & f^*#2(E_1; F')
\end{array}
\]

is a fibration. Since fibrations are defined point-wise, and fibered products are computed point-wise, we have to see that the assertion holds point-wise. Now \( F \to F' \) is a point-wise fibration and \( E_1 \to E'_1 \) is a Reedy cofibration, so by the reasoning in the proof of Lemma 4.1.20 in particular a point-wise cofibration. Hence the assertion holds because of the assumption that the original \( D \to S \) is a bifibration of multi-model-categories \( \underline{4.1.3} \). The requested property for the 0-ary push-forward is easier and is left to the reader.
Proposition 4.1.27. The functor $\mathbb{D}(I) \to S(I)$ defined in 4.1.2 is a bifibration of multicategories whose fibers are equivalent to $\mathcal{D}_F[\mathcal{W}_F^{-1}]$. The pull-back and push-forward functors are given by the left derived functors of $f_*$, and by the right derived functors of $f^{**}j$, respectively.

Proof. We have seen in 4.1.26 that the fibers of $\text{Hom}(I, \mathbb{D}) \to S(I)$ are a bifibration of multi-model-categories in the sense of 4.1.3. Therefore by Proposition 4.1.7 we get that $\mathbb{D}(I) \to S(I)$ are bifibered multicategories with the claimed properties.

Proof of Theorem 4.1.5. (Der1) and (Der2) for $\mathbb{D}$ and $S$ are obvious. (FDer0 left) and (FDer0 right) follow from 4.1.27. (FDer3 left) follows from 4.1.24. (FDer4 left) By construction of $\alpha_!$ the natural base-change

$$\text{colim} S(\mu)_! j^* G \to j^* \alpha_! G$$

is an isomorphism for the non-derived functors. For the derived functors the same follows because all functors in the equation respect cofibrations and trivial cofibrations and all functors which have to be derived in the equation are left Quillen functors and hence can be derived by composing them with cofibrant replacement. (FDer5 left) follows from (FDer0 left) and (FDer0 right).
A  Fibrations of categories

A.1  Grothendieck (op-)fibrations

A.1.1.  (right) Let \( p : D \to S \) be a functor, and let \( f : S \to T \) be a morphism in \( S \). A morphism \( \xi : E' \to E \) over \( f \) is called Cartesian if the composition with \( \xi \) induces an isomorphism

\[
\text{Hom}_g(F, E') \cong \text{Hom}_{f \circ g}(F, E)
\]

for any \( g : R \to S \) and \( F \in D_R \).

The functor \( p \) is called a Grothendieck fibration, if for any \( f : S \to T \) and for every object \( E \) in \( D_T \) (i.e. such that \( p(E) = T \)) there exists a Cartesian morphism \( E' \to E \).

A.1.2.  (left) Let \( p : D \to S \) be a functor, and let \( f : S \to T \) be a morphism in \( S \). A morphism \( \xi : E \to E' \) over \( f \) is called coCartesian if the composition with \( \xi \) induces an isomorphism

\[
\text{Hom}_g(E', F) \cong \text{Hom}_{g \circ f}(E, F)
\]

for any \( g : T \to U \) and \( F \in D_U \).

The functor \( p \) is called a Grothendieck opfibration if for any \( f : S \to T \) and for every object \( E \) in \( D_S \) (i.e. such that \( p(E) = S \)) there exists a coCartesian morphism \( E \to E' \).

The functor \( p : D \to S \) is a Grothendieck opfibration if and only if \( p^{op} : D^{op} \to S^{op} \) is a Grothendieck fibration. We say that \( p \) is a bifibration if is a fibration and an opfibration at the same time.

A.1.3.  If \( p : D \to S \) is a Grothendieck fibration we may choose an associated pseudo-functor, i.e. to each \( S \in S \) we associate the category \( D_S \), and to each \( f : S \to T \) we associate a push-forward functor

\[ f_* : D_S \to D_T \]

such that for each \( E \) in \( D_S \) there is a Cartesian morphism \( E \to f_* E \). Similarly for an opfibration with pull-back \( f^* \) instead of push-forward. If the functor \( p \) is a bifibration, \( f_* \) is left adjoint to \( f^* \).

Situations where this is the opposite can be modeled by considering bifibrations \( D \to S^{op} \).

A.2  Fibered multicategories

A.2.1.  We give a definition of a (op-)fibered multicategory. This is a straightforward generalization of the notion of (op-)fibered category given in the last section. It is very useful to encode the formalism of the Grothendieck six functors. Details about (op-)fibered multicategories can be found, for instance, in [11, 12].

The reader should keep in mind that a multicategory abstracts the properties of multilinear maps, and indeed every monoidal category gives rise to a multicategory setting

\[
\text{Hom}(A_1, \ldots, A_n; B) := \text{Hom}( (A_1 \otimes (A_2 \otimes \cdots)), B).\quad (8)
\]

Definition A.2.2.  A multicategory \( D \) consists of

- a class of objects \( \text{Ob}(D) \),
- for every \( n \in \mathbb{Z}_{\geq 0} \), and objects \( X_1, \ldots, X_n, Y \) a class

\[
\text{Hom}(X_1, \ldots, X_n; Y),
\]

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• an associative composition for objects \(X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z\) and for each integer \(1 \leq i \leq m\):

\[
\text{Hom}(X_1, \ldots, X_n; Y_i) \times \text{Hom}(Y_1, \ldots, Y_m; Z) \to \text{Hom}(Y_1, \ldots, Y_{i-1}, X_1, \ldots, X_n, Y_{i+1}, \ldots, Y_m; Z),
\]

• for each object \(X\) an identity \(\text{id}_X \in \text{Hom}(X; X)\),

satisfying the usual properties. A symmetric (braided) multicategory is given by an action of the symmetric (braid) groups, i.e. isomorphisms

\[
\alpha : \text{Hom}(X_1, \ldots, X_n; Y) \to \text{Hom}(X_{\alpha(1)}, \ldots, X_{\alpha(n)}; Y)
\]

for \(\alpha \in S_n\) (resp. \(\alpha \in B_n\)) forming an action which is compatible with composition in the obvious way (substitution of strings in the braid group).

In some references the composition is defined in a seemingly more general way; in the presence of identities these descriptions are, however, equivalent. We denote a multimorphism in \(f \in \text{Hom}(X_1, \ldots, X_n; Y)\) also by

\[
\begin{array}{c}
X_1 \\
\vdots \\
X_n \\
\end{array}
\xymatrix{
& f \ar[r] & Y \\
\ar[ru] & & \\
}\]

for \(n \geq 1\), or by

\[
\begin{array}{c}
& f \ar[r] & Y \\
\ar[ru] & & \\
\end{array}
\]

for \(n = 0\).

A.2.3. We leave it to the reader to state the obvious definition of a functor between multicategories. Similarly there is a definition of a opmulticategory, in which we have classes

\[
\text{Hom}(X; Y_1, \ldots, Y_n)
\]

and similar data. If \(\mathcal{D}\) is a multicategory, there is a natural opmulticategory \(\mathcal{D}^{\text{op}}\).

The trivial category is considered as a multicategory setting all \(\text{Hom}(\cdot, \ldots, \cdot; \cdot)\) to the 1 element set. It is the final object in the “category” of multicategories.

To clarify the precise relation between multicategories and monoidal categories we have to define Cartesian and coCartesian morphisms. It turns out that we can actually give a definition which is a common generalization of coCartesian morphisms in opfibered categories and those morphisms expressing the existence of a tensor product:

**Definition A.2.4.** Consider a functor of multicategories \(p : \mathcal{D} \to \mathcal{S}\). We call a morphism

\[
\xi \in \text{Hom}(X_1, \ldots, X_n; Y)
\]

in \(\mathcal{D}\) coCartesian w.r.t. \(p\), if for all \(Y_1, \ldots, Y_m, Z\) with \(Y_i = Y\), and for all

\[
f \in \text{Hom}(p(Y_1), \ldots, p(Y_m); p(Z))
\]
the map
\[
\text{Hom}_f(Y_1, \ldots, Y_m; Z) \to \text{Hom}_{f \circ p}(Y_1, \ldots, Y_{i-1}, X_1, \ldots, X_n, Y_{i+1}, \ldots, Y_m; Z)
\]
\[\alpha \mapsto \alpha \circ \xi\]
is bijective.

We call a morphism
\[
\xi \in \text{Hom}(X_1, \ldots, X_n; Y)
\]
in \(\mathcal{D}\) Cartesian w.r.t. \(p\) at the \(i\)-th slot, if for all \(Y_1, \ldots, Y_m, f \in \text{Hom}(p(Y_1), \ldots, p(Y_m); p(X_i))\)
\[
\text{Hom}_f(Y_1, \ldots, Y_m; X_i) \to \text{Hom}_{p(\xi) \circ f}(Y_1, \ldots, X_{i-1}, Y_i, \ldots, Y_m, X_{i+1}, \ldots, X_n; Z).
\]
\[\alpha \mapsto \xi \circ \alpha\]
is bijective.

The functor \(p : \mathcal{D} \to \mathcal{S}\) is called an \textbf{opfibered multicategory} if for every \(g \in \text{Hom}(S_1, \ldots, S_n; T)\) in \(\mathcal{S}\) and for each collection of objects \(X_i\) with \(p(X_i) = S_i\) there is some object \(Y\) over \(T\) and some coCartesian morphism \(\xi \in \text{Hom}(X_1, \ldots, X_n; Y)\) with \(p(\xi) = g\).

The functor \(p : \mathcal{D} \to \mathcal{S}\) is called a \textbf{fibered multicategory} if for every \(1 \leq j \leq n\) and for each \(g \in \text{Hom}(S_1, \ldots, S_n; T)\) in \(\mathcal{S}\), and for each collection of objects \(X_i\) for \(i \neq j\) with \(p(X_i) = S_i\), and for each \(Y\) over \(T\), there is some object \(X_j\) and some Cartesian morphism w.r.t. the \(j\)-th slot \(\xi \in \text{Hom}(X_1, \ldots, X_{j-1}, X_j, X_{j+1}, \ldots, X_n; Y)\) with \(p(\xi) = g\).

The functor \(p : \mathcal{D} \to \mathcal{S}\) is called a \textbf{bifibered multicategory} if it is both fibered and opfibered.

A morphism of (op)fibered multicategories is a commutative diagram of functors
\[
\begin{array}{ccc}
\mathcal{D}_1 & \xrightarrow{F} & \mathcal{D}_2 \\
\downarrow & & \downarrow \\
\mathcal{S}_1 & \xrightarrow{G} & \mathcal{S}_2
\end{array}
\]
such that \(F\) maps (co-)Cartesian morphisms to (co-)Cartesian morphisms.

It turns out that the composition of Cartesian morphisms is Cartesian (and similarly for coCartesian morphisms).

\textbf{Lemma A.2.5.} 1. An opfibered multicategory \(p : \mathcal{D} \to \cdot\) is a monoidal category defining \(X \otimes Y\) to be the target of a coCartesian arrow
\[
\xi \in \text{Hom}(X, Y; X \otimes Y)
\]
over the unique map in \(\text{Hom}(\cdot, \cdot, \cdot)\) of the final multicategory \(\cdot\).

Conversely any monoidal category gives rise to an opfibered multicategory \(p : \mathcal{D} \to \cdot\) (via equation 3). A multicategory \(\mathcal{D}\) is a closed category if and only if it is fibered over \(\cdot\). In particular, the fibers of an (op)fibered multicategory \(p : \mathcal{D} \to \mathcal{S}\) are always closed/monoidal in the following sense: Given any functor of multicategories \(x : \cdot \to \mathcal{S}\), the category \(\mathcal{D}_x\) of objects over \(x\) is closed/monoidal.

---

8 As with fibered categories there are weaker notions of Cartesian which still uniquely determine a Cartesian morphism (up to isomorphism) from given objects over a given multimorphism, however, do not imply stability under composition. Similarly for coCartesian morphisms.

9 This specifies also morphisms in \(\text{Hom}(X, \ldots, X; X)\), for all \(n\), compatible with composition.
2. Given (op)fibered multicategories \( p : C \to D \) and \( q : D \to E \) also the composition \( q \circ p \) is an (op)fibered multicategory. In particular, if we have an opfibered multicategory \( p : C \to S \) and if \( S \to \{\cdot\} \) is opfibered (i.e. \( S \) is monoidal) then also \( C \to \{\cdot\} \) is opfibered (i.e. \( C \) is monoidal). The same holds dually. A morphism \( \alpha \) is (co)Cartesian for \( q \circ p \) if and only if \( \alpha \) is (co)Cartesian for \( q \) and \( q(\alpha) \) is (co)Cartesian for \( p \).

Similarly, the unit 1 is just the target of a coCartesian morphism in \( \text{Hom}(\cdot;1) \) which exists by definition (the existence is also required for the empty set of objects).

The second part of the Lemma encapsulates the distinction between internal and external tensor product in a four (or six) functor context, see [A.2.10]

Example A.2.6. Let \( S \) be a usual category. Then \( S \) may be turned into a symmetric multicategory setting

\[
\text{Hom}(X_1,\ldots,X_n;Y) := \text{Hom}(X_1;Y) \times \cdots \times \text{Hom}(X_n;Y).
\]

If \( S \) has coproducts, then this is opfibered over \( \cdot \). Let \( p : D \to S \) be an opfibered (usual) category. Any object \( X \) induces a canonical functor of multicategories \( x : \cdot \to S \) with image \( X \), hence the fibers of an opfibered multicategory \( p : D \to S \) are monoidal and the datum \( p \) is equivalent to giving a pseudo-functor such that the push-forwards \( f_\ast \) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a covariant monoidal pseudo-functor e.g. in [15, (3.6.7)].

Example A.2.7. \( S^{\text{op}} \) may be turned into a symmetric multicategory (or \( S \) into a symmetric op-multicategory) setting

\[
\text{Hom}(X_1,\ldots,X_n;Y) := \text{Hom}(Y;X_1) \times \cdots \times \text{Hom}(Y;X_n).
\]

If \( S \) has products then \( S^{\text{op}} \), defined like this, is opfibered over \( \cdot \). Let \( p : D \to S^{\text{op}} \) be an opfibered (usual) category. Then an opfibered multicategory structure on \( p \), w.r.t. this multicategory structure on \( S^{\text{op}} \), is equivalent to a monoidal structure on the fibers such that pull-backs \( f^\ast \) (along morphisms in \( S \)) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a contravariant monoidal pseudo-functor e.g. in [15, (3.6.7)].

Definition A.2.8. The point is that the notion of (op)fibered multicategory is not restricted to the situation of Examples A.2.6 and A.2.7. Let \( S \) be a category with fibre products and define \( S_{\text{cor}} \), denoted symmetric multicategory of correspondences in \( S \) to be the symmetric multicategory having the same objects as \( S \), and where a morphism \( \xi \in \text{Hom}(S_1,\ldots,S_n;T) \) is an isomorphism class of objects

\[
\begin{tikzpicture}
  \node (A) at (0,1) {A};
  \node (S1) at (-1,-1) {S_1};
  \node (Sn) at (1,-1) {S_n};
  \node (T) at (0,-2) {T};

  \draw[->] (A) -- (S1);
  \draw[->] (A) -- (Sn);
  \draw[->] (S1) -- (Sn);
  \draw[->] (Sn) -- (T);
  \draw[->] (S1) -- (T);

  \node (g1) at (-1.5,-1) {g_1};
  \node (gn) at (1.5,-1) {g_n};

  \node (f) at (0,-1.5) {f};

\end{tikzpicture}
\]

Composition is given by the isomorphism class of (here depicted for two 1-ary morphism):

\[
\begin{tikzpicture}
  \node (A) at (0,1) {A};
  \node (X) at (-1,-1) {X};
  \node (Y) at (0,-1) {Y};
  \node (B) at (1,-1) {B};
  \node (Z) at (2,-1) {Z};

  \draw[->] (A) -- (X);
  \draw[->] (B) -- (Y);
  \draw[->] (A) -- (B);
  \draw[->] (X) -- (Y);
  \draw[->] (X) -- (B);
  \draw[->] (Y) -- (Z);

\end{tikzpicture}
\]
This multicategory is representable (i.e. opfibered over \(\cdot\)), closed (i.e. fibered over \(\cdot\)) and self-dual, with tensor product and internal hom both given by \(\times\) and having as unit the final object.

**Definition A.2.9.** Let \(S\) be a category with fibre products. A Grothendieck six functor context on \(S\) is a bifibered symmetric multicategory

\[
p : D \to S^{\text{cor}}.
\]

A.2.10. We have a morphism of opfibered (over \(\cdot\)) symmetric multicategories \(S^{\text{op}} \to S^{\text{cor}}\) where \(S^{\text{op}}\) is equipped with the symmetric multicategory structure as in \([A.2.7]\). However there is no reasonable morphism of opfibered multicategories \(S \to S^{\text{cor}}\) (There is no compatibility involving only \(\otimes\) and \(!\)). From a Grothendieck six functor context we get operations \(g_*\), \(g^*\) as pullback and push-forward along the correspondence

\[
\begin{array}{c}
X \\
g \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ww
Lemma A.2.12. Given a Grothendieck six functor context on \( \mathcal{S} \)
\[ p : \mathcal{D} \rightarrow \mathcal{S}^{\text{cor}} \]
for the six operations as extracted in A.2.10 there exist naturally the following compatibility isomorphisms:

<table>
<thead>
<tr>
<th>left adjoints</th>
<th>right adjoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>((<em>,</em>))  ( (fg)^* \rightarrow g^* f^* )</td>
<td>((fg)<em>* \rightarrow f</em>* g_* )</td>
</tr>
<tr>
<td>((!,!))  ( (fg)! \rightsquigarrow f!g! )</td>
<td>((fg)! \rightsquigarrow g! f! )</td>
</tr>
<tr>
<td>((!,<em>))  ( g^</em> f! \rightarrow F!G^* )</td>
<td>( G^* F! \rightarrow f^* g_* )</td>
</tr>
<tr>
<td>((\otimes,<em>)) ( f^</em>(\otimes-) \rightsquigarrow f^* \otimes f^* )</td>
<td>( f_* \mathcal{HOM}(f^* -, -) \rightsquigarrow \mathcal{HOM}(-, f_* -) )</td>
</tr>
<tr>
<td>((\otimes,!)) ( f!(\otimes f^*) \rightarrow (f!)(\otimes -) )</td>
<td>( f_* \mathcal{HOM}(-, f!^<em>) \rightarrow \mathcal{HOM}(f!_</em>, -) )</td>
</tr>
<tr>
<td>((\otimes,\otimes)) ( \otimes - \rightarrow - \otimes (\otimes -) )</td>
<td>( \mathcal{HOM}(-, -) \rightarrow \mathcal{HOM}(-, \mathcal{HOM}(-, -)) )</td>
</tr>
</tbody>
</table>

Here \( f, g, F, G \) are morphisms in \( \mathcal{S} \) which, in the \((!,*)\)-row, are related by a Cartesian diagram:

\[
\begin{array}{ccc}
\ G \\
\ F & \rightarrow & f \\
\downarrow & & \downarrow \\
\ g & \rightarrow & f
\end{array}
\]

Remark A.2.13. In the right hand side column the corresponding adjoint natural transformations are listed. In each case the left hand side natural isomorphism determines the right hand side one and conversely. (In the \((\otimes,!)\)-case there are 2 versions of the commutation between the right adjoints; in this case any of the three isomorphisms determines the other two). The \((!,*)\)-isomorphism (between left adjoints) is called base change, the \((\otimes,!)\)-isomorphism is called the projection formula, and the \((\otimes,\otimes)\)-isomorphism is usually part of the definition of a monoidal functor. The \((\otimes,\otimes)\)-isomorphism is the associativity of the tensor product and part of the definition of a monoidal category. The \((*,*)\)-isomorphism, and the \((!,!)\)-isomorphism, express that the corresponding functors arrange as a pseudo-functor with values in categories.

Proof. The existence of all isomorphisms is a consequences of the fact that compositions of co-Cartesian morphisms are coCartesian. For example, the projection formula \((\otimes,!)\) is derived from the following composition in \( \mathcal{S}^{\text{cor}} \):

\[
\begin{pmatrix}
\begin{array}{cc}
Y & Y \\
\downarrow & \downarrow \\
Y & Y
\end{array}
\end{pmatrix} \circ_1 \begin{pmatrix}
\begin{array}{cc}
X & f \\
\downarrow & \downarrow \\
X & Y
\end{array}
\end{pmatrix} = \begin{pmatrix}
\begin{array}{cc}
X & f \\
\downarrow & \downarrow \\
X & Y
\end{array}
\end{pmatrix}
\]

where \( \circ_1 \) means that we compose w.r.t. the first slot.

The “monoidality of \( f^* \)” \((*,\otimes)\) is derived from the following composition in \( \mathcal{S}^{\text{cor}} \):

\[
\begin{pmatrix}
\begin{array}{cc}
X & f \\
\downarrow & \downarrow \\
Y & X
\end{array}
\end{pmatrix} \circ \begin{pmatrix}
\begin{array}{cc}
Y & Y \\
\downarrow & \downarrow \\
Y & Y
\end{array}
\end{pmatrix} = \begin{pmatrix}
\begin{array}{cc}
Y & f \\
\downarrow & \downarrow \\
Y & X
\end{array}
\end{pmatrix}
\]
Base change \((!, \ast)\) is derived from:

\[
\begin{pmatrix}
\begin{array}{c}
g \\
A
\end{array}
\end{pmatrix} \quad \begin{pmatrix}
\begin{array}{c}
f \\
Y
\end{array}
\end{pmatrix} = \begin{pmatrix}
\begin{array}{c}
F \times_A X \\
Y
\end{array}
\end{pmatrix}
\]

All compatibilities between these isomorphisms can be derived, too. Each of these compatibilities corresponds to an associativity relation in the fibered multicategory. One can also easily axiomatize the properties of the morphism \(f_1 \to f_\ast\) that often accompanies a six functor context. Can one give a finite list of compatibility diagrams from which all the others would follow?

**Proposition A.2.14.** Let \(\mathcal{D}\) be a (symmetric, braided) multicategory and let \(\mathcal{W}\) be a subclass of 1-ary morphisms. Then there exists a (symmetric, braided) multicategory \(\mathcal{D}[\mathcal{W}^{-1}]\) together with a functor \(\iota : \mathcal{D} \to \mathcal{D}[\mathcal{W}^{-1}]\) of (symmetric, braided) multicategories with the property that \(\iota(w)\) is an isomorphism for all \(w \in \mathcal{W}\) and which is universal w.r.t. this property.

**Proof.** This construction is completely analogous to the construction for usual categories. Morphisms \(\text{Hom}(X_1, \ldots, X_n; Y)\) are formal compositions of \(i\)-ary morphisms in \(\mathcal{D}\) and formal inverses of morphisms in \(\mathcal{W}\), for example:

\[
\begin{align*}
X_1 & \quad \xrightarrow{f_1} \quad \cdots \quad \xrightarrow{w_1} \quad \cdots \\
X_2 & \quad \xrightarrow{f_2} \quad \cdots \\
X_3 & \\
X_4 & \quad \xrightarrow{f_3} \quad \cdots \quad \xrightarrow{w_3} \quad \cdots \\
X_5 & \quad \xrightarrow{f_2} \quad \cdots \quad \xrightarrow{w_2} \\
X_6 &
\end{align*}
\]

More precisely:

Morphisms are defined to be the class of lists of \(n_i\)-ary morphisms \(f_i \in \text{Hom}(X_{i,1}, \ldots, X_{i,n_i}; Y_i)\), morphisms \(w_i : Y'_i \to Y_i\) in \(\mathcal{W}\) and integers \(k_i\):

\[
(f_1, w_1, k_1, (f_2, w_2), k_2, \ldots, k_{n-1}, (f_n, w_n))
\]

such that \(Y'_i = X_{i+1,k_i}\), modulo the obvious relations. 

\(\square\)
References


