Derivator Six-Functor-Formalisms — Construction II

Fritz Hörmann
Mathematisches Institut, Albert-Ludwigs-Universität Freiburg

February 8, 2019

2010 Mathematics Subject Classification: 55U35, 14F05, 18D10, 18D30, 18E30, 18G99
Keywords: fibered multiderivators, six-functor-formalisms

Abstract

Starting from very simple and obviously necessary axioms on a (derivator enhanced) four-functor-formalism, we construct derivator six-functor-formalisms using compactifications. This works, for example, for various contexts over topological spaces and algebraic schemes alike. The formalism of derivator six-functor-formalisms not only encodes all isomorphisms between compositions of the six functors (and their compatibilities) but also the interplay with pullbacks along diagrams and homotopy Kan extensions. One could say: a nine-functor-formalism. Such a formalism allows to extend six-functor-formalisms to stacks using (co)homological descent.
The input datum can, for example, be obtained from a fibration of monoidal model categories.

Contents

1 Introduction

For a detailed introduction to classical six-functor-formalisms we refer to the previous article [13]. Recall that those are defined on a base category $\mathcal{S}$ and specify a (usually derived) category $\mathcal{D}_S$ for each object in $\mathcal{S}$, adjoint pairs of functors

$$\begin{array}{llr} f^* & f_* & \textit{for each } f \textit{ in } \mathrm{Mor}(\mathcal{S}) \\ f_! & f^! & \textit{for each } f \textit{ in } \mathrm{Mor}(\mathcal{S}) \\ \otimes & \mathcal{HOM} & \textit{in each fiber } \mathcal{D}_S \end{array}$$

and the following isomorphisms between the left adjoints (all others follow from those by adjunction):

	isomorphisms between left adjoints
$(*,*)$	$(fg)^* \xrightarrow{\sim} g^* f^*$
$(!,!)$	$(fg)_! \xrightarrow{\sim} f_! g_!$
$(!,*)$	$g^* f_! \xrightarrow{\sim} F_! G^*$
$(\otimes,*)$	$f^*(- \otimes -) \xrightarrow{\sim} f^* - \otimes f^* -$
$(\otimes,!)$	$f_!(- \otimes f^* -) \xrightarrow{\sim} (f_! -) \otimes -$
$(\otimes,\otimes)$	$(- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$

as well as isomorphisms $f_! \cong f_*$ for isomorphisms f [1].
Of course these isomorphisms have to fulfill compatibilities as, for example, the pentagon axiom and many more. In [12, 15] it was explained that using the language of (op)fibrations of 2-multicategories one can package all this information into a neat definition:

Definition. *A (symmetric)* **six-functor-formalism** *on $\mathcal{S}$ is a 1-bifibration and 2-bifibration of (symmetric) 2-multicategories with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}.$$

Such a fibration can also be seen as a pseudo-functor of 2-multicategories

$$\mathcal{S}^{\mathrm{cor}} \to \mathcal{CAT}$$

with the property that all multivalued functors in the image have right adjoints w.r.t. all slots. Note that $\mathcal{CAT}$, the "category"[2] of categories, has naturally the structure of a symmetric 2-"multicategory" where the 1-multimorphisms are functors of several variables.

[1] variant: for all proper morphisms, provided such a class has been specified.
[2] having, of course, a *higher class* of objects.

Here $\mathcal{S}^{\mathrm{cor}}$ is the symmetric 2-multicategory whose objects are the objects of $\mathcal{S}$ and in which a 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ is a multicorrespondence

$$\begin{tikzcd}[column sep=small] & & & A \arrow[dlll, "g_1"'] \arrow[dl, "g_n"] \arrow[dr, "f"] & \\ S_1 & \cdots & S_n & ; & T \end{tikzcd} \tag{1}$$

The composition of 1-morphisms is given by forming fiber products and the 2-morphisms are the isomorphisms of such multicorrespondences.
The pseudo-functor maps the correspondence (1) to a functor isomorphic to

$$f_!((g_1^* -) \otimes_A \cdots \otimes_A (g_n^* -))$$

where $\otimes_A$, $f_!$, and g_i^*, are the images of the following correspondences

$$\begin{tikzcd}[column sep=small] & & A \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ A & A & & A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] & \\ A & & T \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "g_i"'] \arrow[dr, equal] & \\ S_i & & A \end{tikzcd}$$

Six-functor-formalisms were first introduced by Grothendieck, Verdier and Deligne [1–3, 25] and there has been continuously increasing interest in them in various contexts in the last decade [4, 5, 7–10, 18–21, 24, 26].
A **derivator six-functor-formalism** $\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}$ (cf. Definition 10.2 for the precise definition) specifies not only a (derived) category for each object in $\mathcal{S}$ but also a (derived) category for each diagram $I \to \mathcal{S}^{\mathrm{cor}}$ of correspondences in $\mathcal{S}$ for each small category I. Over *constant diagrams* with value $S \in \mathcal{S}$ such datum gives back the derivator enhancement of the derived category $\mathcal{D}_S$. Objects in these categories $\mathbb{D}(I)_{p^*S}$ should be seen as coherent versions of diagrams $I \to \mathcal{D}_S$, i.e. they are commutative diagrams of actual complexes up to point-wise weak-equivalences as opposed to diagrams commuting only up to homotopy. The observation of Grothendieck and Heller was, that this datum suffices to reconstruct the triangulated structure and, for example, to construct total complexes, which is not possible in the world of triangulated categories.
What should the category $\mathbb{D}(I)_X$ over an arbitrary diagram of correspondences $X : I \to \mathcal{S}^{\mathrm{cor}}$ be? A functor $\mathcal{E} : I \to \mathcal{D}$ lying over X specifies for each $i \in I$ an object $\mathcal{E}(i)$ over $X(i)$ and for each morphism $\alpha : i \to j$ a morphism

$$f_! g^* \mathcal{E}(i) \to \mathcal{E}(j) \quad \text{or equivalently} \quad \mathcal{E}(i) \to g_* f^! \mathcal{E}(j)$$

in which f and g are the components of the correspondence $X(\alpha)$. Again, objects in $\mathbb{D}(I)_X$ are certain "coherent enhancements" of such objects. Having those at our disposal allows for very general constructions, as for example the extension of six-functor-formalisms to stacks. Very roughly, a stack presented by a simplicial object $(\Delta^{\mathrm{op}}, S)$ in $\mathcal{S}$ gives rise to two categories[3]

$$\mathbb{D}(\Delta)^{\mathrm{cocart}}_{S^{\mathrm{op}}} \quad \text{and} \quad \mathbb{D}(\Delta^{\mathrm{op}})^{\mathrm{cart}}_{S}. \tag{2}$$

Cohomological (resp. homological) descent as developed in [12] shows that these categories do not depend (up to equivalence) on the actual presentation of the stack, under mild assumptions on the

[3]Here $S^{\mathrm{op}} : \Delta \to \mathcal{S}^{\mathrm{cor}}$ is the diagram obtained from S by flipping all the correspondences.

derivator six-functor-formalism. The left hand side version of the category allows easily for the construction of f^*, f_*-functors for morphisms between stacks and the right hand side version of the category allows easily for the construction of $f_!, f^!$-functors. However, one can show that for an algebraic stack we actually have a diagram of correspondences X of shape $\Delta^{\mathrm{op}} \times \Delta$ and morphisms

$$\begin{tikzcd} & (\Delta^{\mathrm{op}} \times \Delta, X) \arrow[dl] \arrow[dr] & \\ (\Delta^{\mathrm{op}}, S) & & (\Delta, S^{\mathrm{op}}) \end{tikzcd}$$

Using the full derivator six-functor-formalism, this allows to show that the two categories (2) are equivalent, and also that base change and projection formula still hold for the combined operations. This will be explained in detail in a forthcoming article.

The purpose of this article is to *construct* such derivator six-functor-formalisms starting from a "derivator four-functor-formalism". The latter is just a fibered multiderivator $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$, where $\mathbb{S}^{\mathrm{op}}$ is the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ considered as multicategory, setting:

$$\mathrm{Hom}(S_1, \dots, S_n; T) := \mathrm{Hom}(T, S_1) \times \cdots \times \mathrm{Hom}(T, S_n).$$

1.1. In contrast to six-functor-formalisms a "derivator four-functor-formalism" is quite easy to construct. For the definition one needs merely a bifibration of multicategories

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$$

equipped with a class of weak equivalences $\mathcal{W}_S \subset \mathrm{Mor}(\mathcal{D}_S)$ for each object S of $\mathcal{S}$.

Definition 1.2. *We define a pre-multiderivator as follows*[4]*. For any $I \in \mathrm{Cat}$:*

$$\mathbb{D}(I) := \mathrm{Fun}(I, \mathcal{D})[\mathcal{W}_I^{-1}]$$

where $\mathcal{W}_I$ is the class of natural transformations which are element-wise in the union $\mathcal{W} := \bigcup_S \mathcal{W}_S$. The functor p obviously induces a morphism of pre-multiderivators

$$\widetilde{p} : \mathbb{D} \to \mathbb{S}^{\mathrm{op}}.$$

Example 1.3. *A basic example for the situation (1.1) is the bifibration of multicategories*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$$

where $\mathcal{S}^{\mathrm{op}}$ is the opposite of the category of ringed spaces*, considered as multicategory as above. The fiber $\mathcal{D}_{(X,\mathcal{O}_X)}$ over a space $(X, \mathcal{O}_X)$ is the category of (unbounded) chain complexes of sheaves of $\mathcal{O}_X$-modules on X, and the class $\mathcal{W}_{(X,\mathcal{O}_X)}$ is the class of quasi-isomorphisms. The push-forward (resp. pull-back) functors are given by a combination of the tensor product (resp. internal hom) and the usual pull-back (resp. push-forward) of sheaves of $\mathcal{O}_X$-modules. Note that, in this example, the multicategory-structure is even the more natural structure because no particular tensor-product, resp. pull-back, has to be chosen a priori to define it.*

Of course, we would like the morphism of pre-multiderivators $\widetilde{p} : \mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ to be a left (resp. right) fibered multiderivator. This is true, provided that the fibers are model categories whose structures are compatible with the structure of bifibration, as follows:

[4] cf. [12, Appendix A.3] for localizations of multicategories

Definition 1.4 ([12, Definition 5.1.3]). *A* **bifibration of multi-model-categories** *is a bifibration of multicategories $p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ together with the collection of a closed model category structure on the fiber*

$$(\mathcal{D}_S, \mathrm{Cof}_S, \mathrm{Fib}_S, \mathcal{W}_S)$$

for any object S in $\mathcal{S}$ such that the following two properties hold:

1. *For any $n \geq 1$ and for every multimorphism $f \in \mathrm{Hom}(S_1, \dots, S_n; T)$, the push-forward $f_\bullet$ and the various pull-backs $f^{\bullet,j}$ define a Quillen adjunction in n-variables*

$$\textstyle\prod_i (\mathcal{D}_{S_i}, \mathrm{Cof}_{S_i}, \mathrm{Fib}_{S_i}, \mathcal{W}_{S_i}) \xrightarrow{\ f_\bullet\ } (\mathcal{D}_T, \mathrm{Cof}_T, \mathrm{Fib}_T, \mathcal{W}_T)$$

$$(\mathcal{D}_T, \mathrm{Cof}_T, \mathrm{Fib}_T, \mathcal{W}_T) \times \textstyle\prod_{i \neq j} (\mathcal{D}_{S_i}, \mathrm{Cof}_{S_i}, \mathrm{Fib}_{S_i}, \mathcal{W}_{S_i}) \xrightarrow{\ f^{\bullet,j}\ } (\mathcal{D}_{S_j}, \mathrm{Cof}_{S_j}, \mathrm{Fib}_{S_j}, \mathcal{W}_{S_j})$$

2. *a technical condition involving units (i.e. push-forwards along 0-ary morphisms) [loc. cit.].*

Remark 1.5. *If $\mathcal{S} = \{\cdot\}$ is the final multicategory, the above notion coincides with the notion of* closed monoidal model-category *in the sense of [16, Definition 4.2.6]. In this case it is enough to claim property 1. for $n = 2$.*

We have the following generalization of a theorem of Cisinski [6]:

Theorem 1.6 ([14, Theorem 5.8]). *Under the conditions of Definition 1.4 the morphism of pre-multiderivators*

$$\widetilde{p} : \mathbb{D} \to \mathbb{S}^{\mathrm{op}}$$

(defined in 1.2) is a (left and right) fibered multiderivator with domain Cat.

Theorem 1.7 (Hovey and Gillespie [11, 17]). *The bifibration of multicategories from Example 1.3*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{op}}$$

can be equipped with the structure of bifibration of multi-model-categories, yielding thus by Theorem 1.6 a "derivator four-functor-formalism", i.e. a fibered multiderivator

$$\mathbb{D} \to \mathbb{S}^{\mathrm{op}}.$$

There is work in progress [23] by our student René Recktenwald extending these results to (a certain class of) ringed sites.

1.8. Coming back to the case of derivator six-functor-formalisms, the procedure carried out in this article is analogous to the construction of the push-forward with proper support $f_!$ in [3, Exposé XVII] using compactifications. We thus assume that $\mathcal{S}$ is a category with compactifications (cf. 2.1), i.e. that we are given abstract classes of (dense) open embeddings and proper morphisms satisfying the usual properties, and such that every morphism in $\mathcal{S}$ can be factored into a dense open embedding followed by a proper map.

We prove the following

Theorem. *Let $\mathcal{S}$ be a category with compactifications and $\mathbb{S}^{\mathrm{op}}$ the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ with domain* Cat. *Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ a fibered multiderivator with stable and perfectly generated* *fibers satisfying axioms (F1–F6) and (F4m–F5m) below. Assume that $\mathbb{D}$ is infinite (i.e. satisfies (Der1) also for infinite coproducts). Then there exists a natural derivator six-functor-formalism*

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$$

with domain Cat *(small categories) such that the pull-back of $\mathbb{E}$ along the natural morphism*

$$\mathbb{S}^{\mathrm{op}} \to \mathbb{S}^{\mathrm{cor}}$$

is equivalent to $\mathbb{D}$ and such that $f_! \cong f_*$ for proper morphisms f and $\iota^* \cong \iota^!$ for embeddings ι. If $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is symmetric then also $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ is in a canonical way.

Actually, it suffices for the construction that $\mathbb{D}$ is defined on Dirlf (directed locally finite small categories). Using the theory of enlargement [14] it even suffices to have $\mathbb{D}$ defined on Dirpos (directed posets). However, in all cases of interest, Theorem 1.6 gives already a fibered multiderivator with domain Cat.

If $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ has moreover well-generated fibers then also $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ has, and the main theorems of cohomological and homological descent of [12] apply.

The actual construction of $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ is very formal. Stable and perfectly generated fibers are only needed to obtain the right-fiberedness of $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ via Brown representability. The left-fiberedness of $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ can even be obtained without assuming stable fibers.

The axioms (F1–F6) and (F4m–F5m) are very mild and obviously necessary. Only two of them, (F1) and (F3) are actually concerned with the derivator enhancement. They require that the pull-back ι^* along a point-wise open embedding ι has a left adjoint $\iota_!$, that ι^* commutes with homotopy limits as well, and that f_* for a proper morphism f commutes with homotopy colimits as well. The other axioms are the usual formulas (proper projection formula and base change, open embedding "coprojection formula" and base change, etc.) and it is sufficient to check them on the level of usual derived categories. Thus, in all cases of interest, they are well-known statements. Only (F4m) and (F5m) are concerned with the multi- (i.e. monoidal) aspect.

The precise formulation is as follows (the notions *proper* and *embedding* refer to the chosen compactification on $\mathcal{S}$):

(F1) For each diagram $I \in$ Dirlf and point-wise embedding $\iota : S \hookrightarrow T$ in $\mathbb{S}(I^{\mathrm{op}})$, the functor ι^* (aka $(\iota^{\mathrm{op}})_\bullet$) commutes with homotopy limits as well, and has a left adjoint

$$\iota_! : \mathbb{D}(I)_S \to \mathbb{D}(I)_T.$$

(F2) For an embedding $\iota : S \hookrightarrow T$ in $\mathcal{S}$ the corresponding functor

$$\iota_! : \mathbb{D}_S(\cdot) \to \mathbb{D}_T(\cdot)$$

is fully faithful.

(F3) For proper morphisms f in $\mathcal{S}$ the functor f_* commutes with homotopy colimits as well.

(F4) For proper morphisms f in $\mathcal{S}$ and any Cartesian square

$$\begin{tikzcd} \cdot \arrow[r, two heads, "F"] \arrow[d, "G"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, two heads, "f"'] & \cdot \end{tikzcd}$$

we have base change, i.e. the natural exchange morphism

$$G^* F_* \cong f_* g^*$$

is an isomorphism.

(F5) For an embedding ι in $\mathcal{S}$ and for any Cartesian square

$$\begin{tikzcd} \cdot \arrow[r, "F"] \arrow[d, hook, "I"'] & \cdot \arrow[d, hook, "\iota"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

we have base change, i.e. the natural exchange morphism

$$\iota^* f_* \cong F_* I^*$$

is an isomorphism.

(F6) For f proper and ι embedding forming a Cartesian square

$$\begin{tikzcd} \cdot \arrow[r, two heads, "F"] \arrow[d, hook, "I"'] & \cdot \arrow[d, hook, "\iota"] \\ \cdot \arrow[r, two heads, "f"'] & \cdot \end{tikzcd}$$

the exchange of the base change isomorphism from (F4) (equivalently from (F5))

$$\iota_! F_* \to f_* I_!$$

is an isomorphism as well.

(F4m) For proper morphisms f in $\mathcal{S}$ we have projection formulas, i.e. the natural exchange morphisms

$$(f_* -) \otimes - \cong f_*(- \otimes f^* -) \qquad - \otimes (f_* -) \cong f_*((f^* -) \otimes -)$$

are isomorphisms[5].

(F5m) For an embedding ι in $\mathcal{S}$ we have "coprojection formulas", i.e. the natural exchange morphisms

$$\iota^* \mathcal{HOM}_l(-,-) \cong \mathcal{HOM}_l(\iota^* -, \iota^* -) \qquad \iota^* \mathcal{HOM}_r(-,-) \cong \mathcal{HOM}_r(\iota^* -, \iota^* -)$$

are isomorphisms[6].

In a subsequent article we will investigate the validity of the axioms in various contexts. This should include the cases of Abelian sheaves on (nice enough) topological spaces, and in algebraic geometry various contexts of (Ind)coherent sheaves, Abelian pro-etale sheaves, motives, the stable homotopy categories, etc.

[5] In case $\mathbb{D}$ is symmetric these two assertions are equivalent.

[6] In case $\mathbb{D}$ is symmetric $\mathcal{HOM}_l = \mathcal{HOM}_r$.

2 Compactification: The axioms

2.1. Let $\mathcal{S}$ be a category with finite limits (equivalently: with final object and pull-backs). We say that $\mathcal{S}$ is a **category with compactifications** if we are given subclasses $\mathcal{S}_i$, $i = 0, 1, 2$ of morphisms of $\mathcal{S}$ satisfying properties (S0)–(S5) below. We call morphisms in $\mathcal{S}_2$ **embeddings**, morphisms in $\mathcal{S}_1$ **dense embeddings** and morphisms in $\mathcal{S}_0$ **proper**.

(S0) $\mathcal{S}_1 \subset \mathcal{S}_2$ and if $\gamma = \beta\alpha$ with $\alpha, \beta \in \mathcal{S}_2$ and $\gamma \in \mathcal{S}_1$ then $\alpha, \beta \in \mathcal{S}_1$.

(S1) $\mathcal{S}_0 \cap \mathcal{S}_1$ is the class of isomorphisms in $\mathcal{S}$.

(S2) If $g \in \mathcal{S}_i$ then $f \in \mathcal{S}_i \Leftrightarrow g \circ f \in \mathcal{S}_i$;

(S3) $\mathcal{S}_0$ and $\mathcal{S}_2$ are stable under pull-back;

(S4) For any object S the diagonal $\Delta : S \to S \times S$ is in $\mathcal{S}_0$;

(S5) Any morphism $f : S \to T$ can be factored as $f = \overline{f} \circ \iota$ with $\overline{f} \in \mathcal{S}_0$ and $\iota \in \mathcal{S}_1$.

In diagrams we will denote embeddings by the symbol $\hookrightarrow$ and proper morphisms by the symbol $\twoheadrightarrow$. A choice of factorization as in (S5) will be called a **compactification** of f. In the relevant examples morphisms in $\mathcal{S}_2$ will be something like open embeddings and morphisms in $\mathcal{S}_1$ dense open embeddings.

Example 2.2. *$\mathcal{S}$ is the category of quasi-compact, separated schemes, $\mathcal{S}_2$ (resp. $\mathcal{S}_1$) the class of (dense) open immersions and $\mathcal{S}_0$ the class of proper morphisms. (S5) is Deligne's extension of Nagata's compactification Theorem*[7].

Example 2.3. *$\mathcal{S}$ is the category of locally compact Hausdorff topological spaces, $\mathcal{S}_2$ (resp. $\mathcal{S}_1$) the class of (dense) open immersions and $\mathcal{S}_0$ the class of proper morphisms.*

We say that a subclass $\mathcal{S}_i$ of morphisms in $\mathcal{S}$ is **stable under limits of shape** I if for each morphism f in the category $\mathrm{Fun}(I, \mathcal{S})$, which is point-wise in $\mathcal{S}_i$, it follows that $\lim_I f$ is in $\mathcal{S}_i$.

Lemma 2.4. *Let $\mathcal{S}$ be a category with compactifications.*

1. *$\mathcal{S}_0$ and $\mathcal{S}_2$ are stable under finite products, i.e. stable under limits of shape a finite set.*
2. *$\mathcal{S}_0$ is stable under fiber products, i.e. stable under limits of shape $\lrcorner$.*
3. *$\mathcal{S}_0$ is stable under taking arbitrary finite limits.*
4. *For a finite diagram $F : I \to \mathcal{S}$ where I has final object, and all morphisms in F are in $\mathcal{S}_0$, the projections $\lim_I F \to F(i)$ are in $\mathcal{S}_0$ for all $i \in I$.*

Proof. 1. Using (S3) the Cartesian diagram

$$\begin{tikzcd} X \times Z \arrow[r, "\alpha\times\mathrm{id}"] \arrow[d] & Y \times Z \arrow[d] \\ X \arrow[r, "\alpha"] & Y \end{tikzcd}$$

[7] For (S5) it is actually sufficient that the destination of the morphism is quasi-separated but in the sequel we will need property (S4).

shows that morphisms of the form $\alpha \times \mathrm{id}$ are in $\mathcal{S}_i$ (and similarly also of the form $\mathrm{id} \times \alpha$) hence also products of morphisms.
2. Since diagonals are in $\mathcal{S}_0$ by (S4), the Cartesian diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, two heads] \arrow[d] & X \times Z \arrow[d] \\ Y \arrow[r, two heads, "\Delta"] & Y \times Y \end{tikzcd}$$

shows that the morphisms $X \times_Y Z \to X \times Z$ are in $\mathcal{S}_0$.
We have a commutative diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, two heads] \arrow[d] & X \times Z \arrow[d, two heads] \\ X' \times_{Y'} Z' \arrow[r, two heads] & X' \times Z' \end{tikzcd}$$

All morphisms except possibly the left vertical one are in $\mathcal{S}_0$. Hence by (S2) also the left vertical one is.
3. Let $f, f' : S \to T$ be two morphisms. The equalizer of f and f' can be computed by the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Eq}(f, f') \arrow[r] \arrow[d] & T \arrow[d, "\Delta"] \\ S \arrow[r, "{(f,f')}"'] & T \times T \end{tikzcd}$$

i.e. as a fiber product.
Since a finite limit can be computed by a finite product and an equalizer, $\mathcal{S}_0$ is stable under finite limits by 1. and 2.
4. Let I have a final object j and let $F : I \to \mathcal{S}$ be a functor. We have $F(j) = \lim_I F(j)$, where $F(j)$, by abuse of notation, also denotes the constant diagram with value $F(j)$. Then the morphism

$$\lim_I F \to \lim_I F(j)$$

is in $\mathcal{S}_0$, as was just shown. For any object $i \in I$, it factors as follows:

$$\lim_I F \to F(i) \to F(j) = \lim_I F(j)$$

and the rightmost morphism is in $\mathcal{S}_0$ by assumption, hence so is the projection $\lim_I F \to F(i)$ by (S2). $\square$

Lemma 2.5. *Any diagram of the form*

$$\begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d, hook] & \cdot \arrow[d, hook] \\ \cdot \arrow[r, two heads] & \cdot \end{tikzcd}$$

in which the left vertical embedding is dense, is Cartesian.

Proof. If we form the pull-back

$$\begin{tikzcd}
{} \arrow[dr, "a"] \arrow[drr, hook, two heads] \arrow[ddr, two heads] & & \\
& \square \arrow[r, two heads] \arrow[d, hook, two heads] & {} \arrow[d, hook] \\
& {} \arrow[r, two heads] & {}
\end{tikzcd}$$

then the morphisms are embeddings (resp. proper) as indicated using (S3). By (S2) the morphism a is proper and an embedding, which is dense by (S0), hence by (S1) an isomorphism. □

2.6. Let $\mathcal{S}$ be a category with compactifications. We say that a square

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & Z \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

in $\mathcal{S}$ is **weakly Cartesian** if the induced morphism $W \to X \times_Y Z$ is proper.

Lemma 2.7. *If in the diagram*

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, hook] & {} \arrow[d, hook] \\
{} \arrow[r] \arrow[d, two heads] & {} \arrow[d, two heads] \\
{} \arrow[r] & {}
\end{tikzcd}$$

the top left vertical embedding is dense and the outer square is weakly Cartesian then the upper square is Cartesian.

Proof. We may form a diagram in which the right squares are Cartesian:

$$\begin{tikzcd}
{} \arrow[r, two heads, "d"] \arrow[d, hook] & \square \arrow[r] \arrow[d, hook, "a"] & {} \arrow[d, hook] \\
{} \arrow[r, two heads, "c"] \arrow[dr, two heads] & \square \arrow[r] \arrow[d, two heads, "b"] & {} \arrow[d, two heads] \\
& {} \arrow[r] & {}
\end{tikzcd}$$

where the morphisms a, b, c, d are an embedding (resp. proper) by (S3) and (S2) and the definition of weakly Cartesian. Now the upper right square is Cartesian by construction and the upper left square is Cartesian because of Lemma 2.5. Hence also the composite square, which is the upper square in the original diagram is Cartesian. □

3 Compactification of morphisms of inverse diagrams

Proposition 3.1. *Let $\mathcal{S}$ be a category with compactifications and let I be an inverse diagram with finite matching diagrams.*

1. *Let $f : F \to G$ be a morphism in $\mathrm{Fun}(I, \mathcal{S})$. The morphism can be factored*

$$f = \overline{f} \circ \iota$$

where $\overline{f}$ is point-wise in $\mathcal{S}_0$ and ι is point-wise in $\mathcal{S}_1$.

2. *Any two such factorizations are dominated by a third in the sense that for two factorizations $f = \overline{f}_i \circ \iota_i, i = 1,2$ we get a third factorization $f = \overline{f}_3 \circ \iota_3$ and a diagram*

$$
\begin{tikzcd}
 & \arrow[ddl, hook', "\iota_1"'] \arrow[d, hook, "\iota_3"] \arrow[ddr, hook, "\iota_2"] & \\
 & \arrow[dl, "\overline{f}_{3,1}"'] \arrow[dr, "\overline{f}_{3,2}"] & \\
 \arrow[dr, "\overline{f}_1"'] & & \arrow[dl, "\overline{f}_2"] \\
 & &
\end{tikzcd}
$$

such that $\overline{f}_3 = \overline{f}_1 \circ \overline{f}_{3,1} = \overline{f}_2 \circ \overline{f}_{3,2}$.

3. *Any given compactification of f restricted to a final subdiagram of I can be extended to a compactification of the whole morphism f.*

Proof. 1. We construct the compactification element-wise using induction on the degree of the element as usual. For degree 1 it follows directly from the compactification axiom. Let now f be a morphism which for elements of degree $< n$ has been factored as required. For each $i \in I$ of degree n we get a diagram:

$$
\begin{tikzcd}
F(i) \arrow[rr, "f(i)"] \arrow[d] & & G(i) \arrow[d] \\
\lim_{M_i} F \arrow[r, "\lim_{M_i} \iota"] & \lim_{M_i} \overline{F} \arrow[r, two heads, "\lim_{M_i} \overline{f}"] & \lim_{M_i} G
\end{tikzcd}
$$

in which M_i is the matching diagram, i.e. the full subcategory of the comma-category $i \times_{/I} I$ consisting of non-identities. By Lemma 2.4, 3. and the assumption that the matching diagrams be finite, $\lim_{M_i} \overline{f}$ is again proper. Hence by forming the pull-back and compactifying the induced morphism using (S5), we get:

$$
\begin{tikzcd}
F(i) \arrow[r, hook, "\iota(i)"] \arrow[d] & \overline{F}(i) \arrow[r, two heads] \arrow[rr, bend left, two heads, "\overline{f}(i)"] & \square \arrow[r, two heads] \arrow[d] & G(i) \arrow[d] \\
\lim_{M_i} F \arrow[rr] & & \lim_{M_i} \overline{F} \arrow[r, two heads, "\lim_{M_i} \overline{f}"] & \lim_{M_i} G
\end{tikzcd}
$$

and define $\iota(i), \overline{f}(i)$ and $\overline{F}(i)$ to be the so denoted objects in this diagram. Note that $\overline{f}(i)$ is proper by axioms (S2) and (S3).

2. The statement is again proved by induction. For degree 1 elements let two compactifications $f = \overline{f}_i \circ \iota_i$ be given. We form the fiber product of the $\overline{f}_i$ and compactify the induced morphism

using (S5):

$$
\begin{tikzcd}
& F(i) \arrow[d, hook, "\iota"] \arrow[ddddl, hook', "\iota_1"'] \arrow[ddddr, hook, "\iota_2"] & \\
& \overline{F}_3(i) \arrow[d, two heads, "\overline{f}"] & \\
& \square \arrow[dl, two heads] \arrow[dr, two heads] & \\
\overline{F}_1(i) \arrow[dr, two heads, "\overline{f}_1"'] & & \overline{F}_2(i) \arrow[dl, two heads, "\overline{f}_2"] \\
& G(i) &
\end{tikzcd}
$$

It is clear how to extract a diagram as in the statement from this. Let now the diagram for elements of degree $< n$ be constructed. We get a diagram

$$
\begin{tikzcd}
& \lim_{M_i} \overline{F}_3 \arrow[dl, two heads] \arrow[dr, two heads] & \\
\lim_{M_i} \overline{F}_1 \arrow[dr, two heads] & & \lim_{M_i} \overline{F}_2 \arrow[dl, two heads] \\
& \lim_{M_i} G &
\end{tikzcd}
$$

in which all the morphisms are proper. Pulling it back along the morphism $G(i) \to \lim_{M_i} G$ and inserting the given compactifications, we arrive at

$$
\begin{tikzcd}
& F(i) \arrow[dl, hook'] \arrow[dr, hook] & \\
\overline{F}_1(i) \arrow[dd, two heads] & & \overline{F}_2(i) \arrow[dd, two heads] \\
& \square \arrow[dl, two heads] \arrow[dr, two heads] & \\
\square \arrow[dr, two heads] & & \square \arrow[dl, two heads] \\
& G(i) &
\end{tikzcd}
$$

Using Lemma 2.4, 4. we see that the limit over the diagram consisting of the proper morphisms fits in a diagram

$$\begin{tikzcd}
& F(i) \arrow[ddl, hook] \arrow[ddr, hook] \arrow[d, dotted, "x"] & \\
& \lim \arrow[dl] \arrow[dr] \arrow[dd] & \\
\overline{F}_1(i) \arrow[dd] & & \overline{F}_2(i) \arrow[dd, two heads] \\
& \square \arrow[dl] \arrow[dr] & \\
\square \arrow[dr] & & \square \arrow[dl] \\
& G(i) &
\end{tikzcd}$$

in which all so indicated morphisms are proper. Now compactify the dotted morphism. It is clear that we may extract from this a diagram as claimed in the assumption.
3. Clear. $\square$

4 The induced compactification of diagrams

Recall the following from [13, 7.3]:

4.1. Let I be a diagram, n a natural number and $\Xi = (\Xi_1, \dots, \Xi_n) \in \{\uparrow, \downarrow\}^n$ be a sequence of arrow directions. We define a diagram

$$^{\Xi}I$$

whose objects are sequences of n objects and $n-1$ morphisms in I

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_n$$

and whose morphisms are commutative diagrams

$$\begin{tikzcd}
i_1 \arrow[r] \arrow[d, leftrightarrow] & i_2 \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & i_n \arrow[d, leftrightarrow] \\
i_1' \arrow[r] & i_2' \arrow[r] & \cdots \arrow[r] & i_n'
\end{tikzcd}$$

in which the j-th vertical arrow goes in the direction indicated by Ξ_j. We call a morphism **of type** j if at most the morphism $i_j \to i_j'$ is *not* an identity.
For a diagram I and an object $i \in I$ we adopt the convention that i denotes also the subcategory of I consisting only of i and its identity. In coherence with this convention $^{\Xi}i$ denotes the subcategory of $^{\Xi}I$ consisting of the sequence $i = \cdots = i$ and its identity.
Examples: $^{\downarrow\downarrow}I = I \times_{/I} I$ is the comma category, $^{\uparrow}I = I^{\mathrm{op}}$, and $^{\downarrow\uparrow}I$ is the twisted arrow category.

4.2. For any ordered subset $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, denoting Ξ' the restriction of Ξ to the subset, we get an obvious restriction functor

$$\pi_{i_1,\dots,i_m} : {}^{\Xi}I \to {}^{\Xi'}I.$$

If $\Xi = \Xi' \circ \Xi'' \circ \Xi'''$, where $\circ$ means concatenation, then the projection

$$\pi_{1,\dots,n'} : \ {}^{\Xi}I \to {}^{\Xi'}I$$

is a *fibration* if the last arrow of Ξ' is $\downarrow$ and an *opfibration* if the last arrow of Ξ' is $\uparrow$ while the projection

$$\pi_{n-n'''+1,\dots,n} : \ {}^{\Xi}I \to {}^{\Xi'''}I$$

is an *opfibration* if the first arrow of Ξ''' is $\downarrow$ and a *fibration* if the first arrow of Ξ''' is $\uparrow$.

For the rest of the section, fix a category $\mathcal{S}$ with compactifications (Definiton 2.1).

Definition 4.3. *Let I be a diagram and $S : I \to \mathcal{S}$ a functor. Any morphism $S \hookrightarrow \overline{S}$ in $\mathrm{Fun}(I, \mathcal{S})$ consisting point-wise of dense embeddings, and such that $\overline{S}$ is a diagram in which all morphisms are proper, is called an* **exterior compactification** *of S. A factorization $S \hookrightarrow \overline{S} \twoheadrightarrow \overline{S}'$, in which the composition is an exterior compactification again, is called a* **refinement***,*

We claim that exterior compactifications exist for I an inverse diagram with finite matching diagrams: First compactify the morphism $F \to \cdot$ using Proposition 3.1, where $\cdot$ is the induced final object of $\mathrm{Fun}(I, \mathcal{S})$:

$$F \xrightarrow{\iota} \overline{F} \xrightarrow{\overline{f}} \cdot$$

Then ι is an exterior compactification because all morphisms in the diagram $\overline{F}$ are automatically proper because of (S2).

Definition 4.4. *Let I be a diagram. A functor $\widetilde{S} : {}^{\downarrow\downarrow}I \to \mathcal{S}$ (see 4.1 for the notation) together with an isomorphism $\Delta^*\widetilde{S} \cong S$ is called an* **interior compactification** *of S if every morphism of type 2 (cf. 4.1) is mapped to a dense embedding and every morphism of type 1 is mapped to a proper morphism. A morphism $\widetilde{S}_1 \to \widetilde{S}_2$ of compactifications (i.e. a morphism compatible with the isomorphisms $\Delta^*\widetilde{S} \cong S$) is called a* **refinement** *if it consists point-wise of proper morphisms.*

Proposition 4.5. *Let I be a diagram and $F \hookrightarrow \overline{F}$ be an exterior compactification in $\mathrm{Fun}(I, \mathcal{S})$. Then there is a canonical* **induced interior compactification** *$\widetilde{F} \in \mathrm{Hom}({}^{\downarrow\downarrow}I, \mathcal{S})$. The association*

$$(F \hookrightarrow \overline{F}) \mapsto \widetilde{F}$$

has the following properties

1. *It is functorial in exterior compactifications, i.e. if*

$$\begin{tikzcd} F \arrow[r] \arrow[d, hook] & G \arrow[d, hook] \\ \overline{F} \arrow[r] & \overline{G} \end{tikzcd}$$

is a commutative diagram in which the vertical morphisms are exterior compactifications then there is an induced morphism

$$\widetilde{F} \to \widetilde{G}.$$

This association is functorial. Note that in the diagram above, $\overline{F} \to \overline{G}$ is automatically point-wise proper.

2. For a refinement of exterior compactifications consider the diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, hook] & F \arrow[d, hook] \\ \overline{F} \arrow[r, two heads] & \overline{F}' \end{tikzcd}$$

Then the induced morphism $\widetilde{F} \to \widetilde{F}'$ is a refinement.

Proof. Note that the comma category ${}^{\downarrow\downarrow}I = I \times_{/I} I$ comes equipped with the following 2-commutative diagram

$$\begin{tikzcd} {}^{\downarrow\downarrow}I \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\nearrow\mu"] & I \arrow[d, equal] \\ I \arrow[r, equal] & I \end{tikzcd}$$

We let $\widetilde{F} : {}^{\downarrow\downarrow}I \to \mathcal{S}$ be the pull-back:

$$\begin{tikzcd} \widetilde{F} \arrow[r] \arrow[d] & \pi_2^* F \arrow[d, "\pi_2^*\iota"] \\ \pi_1^*\overline{F} \arrow[r] & \pi_2^*\overline{F} \end{tikzcd}$$

where the bottom horizontal morphism is induced by the natural transformation $\mu : \pi_1 \Rightarrow \pi_2$. We have to see that $\widetilde{F}$ is an interior compactification. First of all taking Δ^* of the diagram for $\Delta : I \to {}^{\downarrow\downarrow}I$ being the diagonal, we arrive at:

$$\begin{tikzcd} \Delta^*\widetilde{F} \arrow[r] \arrow[d] & F \arrow[d, "\iota"] \\ \overline{F} \arrow[r, equal] & \overline{F} \end{tikzcd}$$

hence there is a canonical isomorphism $\Delta^*\widetilde{F} \cong F$.
By definition a morphism

$$\begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ i' \arrow[r] & j' \end{tikzcd}$$

in ${}^{\downarrow\downarrow}I$ is mapped by $\widetilde{F}$ to the morphism

$$\widetilde{F}(i \to j) = \overline{F}(i) \times_{\overline{F}(j)} F(j) \to \overline{F}(i') \times_{\overline{F}(j')} F(j') = \widetilde{F}(i' \to j')$$

Since the morphism $\overline{F}(i) \to \overline{F}(i')$ is proper this morphism is proper if $j = j'$ by Lemma 2.4, 2. If $i = i'$ look at the following commuative diagram:

$$\begin{tikzcd} F(i) \arrow[r, hook] \arrow[d, equal] & \widetilde{F}(i \to j) = \overline{F}(i) \times_{\overline{F}(j)} F(j) \arrow[r, hook] \arrow[d] & \overline{F}(i) \arrow[d, equal] \\ F(i) \arrow[r, hook] & \widetilde{F}(i \to j') = \overline{F}(i) \times_{\overline{F}(j')} F(j') \arrow[r, hook] & \overline{F}(i) \end{tikzcd}$$

The horizontal morphisms are all embeddings by construction, by (S2), and by (S3). Since the composition $F(i) \hookrightarrow \overline{F}(i)$ is dense by construction, the horizontal embeddings are all dense by (S0) and hence so is the the middle vertical one by (S2). The observations together imply that $\widetilde{\overline{F}}$ is an interior compactification of F. The claimed functoriality is clear. $\square$

5 Compactifications of diagrams of correspondences

We need a Lemma on diagram categories:

Lemma 5.1. *1. If I is in* Catlf *(locally finite diagrams*[8]*) then ${}^{\downarrow\uparrow}I$ is inverse and locally finite and has finite matching diagrams.*

2. If I is in Dirlf *then also ${}^{\uparrow\uparrow\downarrow}I$ and ${}^{\downarrow\uparrow\uparrow\downarrow}I$ are in* Dirlf*.*

Proof. 1. We define a functor $\nu : {}^{\downarrow\uparrow}I \to (\mathbb{N}_0)^{\mathrm{op}}$ with maps a morphism $\alpha : i \to J$ to the maximum number of (non-identity) morphisms into which α can be factored or to 0 if α is an identity. The property of ν being a functor and that pre-images of identities are identities is clear. Any morphism from α to another β in ${}^{\downarrow\uparrow}I$ corresponds to a factorization of α. Since I is locally finite the matching category of α is thus finite.
2. If I is in Catlf also ${}^{\downarrow\downarrow}I$ is in Catlf. By 1. thus ${}^{\uparrow\uparrow\downarrow}I$ (which is a subcategory of $({}^{\downarrow\uparrow}({}^{\downarrow\downarrow}I))^{\mathrm{op}}$) is in Dirlf. Therefore also $I \times {}^{\uparrow\uparrow\downarrow}I$ and finally ${}^{\downarrow\uparrow\uparrow\downarrow}I$ are in Dirlf. $\square$

5.2. Let $\Xi \in \{\downarrow\uparrow\}^n$ be a sequence of arrow directions. Recall that a morphism $f : S \to T$ in $\mathrm{Fun}({}^{\Xi}I, \mathcal{S})$ is called **type i admissible** if for every type i morphism $\alpha \to \beta$ the square

$$\begin{tikzcd} S(\alpha) \arrow[r, "f(\alpha)"] \arrow[d] & T(\alpha) \arrow[d] \\ S(\beta) \arrow[r, "f(\beta)"'] & T(\beta) \end{tikzcd}$$

is Cartesian. Having chosen a class $\mathcal{S}_0$ of proper morphisms in $\mathcal{S}$, we will call a morphism $f : S \to T$ in $\mathrm{Fun}({}^{\Xi}I, \mathcal{S})$ **weakly type i admissible** if the squares above are weakly Cartesian (cf. 2.6). Of course this definition depends on the chosen class of proper morphisms. Note that the analog of [13, Lemma 7.10] holds true for *weakly type i admissible.*
Let $\mathcal{S}$ now be an opmulticategory (in this article always a usual category $\mathcal{S}$ equipped with the opmulticategory structure encoding the product). In the same way, if $f : S \to T_1, \dots, T_n$ is a multimorphism in $\mathrm{Fun}({}^{\Xi}I, \mathcal{S})$ then we say that f is **type i admissible** if for every type i morphism $\alpha \to \beta$ the square

$$\begin{tikzcd} S(\alpha) \arrow[r, "f(\alpha)"] \arrow[d] & T_1(\alpha), \dots, T_n(\alpha) \arrow[d] \\ S(\beta) \arrow[r, "f(\beta)"'] & T_1(\beta), \dots, T_n(\beta) \end{tikzcd}$$

is Cartesian (i.e. a multi-pullback). Similarly for weakly type i admissible.

[8] A diagram (i.e. a small category) is called **locally finite** if any morphism can be factored only in a finite number of ways into non-identity morphisms.

5.3. Let $\mathcal{S}^{\mathrm{cor}}$ (resp. $\mathcal{S}^{\mathrm{cor},0}$) be the symmetric 2-multicategory of multicorrespondences in $\mathcal{S}$ in which the 2-morphisms are given by the isomorphisms (resp. proper morphisms — same class as in the definition of compactification on $\mathcal{S}$), cf. [15, 3.6]. The objects of both categories are the same as the objects of $\mathcal{S}$, and 1-morphisms $S_1, \dots, S_n \to T$ are multicorrespondences

$$\begin{tikzcd} & & & A \arrow[llld, "g_1"'] \arrow[ld, "g_n"] \arrow[rd, "f"] & \\ S_1 & \cdots & S_n & ; & T \end{tikzcd} \tag{3}$$

The composition of 1-morphisms is given by forming fiber products and the 2-morphisms are the isomorphisms (in $\mathcal{S}^{\mathrm{cor}}$), or proper morphisms (in $\mathcal{S}^{\mathrm{cor},0}$), of such multicorrespondences. The operation of the symmetric groups is the obvious one.

In reality the above definition gives only bimulticategories because the formation of fiber products is only associative up to isomorphism. One can, however, enlarge the class of objects adjoining strictly associative fiber products. We sketch the precise construction of equivalent 2-multicategories with are also strictly symmetric.

Consider the following class of objects, called **abstract fiber products**. An abstract fiber product is a finite unoriented tree (in the sense of graph theory). Each vertex v and each edge e has an associated object X_v, resp. X_e, in $\mathcal{S}$. For each edge e there are morphisms $X_v \to X_e \leftarrow X_{v'}$ from the objects corresponding to the vertices of the edge. A *morphism* from such on object X to an object $S \in \mathcal{S}$ is given by the choice of a vertex v and a morphism $X_v \to S$. From a diagram $X \to S \leftarrow Y$ where X and Y are abstract fiber products and $S \in \mathcal{S}$, an abstract fiber product (called concatenation) can be formed, adding a new edge with associated object S to the disjoint union of X and Y. Each such abstract fiber product has a *reduced form* in which for each identity, say in an edge $X_v \to X_e = X_{v'}$, the following operation is executed: The edge e is removed, identifying the vertices v and v', and each morphism going out of $X_{v'}$ is composed with the morphism $X_v \to X_e = X_{v'}$.

With each abstract fiber product X one can associate an actual fiber product "$\lim X$" in $\mathcal{S}$ (the limit over X seen as the obvious diagram of the X_e and X_v), and with a morphism $X \to S$ the obvious projection $\lim X \to S$. Concatenation corresponds (up to unique isomorphism) to the formation of fiber product.

Now define the symmetric 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ as follows: The objects are the objects of $\mathcal{S}$, 1-morphisms $S_1, \dots, S_n \to T$ are reduced abstract fiber products X together with morphisms (in the sense above) to $S_1, \dots, S_n$, and T. There is an obvious composition given by concatenation and subsequent reduction which is strictly associative and has units. There is also an operation of the symmetric groups compatible with composition. To define the sets of 2-morphisms we choose an actual fiber product for each abstract fiber product. This maps each 1-morphism to a multicorrespondence like (3) and the sets of 2-morphisms are defined to be the isomorphisms between these multicorrespondences. Similarly the category $\mathcal{S}^{\mathrm{cor},0}$ is defined allowing all proper morphisms between these multicorrespondences.

Finally, there are obvious strict functors $\mathcal{S} \to \mathcal{S}^{\mathrm{cor}}$, and $\mathcal{S}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{cor}}$, and for each $S \in \mathcal{S}$ a strict functor $\{\cdot\} \to \mathcal{S}^{\mathrm{cor}}$ from the final multicategory with value S.

5.4. Note that a diagram of correspondences $X : I \to \mathcal{S}^{\mathrm{cor}}$ (or equivalently $X : I \to \mathcal{S}^{\mathrm{cor},0}$) can be (up to isomorphism; thus the strictification discussed above does not matter) equivalently given by a diagram

$${}^{\downarrow\uparrow}I \to \mathcal{S}$$

in which

- every square in which the horizontal morphisms are of type 2 and the vertical morphisms are of type 1 is mapped to a Cartesian square.

We call such functors ${}^{\downarrow\uparrow}I \to \mathcal{S}$ **admissible** or, by abuse of notation, **diagrams of correspondences**. We denote the corresponding full subcategory of the functor category by $\mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})^{\mathrm{adm}}$. We will also call a functor ${}^{\downarrow\uparrow}I \to \mathcal{S}$ **weakly admissible**, if all squares as above are mapped to weakly Cartesian squares (cf. 2.6). A multimorphism $\xi : X_1, \dots, X_n \to Y$ in the multicategory $\mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$ can be given by a diagram in $\mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})^{\mathrm{adm}}$

$$\begin{tikzcd} & & A \arrow[lld, "g_1"'] \arrow[ld, "g_n"] \arrow[rd, "f"] & \\ X_1 \quad \dots & X_n & ; & Y \end{tikzcd} \tag{4}$$

in which f is type 2 admissible, and g *as a multimorphism in* $\mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})$ is type 1 admissible. A lax multimorphism $X_1, \dots, X_n \to Y$, i.e. a multimorphism in $\mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0})$ is given by a diagram of the same shape, in which, however the morphism f is *weakly type 2 admissible.* Similarly an oplax multimorphism $X_1, \dots, X_n \to Y$ is given by the same diagram, in which, however the multimorphism g is only *weakly type 1 admissible.* A 2-morphism $\mu : \xi \to \xi'$ can be given by a morphism of multicorrespondences

$$\begin{tikzcd} & & A \arrow[lld, "g_1"'] \arrow[ld, "g_n"] \arrow[dd, "h"] \arrow[rd, "f"] & \\ X_1 \quad \dots & X_n & & Y \\ & & A \arrow[llu, "g_1'"] \arrow[lu, "g_n'"'] \arrow[ru, "f'"'] & \end{tikzcd} \tag{5}$$

in which the morphism h is an isomorphism (or, in the lax and oplax case, any proper map). It is (similarly to the case of $\mathcal{S}^{\mathrm{cor},G}$, cf. [15, Definition 3.2]) automatically type 1 admissible and weakly type 2 admissible in the lax case and weakly type 1 admissible and type 2 admissible in the oplax case.

Definition 5.5. *Let I be a locally finite diagram. We define a 2-multicategory $\mathbb{S}^{\mathrm{cor,comp}}(I)$ (resp. $\mathbb{S}^{\mathrm{cor,0,comp,lax}}(I)$ and $\mathbb{S}^{\mathrm{cor,0,comp,oplax}}(I)$) as the following: Objects are exterior compactifications (cf. Definition 4.3) $X \hookrightarrow \overline{X}$ of diagrams $\mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})$ where X is admissible, i.e. associated with an object in $\mathbb{S}^{\mathrm{cor}}(I)$[9] as described in 5.4. Note that $\overline{X}$ is* not *assumed to be admissible.*
For $X_1, \dots, X_n, Y \in \mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})$ diagrams of correspondences of shape I with exterior compactifications $X_i \hookrightarrow \overline{X}_i$ and $Y \hookrightarrow \overline{Y}$ we define morphism categories

$$\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,comp}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y})$$
$$\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,0,comp,lax}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y})$$
$$\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,0,comp,oplax}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y})$$

whose 1- and 2-morphisms are diagrams (4) and (5) as in 5.4 equipped with an exterior compactification restricting to the respective compactifications given on source and destinations. Finally we

[9]Note that $\mathbb{S}^{\mathrm{cor}}(I)$, $\mathbb{S}^{\mathrm{cor,0,lax}}(I)$, resp. $\mathbb{S}^{\mathrm{cor,0,oplax}}(I)$ all have the same class of objects.

formally invert all 2-morphisms given by an exterior compactification of a diagram (5) in which h is an identity. We call such 2-morphisms **refinements**. *Note that for each $\sigma \in S_n$ there is an isomorphism of categories*

$$\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,comp}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y}) \to \mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,comp}}(I)}(\overline{X}_{\sigma(1)}, \dots, \overline{X}_{\sigma(n)}; \overline{Y})$$

forming an action. Similarly for the other categories.

We denoted the morphism categories with a prime because they are not yet the final ones. One them one cannot define a *strictly* associative composition of 1-multimorphisms. First, we need the following Lemmas. We speak of the objects in the above morphism categories already as 1-morphisms and of the morphisms as 2-morphisms to not get confused.

Lemma 5.6. *Each 2-morphism in the morphism categories defined in Definition 5.5 can be represented by a roof $\widetilde{\mu} \circ \widetilde{\nu}^{-1}$ where the underlying 2-morphism of $\widetilde{\nu}$ is an identity.*

Proof. For this, is suffices to show that each composition of the form $(\widetilde{\nu}')^{-1} \circ \widetilde{\mu}'$, where the underlying 2-morphism of $\widetilde{\nu}'$ is an identity can be represented this way. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & & A \arrow[dl, equal] \arrow[dr, "\mu"] & \\
 & & A \arrow[dr, "\mu"] & & A' \arrow[dl, equal] \\
 & & & A' \arrow[dlll] \arrow[dl] \arrow[dr] & \\
X_1 & \dots & X_n & ; & Y
\end{tikzcd}$$

The lower part (except the topmost A) is compactified by means of $\widetilde{\mu}'$ and $\widetilde{\nu}'$. Hence extending the compactification to the whole diagram (cf. Proposition 3.1, 3.), we get a representation of the form $\widetilde{\mu} \circ \widetilde{\nu}^{-1}$ of the same 2-morphism. □

Lemma 5.7. *For two* parallel *morphisms in the categories defined in Definition 5.5 represented by roofs $\widetilde{\mu}_i \circ \widetilde{\nu}_i^{-1}$, $i = 1, 2$, we have $\mu_1 = \mu_2 \Rightarrow \widetilde{\mu}_1 \circ \widetilde{\nu}_1^{-1} = \widetilde{\mu}_2 \circ \widetilde{\nu}_2^{-1}$.*

Proof. Denote $\mu := \mu_1 = \mu_2$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & A \arrow[dl, equal] \arrow[dr, equal] & & \\
 & A \arrow[d, equal] \arrow[drr, "\mu"] & & A \arrow[dll, equal] \arrow[d, "\mu"] & \\
 & A \arrow[dl] \arrow[dr] \arrow[drrr] & & A' \arrow[dlll] \arrow[dl] \arrow[dr] & \\
X_1 & \dots & X_n & ; & Y
\end{tikzcd}$$

The lower part (except the topmost A) is compactified by means of $\widetilde{\nu}_1, \widetilde{\mu}_1, \widetilde{\nu}_2, \widetilde{\mu}_2$ in this order. Note that, by assumption, the sources and destinations are given in each case by the same compactification of A resp. A'. Now extend the compactification to the whole diagram (cf. Proposition 3.1, 3.), and call the resulting compactifications of the top diagonal morphisms $\widetilde{\nu}_3, \widetilde{\nu}_4$ in this order. We get the following commutative diagram of 2-morphisms:

$$\begin{tikzcd}
 & \cdot \arrow[dl, "\widetilde{\nu}_3"'] \arrow[dr, "\widetilde{\nu}_4"] & \\
\cdot \arrow[d, "\widetilde{\nu}_1"'] \arrow[drr, "\widetilde{\mu}_1" near start] & & \cdot \arrow[d, "\widetilde{\mu}_2"] \arrow[dll, "\widetilde{\nu}_2"' near start] \\
\cdot & & \cdot
\end{tikzcd}$$

We get thus

$$\widetilde{\nu}_1 \circ \widetilde{\nu}_3 = \widetilde{\nu}_2 \circ \widetilde{\nu}_4$$
$$\widetilde{\mu}_1 \circ \widetilde{\nu}_3 = \widetilde{\mu}_2 \circ \widetilde{\nu}_4$$

hence

$$\widetilde{\mu}_1 \circ \widetilde{\nu}_1^{-1} = \widetilde{\mu}_2 \circ \widetilde{\nu}_2^{-1}.$$

□

Proposition 5.8. *The forgetful functors (forgetting the exterior compactification) define equivalences of 1-categories:*

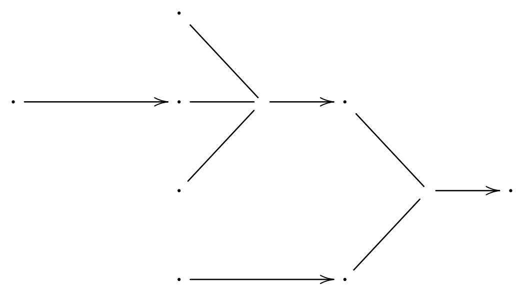
$$\begin{aligned}
\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,comp}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y}) &\cong \mathrm{Hom}_{\mathbb{S}^{\mathrm{cor}}(I)}(X_1, \dots, X_n; Y) \\
\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,0,comp,lax}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y}) &\cong \mathrm{Hom}_{\mathbb{S}^{\mathrm{cor,0,lax}}(I)}(X_1, \dots, X_n; Y) \\
\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,0,comp,oplax}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y}) &\cong \mathrm{Hom}_{\mathbb{S}^{\mathrm{cor,0,oplax}}(I)}(X_1, \dots, X_n; Y)
\end{aligned}$$

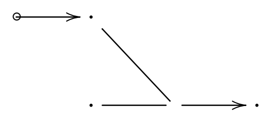
Furthermore the forgetful functors are strictly compatible with the operation of the symmetric groups.

Proof. First note that the forgetful functor maps refinements to identities so it is well-defined on the localization. Lemmas 5.6–5.7 imply that it is faithful. To see that it is full consider a 2-morphism $\mu : \xi \Rightarrow \xi'$ given by a diagram (5) with given compactifications $\widetilde{\xi}$ and $\widetilde{\xi}'$ of source and destination. First, we chose an exterior compactification $\widetilde{\mu}$ of the diagram (5) in which the destination is $\widetilde{\xi}'$ of the multicorrespondence. This is possible (cf. Proposition 3.1, 3.) because the subdiagram consisting of the destination multicorrespondence is final. On ξ this induces an exterior compactification $\widetilde{\widetilde{\xi}}$. Then, by Proposition 3.1, 2., the pair of compactifications $\widetilde{\widetilde{\xi}}, \widetilde{\xi}$ has a common refinement $\widetilde{\widetilde{\widetilde{\xi}}}$. The common refinements induce refinement morphisms $\widetilde{\nu}_1 : \widetilde{\widetilde{\widetilde{\xi}}} \Rightarrow \widetilde{\widetilde{\xi}}$ and $\widetilde{\nu}_2 : \widetilde{\widetilde{\widetilde{\xi}}} \Rightarrow \widetilde{\xi}$ whose underlying 2-morphisms $\xi \Rightarrow \xi$ are, in both cases, the identity. The composition $\widetilde{\nu}_2 \widetilde{\nu}_1^{-1} \widetilde{\mu}$ is a preimage of μ. The functor is essentially surjective because any object given by a diagram (4) can be compactified such that it induces the compactifications $\overline{X}_1, \dots, \overline{X}_n; \overline{Y}$ given, because the subdiagram consisting of the union of the X_i and Y is final. □

5.9. Recall that a **tree** is a finite connected multicategory freely generated by a set of multimorphisms such that each object occurs at most once as a source and at most once as a destination of one of these generating multimorphisms. The generating multimorphisms are allowed to be 0-ary.

Examples:





A **symmetric tree** is obtained from a tree adding images (in the most free way possible) of the multimorphisms (not only the generating ones) under the respective symmetric groups. Observe that there is an obvious composition turning a symmetric tree into a symmetric multicategory. Giving a functor (of multicategories) from a tree to a symmetric multicategory $\mathcal{S}$ is the same as giving a functor (of symmetric multicategories) from its symmetric variant to $\mathcal{S}$.
The most basic tree is $\Delta_{1,n}$ consisting of $n+1$ objects and one n-ary morphism connecting them. Each tree has a well-defined destination object and a number (possibly zero) of source objects. Two trees $\Delta_{T,1}$ and $\Delta_{T,2}$ can by concatenated to a tree $\Delta_{T,2} \circ_i \Delta_{T,1}$ choosing any source object i of the tree $\Delta_{T,2}$.

5.10. We now define *equivalent* morphism categories

$$\mathrm{Hom}_{\mathbb{S}^{\mathrm{cor,comp}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y})$$
$$\mathrm{Hom}_{\mathbb{S}^{\mathrm{cor,0,comp,lax}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y})$$
$$\mathrm{Hom}_{\mathbb{S}^{\mathrm{cor,0,comp,oplax}}(I)}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y})$$

for which a strictly associative composition of 1-morphisms can be defined. We discuss the plain case, the lax and oplax cases being similar. 1-morphisms are trees of compactified correspondences $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$ in which Δ_T is a tree (cf. 5.9), $S_o \hookrightarrow \overline{S}_o$ is an exterior compactification for each object o of Δ_T and $\overline{\xi}_m$ for all *generating* morphisms $m : o_1, \dots, o_k \to o'$ in Δ_T are morphisms in $\mathrm{Hom}'_{\mathbb{S}^{\mathrm{cor,comp}}(I)}(\overline{S}_{o_1}, \dots, \overline{S}_{o_k}; \overline{S}_{o'})$ and where the total sources are $\overline{X}_1, \dots, \overline{X}_n$ and the destination is $\overline{Y}$. To each such object the following Lemma 5.11 associates an exterior compactification of the corresponding functor ${}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$. There is an embedding $c : {}^{\downarrow\uparrow}(\Delta_{1,n} \times I) \to {}^{\downarrow\uparrow}(\Delta_T \times I)$ corresponding to the multimorphism of maximal length. The set of 2-morphisms between two such $\overline{\xi}$ we set equal to the morphisms between the pull-back along c. It is clear that these categories are equivalent to the previous ones. There is a composition functor:

$$\mathrm{Hom}(\overline{X}_1, \dots, \overline{X}_n; \overline{Y}_i) \times \mathrm{Hom}(\overline{Y}_1, \dots, \overline{Y}_m; \overline{Z}) \to \mathrm{Hom}(\overline{Y}_1, \dots, \overline{X}_1, \dots, \overline{X}_n, \dots, \overline{Y}_m; \overline{Z})$$

defined on 1-morphisms by concatenation (clearly strictly associative) and on 2-morphisms is inherited by the composition in $\mathcal{S}^{\mathrm{cor}}$ using the equivalences of Proposition 5.8.
It would have been possible to define the sets of 2-morphisms using any compactification for the composition in $\mathcal{S}^{\mathrm{cor}}$. However a compatible choice as given by the Lemma will turn out to be very suitable for the construction of the derivator six-functor-formalism.
Note that there is no operation of the symmetric groups on these morphism categories anymore.

Lemma 5.11. *On $\mathbb{S}^{\mathrm{cor,comp}}(I)$ (resp. $\mathbb{S}^{\mathrm{cor,0,comp,lax}}$, resp. $\mathbb{S}^{\mathrm{cor,0,comp,oplax}}$) we can compactify compositions as follows. For each tree of compactified correspondences $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$ as above we can choose an exterior compactification of the underlying morphism ${}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ compatible with composition of trees. I.e. if $\Delta_T = \Delta_{T,1} \circ_i \Delta_{T,2}$ (concatenation of trees), with $\Delta_{T,1}$ and $\Delta_{T,2}$ non-trivial, consider the diagram*

$$\begin{tikzcd} & {}^{\downarrow\uparrow}(\Delta_T \times I) & \\ {}^{\downarrow\uparrow}(\Delta_{T,1} \times I) \arrow[ur, hook] & & {}^{\downarrow\uparrow}(\Delta_{T,2} \times I) \arrow[ul, hook'] \end{tikzcd}$$

Then the pull-back of the chosen compactification for $\overline{\xi} = (\overline{\xi}_i)$ to the two diagrams below coincides with the chosen compactifications for them, and for $\Delta_T = \Delta_0$ given by the chosen $S_o \hookrightarrow \overline{S}_o$.

Proof. The subdiagram of morphisms of length m in $({}^{\downarrow\uparrow}\Delta_T)^{\circ}$ (cf. 8.6 for the notation) is final. Hence this follows by induction on the number of generating multimorphisms in Δ_T from Proposition 3.1, 3. $\square$

Proposition 5.12. *The association*

$$I \mapsto \mathbb{S}^{\mathrm{cor,comp}}(I) \quad (\textit{resp. } \mathbb{S}^{\mathrm{cor,0,comp,lax}}(I), \textit{resp. } \mathbb{S}^{\mathrm{cor,0,comp,oplax}}(I))$$

is naturally a pre-2-multiderivator (resp. lax pre-2-multiderivator, resp. oplax pre-2-multiderivator) with domain Catlf*. The forgetful morphisms of (lax, oplax) pre-2-multiderivators*

$$\begin{array}{rcl} \mathbb{S}^{\mathrm{cor,comp}} & \to & \mathbb{S}^{\mathrm{cor}} \\ \mathbb{S}^{\mathrm{cor,0,comp,lax}} & \to & \mathbb{S}^{\mathrm{cor,0,lax}} \\ \mathbb{S}^{\mathrm{cor,0,comp,oplax}} & \to & \mathbb{S}^{\mathrm{cor,0,oplax}} \end{array}$$

are equivalences of (lax, oplax) pre-2-multiderivators.

Proof. First note that by Lemma 5.1, 1. the category ${}^{\downarrow\uparrow}I$ is inverse with finite matching categories. Hence the compactification techniques of this section apply. We have to specify images of functors between diagrams $\alpha : I \to J$ and natural transformations $\mu : \alpha \Rightarrow \beta$ between those. A functor α is mapped to the pull-back $({}^{\downarrow\uparrow}\alpha)^*$ between diagrams ${}^{\downarrow\uparrow}J \to \mathcal{S}$ and ${}^{\downarrow\uparrow}I \to \mathcal{S}$ of correspondences. Observe that this pull-back also operates on exterior compactifications. The same is true for 1-morphisms and their compactifications. This is strictly compatible with the composition of αs. The sets of 2-morphisms are isomorphic to the ones in $\mathbb{S}^{\mathrm{cor}}$ etc. and thus α^* on them is also strictly compatible with composition of αs.
A natural transformation $\mu : \alpha \Rightarrow \beta$ can be given as a functor $\mu' : \Delta_1 \times I \to J$. The compactification $(\mu')^*\overline{S}$ of the diagram $(\mu')^*S : {}^{\downarrow\uparrow}(\Delta_1 \times I) \to \mathcal{S}$, by definition, gives a 1-morphism from $\alpha^*\overline{S} \to \beta^*\overline{S}$ in $\mathbb{S}^{\mathrm{cor,comp}}(I)$. This defines the 2-functoriality. One checks that the resulting association

$$\mathrm{Fun}(I, J) \to \mathrm{Fun}(\mathbb{S}^{\mathrm{cor,comp}}(J), \mathbb{S}^{\mathrm{cor,comp}}(I))$$

defines a pseudo-functor in a natural way, and likewise for the (op)lax case. One also checks that one has an equality of pseudo-functors

$$\beta^* \circ \mathbb{S}^{\mathrm{cor,comp}}(-) = \mathbb{S}^{\mathrm{cor,comp}}(\beta \circ -).$$

By Proposition 5.8 the forgetful functor induces equivalences of the morphism categories. Therefore it suffices to see that the forgetful functor is surjective on objects. However any diagram of correspondences of shape I, with I locally finite, has an exterior compactification. $\square$

6 The input for the construction of derivator six-functor-formalisms

6.1. Let $\mathcal{S}$ be a category with compactifications, and $\mathbb{S}^{\mathrm{op}}$ the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ with its natural multicategory structure encoding the product. We consider a (symmetric) fibered multiderivator $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ with domain Dirlf. This might be seen as a (symmetric) derivator four-functor-formalism encoding $f_*, f^*, \otimes, \mathcal{HOM}$ and their properties.
More precisely, for a morphism f in $\mathrm{Fun}(I^{\mathrm{op}}, \mathcal{S})$ we denote by f^* a push-forward functor $(f^{\mathrm{op}})_\bullet$, by f_* a pull-back functor $(f^{\mathrm{op}})^\bullet$, and for a diagram $S \in \mathrm{Fun}(I^{\mathrm{op}}, \mathcal{S})$ by $\otimes$ (S being understood) a push-forward along the multimorphism $(\mathrm{id}_S, \mathrm{id}_S)$, and by $\mathcal{HOM}_l$, resp. $\mathcal{HOM}_r$, pull-back functors w.r.t. the first, resp. second slot along the multimorphism $(\mathrm{id}_S, \mathrm{id}_S)$. All pull-back and push-forward functors exist by (FDer0 left), resp. (FDer0 right).
We consider the following axioms

(F1) For each diagram $I \in \mathrm{Dirlf}$ and point-wise embedding $\iota : S \hookrightarrow T$ in $\mathbb{S}(I^{\mathrm{op}})$, the functor ι^* (aka $(\iota^{\mathrm{op}})_\bullet$) commutes with homotopy limits as well, and has a left adjoint

$$\iota_! : \mathbb{D}(I)_S \to \mathbb{D}(I)_T.$$

(F2) For an embedding $\iota : S \hookrightarrow T$ in $\mathcal{S}$ the corresponding functor

$$\iota_! : \mathbb{D}_S(\cdot) \to \mathbb{D}_T(\cdot)$$

is fully faithful.

(F3) For proper morphisms f in $\mathcal{S}$ the functor f_* commutes with homotopy colimits as well.

(F4) For proper morphisms f in $\mathcal{S}$ and any Cartesian square

$$\begin{tikzcd} \cdot \arrow[r, two heads, "F"] \arrow[d, "G"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, two heads, "f"'] & \cdot \end{tikzcd}$$

we have base change, i.e. the natural exchange morphism

$$G^* F_* \cong f_* g^*$$

is an isomorphism.

(F5) For an embedding ι in $\mathcal{S}$ and for any Cartesian square

$$\begin{tikzcd} \cdot \arrow[r, "F"] \arrow[d, hook, "I"'] & \cdot \arrow[d, hook, "\iota"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

we have base change, i.e. the natural exchange morphism

$$\iota^* f_* \cong F_* I^*$$

is an isomorphism.

(F6) For f proper and ι embedding forming a Cartesian square

$$\begin{tikzcd} \cdot \arrow[r, two heads, "F"] \arrow[d, hook, "I"'] & \cdot \arrow[d, hook, "\iota"] \\ \cdot \arrow[r, two heads, "f"'] & \cdot \end{tikzcd}$$

the exchange of the base change isomorphism from (F4) (equivalently from (F5))

$$\iota_! F_* \to f_* I_!$$

is an isomorphism as well.

(F4m) For proper morphisms f in $\mathcal{S}$ we have projection formulas, i.e. the natural exchange morphisms

$$(f_* -) \otimes - \cong f_*(- \otimes f^* -) \qquad - \otimes (f_* -) \cong f_*((f^* -) \otimes -)$$

are isomorphisms[10].

(F5m) For an embedding ι in $\mathcal{S}$ we have "coprojection formulas", i.e. the natural exchange morphisms

$$\iota^* \mathcal{HOM}_l(-,-) \cong \mathcal{HOM}_l(\iota^* -, \iota^* -) \qquad \iota^* \mathcal{HOM}_r(-,-) \cong \mathcal{HOM}_r(\iota^* -, \iota^* -)$$

are isomorphisms[11].

Remark 6.2. *Except for (F1) and (F3) these axioms only involve the underlying bifibration*

$$\mathbb{D}(\cdot) \to \mathcal{S}^{\mathrm{op}}$$

and have thus nothing to do with the derivator enhancement.

Remark 6.3. *If ι^* has a left adjoint $\iota_!$ for any embedding ι in $\mathcal{S}$ (e.g. if (F1) holds true) then (F5), resp. (F5m), is equivalent to the condition that*

$$I_! F^* \cong f^* \iota_! \qquad \iota_!(- \otimes (\iota^* -)) \cong (\iota_! -) \otimes - \qquad \iota_!((\iota^* -) \otimes -) \cong - \otimes (\iota_! -)$$

are isomorphisms.

6.4. Assume (F1) and (F2). Then a morphism $\mathcal{E} \to \mathcal{F}$ in $\mathbb{D}(\cdot)$ over an embedding is called **strongly coCartesian**, if it is coCartesian, i.e. if it induces an isomorphism

$$\iota^* \mathcal{E} \to \mathcal{F}$$

whose inverse

$$\mathcal{F} \to \iota^* \mathcal{E}$$

induces an isomorphism

$$\iota_! \mathcal{F} \xrightarrow{\sim} \mathcal{E}$$

Let $\iota : U \hookrightarrow S$ be an embedding. We say that an object $\mathcal{E}$ in $\mathbb{D}(\cdot)_S$ has **support in** U if it lies in the essential image of the fully-faithful functor $\iota_!$. A coCartesian morphism

$$\mathcal{E} \to \mathcal{F}$$

over ι^{op} is *strongly* coCartesian if and only if $\mathcal{E}$ has support in U. We use the notation cocart* for strongly coCartesian. It will only be used over embeddings.

[10] In case $\mathbb{D}$ is symmetric these two assertions are equivalent.

[11] In case $\mathbb{D}$ is symmetric $\mathcal{HOM}_l = \mathcal{HOM}_r$.

Lemma 6.5. *Axioms (F4) and (F4m) are equivalent to the following statement: For all Cartesian squares*

$$\begin{tikzcd} W \arrow[r, "G"] \arrow[d, "F"'] & Z_1, \dots, Z_n \arrow[d, "f_1,\dots,f_n"] \\ Y \arrow[r, "g"'] & X_1, \dots, X_n \end{tikzcd} \tag{6}$$

in which the f_i and F are proper, for a commutative square

$$\begin{tikzcd} \mathcal{H} & \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"'] \end{tikzcd}$$

in $\mathbb{D}(\cdot)$ above it, the following holds: If the α_i are Cartesian ($\mathcal{F}_i \cong f_{i,*}\mathcal{E}_i$) and δ is coCartesian ($G^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is Cartesian or, in other words, the natural exchange

$$g^*(f_{1,*}-, \dots, f_{n,*}-) \to F_* G^*(-, \dots, -)$$

is an isomorphism.

Proof. Exercise. $\square$

Lemma 6.6. *Axioms (F5) and (F5m) are (in the presence of (F1–2)) equivalent to the following statement: For all Cartesian squares*

$$\begin{tikzcd} W \arrow[r, "G"] \arrow[d, hook, "I"'] & Z_1, \dots, Z_n \arrow[d, hook, "\iota_1,\dots,\iota_n"] \\ Y \arrow[r, "g"'] & X_1, \dots, X_n \end{tikzcd} \tag{7}$$

in which the ι_i, and I are embeddings, for a commutative square

$$\begin{tikzcd} \mathcal{H} & \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"'] \end{tikzcd}$$

in $\mathbb{D}(\cdot)$ above it, the following holds: If the α_i are strongly coCartesian ($\iota_i^*\mathcal{F}_i \cong \mathcal{E}_i$ inducing $\iota_{i,!}\mathcal{E}_i \cong \mathcal{F}_i$) and δ is coCartesian ($G^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange

$$I_! G^*(-, \dots, -) \to g^*(\iota_{1,!}-, \dots, \iota_{n,!}-)$$

is an isomorphism.

Proof. Exercise, cf. also Remark 6.3. $\square$

6.7. There are two possibilities of constructing the left adjoint $\iota_!$ required by (F1). One possibility is to use Brown representability:

Proposition 6.8. *Assume that $\mathbb{D}$ is infinite, and ι^* for all morphisms ι as in (F1) commutes with homotopy limits as well. Assume furthermore that $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ has stable,* compactly generated *fibers. Then a left adjoint $\iota_!$ to ι^* exists for all morphisms ι as in (F1).*

Proof. cf. [12, Theorem 4.2.2]. □

Another possibility by direct construction is available if $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ has been constructed from a bifibration of multi-model categories. Let $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be a bifibration of multi-model categories as in [12, Definition 5.1.3]. In [14] it was shown that the associated morphism of pre-multiderivators, i.e.

$$\begin{tikzcd} \mathbb{D}(I) := \mathrm{Fun}(I, \mathcal{D})[\mathcal{W}_I^{-1}] \arrow[d] \\ \mathbb{S}^{\mathrm{op}}(I) := \mathrm{Fun}(I, \mathcal{S}^{\mathrm{op}}) \end{tikzcd}$$

is a left and right fibered multiderivator *with domain* Cat.

Proposition 6.9. *Assume that for any embedding $\iota : S \hookrightarrow T$ the functor $\iota^* : \mathcal{D}_T \to \mathcal{D}_S$ has a left adjoint $\iota_!$ which is left Quillen as well, then $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ satisfies (F1)* when restricted to Dirlf*. If $\iota_!$ is fully-faithful then also (F2) holds true.*

Proof. Let I be a diagram in Dirlf. Consider a point-wise embedding $\iota : S \hookrightarrow T$ in $\mathbb{S}(I^{\mathrm{op}})$. In [12, 5.1.18] model category structures have been constructed on the categories

$$\mathrm{Fun}(I, \mathcal{D})_S \quad \mathrm{Fun}(I, \mathcal{D})_T$$

in which fibrations and weak equivalences are the point-wise ones, turning ι^* (aka $(\iota^{\mathrm{op}})_\bullet$) into a left Quillen functor. We show that ι^* has a left adjoint $\iota_!$ which preserves cofibrant objects and weak equivalences between cofibrant objects. $(\iota_!\mathcal{E})(i)$ is defined by induction on the degree n of i by the coCartesian square:

$$\begin{tikzcd} \iota_! L_i \mathcal{E} \arrow[r] \arrow[d] & \iota_!(\mathcal{E}(i)) \arrow[d] \\ L_i \iota_! \mathcal{E} \arrow[r] & (\iota_!\mathcal{E})(i) \end{tikzcd}$$

where L_i is the latching object functor (cf. [12, p. 74]). L_i respects cofibrations and trivial cofibrations because it is a composition of three functors, two of which are left Quillen and one respects cofibrations and trivial cofibrations by [12, Lemma 5.1.24].

For a cofibrant object $\mathcal{E}$ the top horizontal morphism is a cofibration and therefore also the bottom horizontal morphism is a cofibration. By induction $\iota_!\mathcal{E}$ is cofibrant when restricted to objects of degree $< n$ and thus also $L_i\iota_!\mathcal{E}$ is cofibrant and hence the so extended $\iota_!\mathcal{E}$ is. Summarizing, all entries in the above square are cofibrant and the top horizontal morphism is a cofibration. In particular, the diagram is also a homotopy push-out. For a weak equivalence $\mathcal{E} \to \mathcal{F}$ between cofibrant objects, we get a morphism of diagrams of shape $\ulcorner$ which consists point-wise of weak equivalences: For the lower left entry use induction and the fact that $\iota_!\mathcal{E}$ is cofibrant again, for the upper entries by the assumption and Ken Browns Lemma $\iota_!$ maps weak equivalences between cofibrant objects to weak equivalences. Therefore also the induced morphism $(\iota_!\mathcal{E})(i) \to (\iota_!\mathcal{F})(i)$ is a weak equivalence. We obtain functors $\iota_!$ and ι^*

$$\begin{tikzcd} \mathrm{Fun}(I, \mathcal{D})_S^{\mathrm{Cof}} \arrow[r, shift left, "\iota_!"] & \mathrm{Fun}(I, \mathcal{D})_T^{\mathrm{Cof}} \arrow[l, shift left, "\iota^*"] \end{tikzcd}$$

which both preserve (pointwise) weak equivalences. By construction they are adjoint. They thus induce morphisms between the respective localizations $\mathbb{D}(I)_S$ and $\mathbb{D}(I)_T$ which are adjoint again. Lastly, over a point $I = \cdot$, if the original $\iota_!$ is fully-faithful, then the unit of the adjunction is an isomorphism and thus also still when passing to the localizations, which is equivalent to the induced $\iota_!$ on the localization being fully-faithful.
It remains to be shown that ι^* commutes with homotopy limits or, equivalently, that the functor $\iota_!$ constructed above is computed point-wise *on constant diagrams*. By definition of $\iota_!$ this is the case if the morphism

$$\iota_! L_i \mathcal{E} \to L_i \iota_! \mathcal{E}$$

is a weak equivalence. However, by induction, $\iota_!$ is computed point-wise when restricted to objects of degree $< n$ and hence the statement follows from the fact that $\iota_!$, being a left adjoint, commutes with colimits. □

Later we will need more information about the functors $\iota_!$:

Definition 6.10. *Let $\mathcal{S}$ be a category and let $\mathbb{S}^{\mathrm{op}}$ be the pre-derivator represented by $\mathcal{S}^{\mathrm{op}}$. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a fibered derivator with domain* Dia *such that for all $I \in$* Dia *the functors*

$$\iota^* := (\iota^{\mathrm{op}})_\bullet : \mathbb{D}(I)_T \to \mathbb{D}(I)_S$$

have left adjoints $\iota_!$ for all morphisms $\iota : S \to T$ in $\mathbb{S}(I^{\mathrm{op}})$, which are point-wise is some subclass $\mathcal{S}_2$ of morphisms of $\mathcal{S}$. We will say that $\iota_!$ **commutes with** *α^* on (J, S) for a functor $\alpha : I \to J$ if for all objects $\mathcal{E}$ in $\mathbb{D}(I)_S$ the natural exchange morphism*

$$(\alpha^* \iota)_! \alpha^* \mathcal{E} \to \alpha^* \iota_! \mathcal{E}$$

is an isomorphism. We will say that $\iota_!$ is **computed point-wise** *on (J, S), if it commutes with j^* for all $j \in J$.*

Note that it does *not* automatically follow from (F1), i.e. the commutation of ι^* with homotopy colimits, that $\iota_!$ is computed point-wise. This is true only over constant diagrams in $\mathbb{S}(I)$ — then it is a well-known and quite trivial statement about usual derivators.

Lemma 6.11. *Let $\mathcal{S}$ be a category with compactifications, $\mathbb{S}$ the pre-derivator represented by $\mathcal{S}$, and let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a fibered derivator on some diagram category* Dia *satisfying axioms (F1–F6).*

1. *Let $\alpha : I \to J$ be an opfibration in* Dia *and $\iota : S \to T$ a morphism in $\mathbb{S}(I^{\mathrm{op}})$. If for all coCartesian morphisms $\mu : i \to i'$ in I the square*

$$\begin{tikzcd} S_i \arrow[d, "\iota_i"'] & S_{i'} \arrow[l, "S(\mu)"'] \arrow[d, "\iota_{i'}"] \\ T_i & T_{i'} \arrow[l, "T(\mu)"] \end{tikzcd}$$

   is Cartesian, then $\iota_!$ commutes with e_j^* for the inclusions $e_j : I_j \hookrightarrow I$ of the fibers.

2. *Consider $I \times J$ in* Dia *and a morphism $\iota : S \to T$ in $\mathbb{S}(I^{\mathrm{op}})$. Then $(\mathrm{pr}_1^* \iota)_!$ commutes with the inclusion $I \times j \hookrightarrow I \times J$ for any $j \in J$ on $(I \times J, \mathrm{pr}_1^* S)$. In particular $\iota_!$ is computed point-wise over constant diagrams in $\mathbb{S}(J^{\mathrm{op}})$.*

3. *Assume $\mathbb{D}$ is infinite, or that I is a diagram in* Dia *with finite* Hom*-sets. Let $\iota : S \to T$ be a morphism in $\mathbb{S}(I^{\mathrm{op}})$. Then the functor*

$$\iota_! : \mathbb{D}(I)_S \to \mathbb{D}(I)_T$$

is computed point-wise on an object $\mathcal{E}$, if for any pair of morphisms $\alpha : i \to j$ and $\mu : j \to k$ in I we have that

$$\iota_{k,!} S(\mu)^* S(\alpha)^* i^* \mathcal{E} \to T(\mu)^* \iota_{j,!} S(\alpha)^* i^* \mathcal{E} \tag{8}$$

is an isomorphism.

Hence $\iota_!$ is computed point-wise on an (absolutely) coCartesian *object $\mathcal{E}$, if for any $\mu : j \to k$ in I we have that*

$$\iota_{k,!} S(\mu)^* j^* \mathcal{E} \to T(\mu)^* \iota_{j,!} j^* \mathcal{E} \tag{9}$$

is an isomorphism. Note that (9) implies (8) for a coCartesian object.

Proof. 1. The statement is equivalent to the natural exchange morphism

$$\iota^* e_{j,*} \mathcal{E} \to e_{j,*} (e_j^* \iota)^* \mathcal{E}$$

being an isomorphism. This can be checked point-wise at an object $i \in I$. Consider the homotopy exact square:

$$\begin{tikzcd}
\alpha(i) \times_{/J} j \arrow[r, "\rho"] \arrow[d] \arrow[dr, phantom, "\nearrow\mu"] & I_j \arrow[d, "e_j"] \\
i \arrow[r] & I
\end{tikzcd}$$

where the ρ maps a morphism $\nu : \alpha(i) \to j$ to $\nu_\bullet(i)$ and $\mu(\nu)$ is given by the coCartesian morphism $\widetilde{\nu} : i \to \nu_\bullet(i)$. It shows that we have

$$\begin{aligned}
(\iota_i)^* i^* e_{j,*} \mathcal{E} &\cong (\iota_i)^* \operatorname*{holim}_{\alpha(i) \times_{/J} j} S(\mu)_* \rho^* \mathcal{E} \\
&\cong \operatorname*{holim}_{\alpha(i) \times_{/J} j} S(\mu)_* \rho^* (e_j^* \iota)^* \mathcal{E} \\
&\cong i^* e_{j,*} (e_j^* \iota)^* \mathcal{E}
\end{aligned}$$

because ι^* commutes with homotopy limits by (F1), and with $S(\mu)_*$ by (F5) and the assumption. Note that the latter can be checked point-wise.
2. A special case of 1.
3. Look at the following diagram

$$\begin{tikzcd}
{}^{\downarrow\uparrow\downarrow}I \arrow[r, "\pi_1"] \arrow[d, "\rho"'] & I \arrow[dd, equal] \\
{}^{\downarrow\uparrow}I \times I \arrow[d, "\pi_3"'] \arrow[dr, phantom, "\swarrow\mu"] & \\
I \arrow[r, equal] & I
\end{tikzcd}$$

in which $\rho = (\pi_{12}, \pi_3)$ forgets the second morphism. The outer square is homotopy exact, hence

$$\mathcal{E} \cong \pi_{3,!} \rho_! S(\mu)^* \pi_1^* \mathcal{E}$$

for any $\mathcal{E} \in \mathbb{D}(I)_S$. Furthermore ρ and π_3 are opfibrations, and π_1 is a fibration. ρ has disrete fibers with fiber over $(\alpha : i \to j, k)$ equal to $\mathrm{Hom}(j,k)$.
We will later show that $(\pi_3^* \iota)_!$ is computed point-wise on $({}^{\downarrow\uparrow}I \times I, \pi_3^* S)$ on objects of the form

$$\mathcal{F} := \rho_! S(\mu)^* \pi_1^* \mathcal{E}$$

for any object $\mathcal{E}$ satisfying (8). We claim that the statement follows from this.
First, by 2., $(\pi_3^* \iota)_!$ commutes with e_α^* for the inclusions $e_\alpha : \alpha \times I \hookrightarrow {}^{\downarrow\uparrow}I \times I$ on $\pi_3^* S$, and $(\iota_k)_!$ commutes with $(\alpha)^*$ for $(\alpha) : \alpha \hookrightarrow {}^{\downarrow\uparrow}I$ on any constant object S_k for any $k \in I$. Denote $e_k : {}^{\downarrow\uparrow}I \times k \to {}^{\downarrow\uparrow}I \times I$ the inclusion. Consider the following exchange morphisms

$$(\iota_k)_!(\alpha)^* e_k^* \mathcal{F} \to (\alpha)^* (\iota_k)_! e_k^* \mathcal{F} \to (\alpha)^* e_k^* (\pi_3^* \iota)_! \mathcal{F}.$$

By assumption, the composition is an isomorphism because $\iota_!$ is computed point-wise on $\mathcal{F}$. Also the left morphism is an isomorphism because $\iota_{k,!}$ commutes with $(\alpha)^*$. Hence by (Der2) $(\pi_3^* \iota)_!$ commutes also with e_k^*.
Then

$$\begin{aligned}
\iota_{k,!} k^* \pi_{3,!} \mathcal{F} &\cong \iota_{k,!} \operatorname*{hocolim}_{{}^{\downarrow\uparrow}I} e_k^* \mathcal{F} \\
&\cong \operatorname*{hocolim}_{{}^{\downarrow\uparrow}I} \iota_{k,!} e_k^* \mathcal{F} \\
&\cong \operatorname*{hocolim}_{{}^{\downarrow\uparrow}I} e_k^* (\pi_3^* \iota)_! \mathcal{F} \\
&\cong k^* \pi_{3,!} (\pi_3^* \iota)_! \mathcal{F} \\
&\cong k^* \iota_! \pi_{3,!} \mathcal{F}
\end{aligned}$$

using that $\iota_{k,!}$ commutes with arbitrary homotopy left Kan extensions. Hence $\iota_!$ is also computed point-wise on (I, S) for $\mathcal{E} \cong \pi_{3,!} \mathcal{F}$.
Hence we are left to show that $(\pi_3^* \iota)_!$ is computed point-wise on $\mathcal{F}$ on ${}^{\downarrow\uparrow}I \times I$. For a morphism $\alpha : i \to j$ in I we have using (Der1)

$$\begin{aligned}
\iota_{k,!} (\alpha, k)^* \mathcal{F} = \iota_{k,!} (\alpha, k)^* \rho_! S(\mu)^* \pi_1^* \mathcal{E} &\cong \iota_{k,!} k^* j_! S(\alpha)^* i^* \mathcal{E} \\
&\cong \iota_{k,!} \bigoplus_{\beta \in \mathrm{Hom}(j,k)} S(\beta)^* S(\alpha)^* i^* \mathcal{E} \\
&\cong \bigoplus_{\beta \in \mathrm{Hom}(j,k)} T(\beta)^* \iota_{j,!} S(\alpha)^* i^* \mathcal{E}
\end{aligned}$$

because of the assumption (8) and commutation of $(-)_!$ with homotopy colimits. Furthermore, because of (Der1), $(-)_!$ is clearly computed point-wise on the discrete diagram $\mathrm{Hom}(j,k)$ over any object in $\mathbb{S}(\mathrm{Hom}(j,k))$. If $\mathrm{Hom}(j,k)$ is infinite we need (Der1) also for infinite sets, that is, $\mathbb{D}$ has to be infinite. Then this is isomorphic to

$$\begin{aligned}
&\cong k^* j_! \iota_{j,!} S(\alpha)^* i^* \mathcal{E} \\
&\cong k^* \iota_! j_! S(\alpha)^* i^* \mathcal{E}
\end{aligned}$$

using that $j_!$ commutes with $\iota_!$ because j^* commutes with ι^*. And finally to

$$\begin{aligned}
&\cong k^* \iota_! e_\alpha^* \rho_! S(\mu)^* \pi_1^* \mathcal{E} \\
&\cong (\alpha, k)^* (\pi_3^* \iota)_! \rho_! S(\mu)^* \pi_1^* \mathcal{E} = (\alpha, k)^* (\pi_3^* \iota)_! \mathcal{F}
\end{aligned}$$

using that $(\pi_3^* \iota)_!$ commutes with e_α^*. A tedious check shows that this composition of isomorphisms is the exchange morphism associated with the commutation of $(\pi_3^* \iota)^*$ and $(\alpha, k)^*$. $\square$

7 Preliminaries for the construction of the derivator six-functor-formalism (non-multi-case)

We will neglect the multi-aspect in this section and work with a fibered derivator (not multiderivator) $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ satisfying the axioms (F1)–(F6). In the next section the results are generalized to the multi-case. This is straightforward, but a bit more technical, hence it has been moved to the next section for the convenience of the reader.

7.1. Let $\mathcal{S}$ be a category with compactifications as in 2.1. Let I be locally finite and $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ admissible in the sense of 5.4, i.e. induced by a diagram of correspondences $I \to \mathcal{S}^{\mathrm{cor}}$. For any exterior compactification $X \hookrightarrow \overline{X}$ we get an induced interior compactification ${}^{\downarrow\downarrow}({}^{\downarrow\uparrow}I) \to \mathcal{S}$ (cf. Proposition 4.5). We are rather interested in its pullback along the following functor

$$ {}^{\downarrow\downarrow\uparrow}I \to {}^{\downarrow\downarrow}({}^{\downarrow\uparrow}I) $$

mapping $i \to j \to k$ to the diagram

$$\begin{tikzcd} i \arrow[r] \arrow[d] & k \\ j \arrow[r] & k \arrow[u, "\mathrm{id}_k"'] \end{tikzcd}$$

The reason is that we need a compactification only for the morphism f going to the right in a correspondence (4). The above functor forgets the interior compactification on the other morphism g going to the left. We will therefore always denote by $\widetilde{X}$ the pull-back of the induced interior compactification to ${}^{\downarrow\downarrow\uparrow}I$ and will call it an interior compactification of X. It has the property that a type 1 morphism (4.1) is mapped to a proper morphism, and a type 2 morphism is mapped to a dense embedding.
We will also need this w.r.t. a morphism $\mathrm{Fun}(\Delta_n, \mathcal{S}^{\mathrm{cor},0,\mathrm{lax}}(I))$, resp. $\mathrm{Fun}(\Delta_n, \mathcal{S}^{\mathrm{cor},0,\mathrm{oplax}}(I))$. We get, in each case, a diagram $X : {}^{\downarrow\uparrow}(\Delta_n \times I) \to \mathcal{S}$ which is, however, only weakly admissible in the sense of 5.4. For this diagram we can construct in the same way exterior and interior compactifications.

Definition 7.2. *A morphism of squares in $\mathcal{S}$*

$$\begin{tikzcd} & W \arrow[rr] \arrow[dl] \arrow[dd, hook, "\iota_W"] & & Z \arrow[dl] \arrow[dd, hook, "\iota_Z"] \\ Y \arrow[rr, crossing over] \arrow[dd, hook, "\iota_Y"] & & X & \\ & \overline{W} \arrow[rr] \arrow[dl] & & \overline{Z} \arrow[dl] \\ \overline{Y} \arrow[rr] & & \overline{X} \arrow[from=uu, hook, crossing over, "\iota_X"'] & \end{tikzcd}$$

such that all embeddings are dense, the top, the front and the back are Cartesian is called a **weak compactification** *of the top Cartesian square. The bottom square does not need to be Cartesian, and neither the objects nor the morphisms in the bottom square are assumed to be proper.*

This rather ad hoc definition will only be used in this section. Note that the orientation of the top square matters. To draw it in the plane we will always rotate the cube by 90° in such a way that it becomes the front face.

Lemma 7.3. *Let $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be weakly admissible and let $\widetilde{X}$ be any interior compactification. Any square of the form*

$$\begin{tikzcd}
\widetilde{X}(i \to j \to k) \arrow[r, two heads] \arrow[d, hook] & \widetilde{X}(i' \to i \to k) \arrow[d, hook] \\
\widetilde{X}(i \to j' \to k) \arrow[r, two heads] & \widetilde{X}(i' \to j' \to k)
\end{tikzcd}$$

is Cartesian.

Proof. Cf. Lemma 2.5. □

Lemma 7.4. *Let $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be weakly admissible and let $\widetilde{X}$ be any interior compactification. Any square of the form*

$$\begin{tikzcd}
\widetilde{X}(i = i \to k) \arrow[r] \arrow[d, hook] & \widetilde{X}(i = i \to k') \arrow[d, hook] \\
\widetilde{X}(i \to j \to k) \arrow[r] & \widetilde{X}(i \to j \to k')
\end{tikzcd}$$

is Cartesian.

Proof. We may extend the diagram as follows

$$\begin{tikzcd}
\widetilde{X}(i = i \to k) \arrow[r] \arrow[d, hook] & \widetilde{X}(i = i \to k') \arrow[d, hook] \\
\widetilde{X}(i \to j \to k) \arrow[r] \arrow[d] & \widetilde{X}(i \to j \to k') \arrow[d] \\
\widetilde{X}(j = j \to k) \arrow[r] & \widetilde{X}(j = j \to k')
\end{tikzcd}$$

in which the outer square is weakly Cartesian, because X is weakly admissible. The statement follows therefore from Lemma 2.7 □

Lemma 7.5. *Let $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be weakly admissible and let $\widetilde{X}$ be any interior compactification. The cube*

$$\begin{tikzcd}
 & \widetilde{X}(i = i \to k) \arrow[rr] \arrow[dl, equal] \arrow[dd, hook] & & \widetilde{X}(i = i \to k') \arrow[dl, equal] \arrow[dd, hook] \\
\widetilde{X}(i = i \to k) \arrow[rr] \arrow[dd, hook] & & \widetilde{X}(i = i \to k') \arrow[dd, hook] & \\
 & \widetilde{X}(i \to j \to k) \arrow[rr, "\overline{G}"] \arrow[dl, hook] & & \widetilde{X}(i \to j \to k') \arrow[dl, hook] \\
\widetilde{X}(i \to j' \to k) \arrow[rr, "\overline{g}"'] & & \widetilde{X}(i \to j' \to k') &
\end{tikzcd}$$

is a weak compactification of the top (trivially Cartesian) square. If S is admissible, the cube

$$\begin{tikzcd}[row sep=small, column sep=small]
& \widetilde{X}(i = i \to k) \arrow[rr] \arrow[dl, two heads] \arrow[dd, hook] & & \widetilde{X}(i = i \to k') \arrow[dl, two heads] \arrow[dd, hook] \\
\widetilde{X}(i' = i' \to k) \arrow[rr, crossing over] \arrow[dd, hook] & & \widetilde{X}(i' = i' \to k') & \\
& \widetilde{X}(i \to j \to k) \arrow[rr, "\overline{G}"] \arrow[dl, two heads] & & \widetilde{X}(i \to j \to k') \arrow[dl, two heads] \\
\widetilde{X}(i' \to j \to k) \arrow[rr, "\overline{g}"'] & & \widetilde{X}(i' \to j \to k') \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

is a weak compactification of the top Cartesian square.

Proof. We need to show that in each case the front and back squares are Cartesian. This is a consequence of Lemma 7.4. □

Lemma 7.6. *Consider a square in $\mathcal{S}$*

$$\begin{tikzcd}
W \arrow[r, two heads, "\overline{F}"] \arrow[d, hook, "I"'] & X \arrow[d, hook, "\iota"] \\
Z \arrow[r, two heads, "\overline{f}"'] & Y
\end{tikzcd}$$

in which I is dense and a square

$$\begin{tikzcd}
\mathcal{H} & \mathcal{E} \arrow[l, "\delta"'] \\
\mathcal{G} \arrow[u, "\gamma"] & \mathcal{F} \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

above it. If γ is strongly coCartesian ($I^*\mathcal{G} \cong \mathcal{H}$ inducing $I_!\mathcal{H} \cong \mathcal{G}$) and δ is Cartesian ($\mathcal{E} \cong \overline{F}_*\mathcal{H}$) then β is Cartesian if and only if α is strongly coCartesian, or in other words the natural exchange

$$\iota_!\overline{F}_* \to \overline{f}_* I_!$$

is an isomorphism.

Proof. By Lemma 2.5 the square is actually Cartesian and hence by (F6) the statement holds. □

Remark 7.7. *A posterori the conclusion will hold regardless of I being dense.*

Lemma 7.8. *Consider a weak compactification of a Cartesian square in $\mathcal{S}$*

$$\begin{tikzcd}
& W \arrow[rr] \arrow[dl] \arrow[dd, hook] & & Z \arrow[dl] \arrow[dd, hook] \\
Y \arrow[rr, crossing over] \arrow[dd, hook] & & X & \\
& \overline{W} \arrow[rr, "\overline{G}"] \arrow[dl, "\overline{F}"'] & & \overline{Z} \arrow[dl, "\overline{f}"] \\
\overline{Y} \arrow[rr, "\overline{g}"'] & & \overline{X} \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

in which $\overline{f}$ and $\overline{F}$ are proper. Then for a square

$$\begin{tikzcd}
\mathcal{H} & \mathcal{E} \arrow[l, "\delta"'] \\
\mathcal{G} \arrow[u, "\gamma"] & \mathcal{F} \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

in $\mathbb{D}(\cdot)$ above the bottom square the following holds: If $\underline{\mathcal{E} \text{ has support in } Z}$ and α is Cartesian ($\mathcal{F} \cong \overline{f}_* \mathcal{E}$) and δ is coCartesian ($\overline{G}^* \mathcal{E} \cong \mathcal{H}$) then β is coCartesian if and only if γ is Cartesian or, in other words, the natural exchange

$$\overline{g}^* \overline{f}_* \to \overline{F}_* \overline{G}^*$$

is an isomorphism on objects with support in Z.

Proof. We look at the following diagram

$$\begin{tikzcd}
W \arrow[r, equal] \arrow[d, hook, "\iota_W"'] \arrow[dr, phantom, "1"] & W \arrow[r, "G"] \arrow[d, hook, "\iota"] \arrow[dr, phantom, "2"] & Z \arrow[d, hook, "\iota_Z"] \\
\overline{W} \arrow[r, two heads, "\overline{f}''"'] \arrow[d, "\overline{F}"'] & \square \arrow[r, "G'"] \arrow[d, "\overline{f}'"] \arrow[dr, phantom, "3"] & \overline{Z} \arrow[d, "\overline{f}"] \\
\overline{Y} \arrow[r, equal] & \overline{Y} \arrow[r, "\overline{g}"'] & \overline{X}
\end{tikzcd}$$

in which squares 2 and 3 are Cartesian. The middle left horizontal morphism is proper because of (S2). Because of the support condition, we have to show that the natural exchange

$$\overline{g}^* \overline{f}_* \iota_{Z,!} \to \overline{F}_* \overline{G}^* \iota_{Z,!}$$

is an isomorphism. Elementary properties of exchange morphisms imply that the morphism is the composition of the following (exchange) morphisms which are all isomorphisms because of the indicated reason:

$$\begin{array}{rcll}
\overline{g}^* \overline{f}_* \iota_{Z,!} & \xrightarrow{\sim} & \overline{f}'_* (G')^* \iota_{Z,!} & \text{because 3 is Cartesian and (F4)} \\
& \xleftarrow{\sim} & \overline{f}'_* \iota_! G^* & \text{because 2 is Cartesian and (F5)} \\
& \xrightarrow{\sim} & \overline{f}'_* \overline{f}''_* \iota_{W,!} G^* & \text{applying Lemma 7.6 for 1} \\
& \xrightarrow{\sim} & \overline{F}_* \iota_{W,!} G^* & \\
& \xrightarrow{\sim} & \overline{F}_* \overline{G}^* \iota_{Z,!} & \text{because the composite of 1 and 2 is Cartesian and (F5).}
\end{array}$$

$\square$

Lemma 7.9. *Consider a weak compactification of a Cartesian square in $\mathcal{S}$*

$$\begin{tikzcd}
& W \arrow[rr] \arrow[dl, hook', "I"'] \arrow[dd, hook] & & Z \arrow[dl, hook', "\iota"'] \arrow[dd, hook] \\
Y \arrow[rr] \arrow[dd, hook] & & X \arrow[dd, hook] & \\
& \overline{W} \arrow[rr, "\overline{G}"] \arrow[dl, hook', "\overline{I}"'] & & \overline{Z} \arrow[dl, hook', "\overline{\iota}"] \\
\overline{Y} \arrow[rr, "\overline{g}"'] & & \overline{X} &
\end{tikzcd}$$

in which $\overline{\iota}$, ι, I and $\overline{I}$ are embeddings. Then for a square

$$\begin{tikzcd}
\mathcal{H} & \mathcal{E} \arrow[l, "\delta"'] \\
\mathcal{G} \arrow[u, "\gamma"] & \mathcal{F} \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

in $\mathbb{D}(\cdot)$ above the bottom square the following holds: If <u>$\mathcal{E}$ has support in Z</u> and α is strongly co-Cartesian ($\overline{\iota}^*\mathcal{F} \cong \mathcal{E}$ inducing $\overline{\iota}_!\mathcal{E} \cong \mathcal{F}$) and δ is coCartesian ($\overline{G}^*\mathcal{E} \cong \mathcal{H}$) then β is coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange

$$\overline{I}_!\overline{G}^* \to \overline{g}^*\overline{\iota}_!$$

is an isomorphism on objects with support in Z.

Proof. Because of the support condition it suffices to see that the natural exchange

$$\overline{I}_!\overline{G}^*\iota_{Z,!} \to \overline{g}^*\overline{\iota}_!\iota_{Z,!}$$

is an isomorphism. Elementary properties of exchange morphisms imply that the morphism is the composition of the following (exchange) morphisms which are all isomorphisms because of pseudofunctoriality and because of the indicated reason:

$$\begin{array}{rcll}
\overline{I}_!\overline{G}^*\iota_{Z,!} & \overset{\sim}{\longleftarrow} & \overline{I}_!\iota_{W,!}G^* & \text{because the back square in Cartesian and (F5)} \\
& \overset{\sim}{\longrightarrow} & \iota_{Y,!}I_!G^* & \\
& \overset{\sim}{\longrightarrow} & \iota_{Y,!}g^*\iota_! & \text{because the top square is Cartesian and (F5)} \\
& \overset{\sim}{\longrightarrow} & \overline{g}^*\iota_{X,!}\iota_! & \text{because the front square is Cartesian and (F5)} \\
& \overset{\sim}{\longrightarrow} & \overline{g}^*\overline{\iota}_!\iota_{Z,!} &
\end{array}$$

$\square$

We summarize the discussion in the following

Proposition 7.10. *Let $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be weakly admissible and let $\widetilde{X} : {}^{\downarrow\downarrow\uparrow}I \to \mathcal{S}$ be any interior compactification of it. Consider a diagram*

$$\begin{tikzcd} w \arrow[r, "d"] \arrow[d, "c"'] & z \arrow[d, "a"] \\ y \arrow[r, "b"'] & x \end{tikzcd}$$

in ${}^{\downarrow\downarrow\uparrow}I$. Let

$$\begin{tikzcd} \mathcal{H} & \mathcal{E} \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F} \arrow[l, "\beta"] \arrow[u, "\alpha"'] \end{tikzcd}$$

be a diagram in $\mathbb{D}(\cdot)$ above $\widetilde{X}^{\mathrm{op}}$ applied to the top square. *Then the following holds:*

1. *If $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ is admissible, let a and c be of type 1 and b and d of type 3.*

   If $\mathcal{E}$ has support in $X(\pi_{13}(z))$ and α is Cartesian ($\mathcal{F} \cong \widetilde{X}(a)_*\mathcal{E}$) and δ is coCartesian ($\widetilde{X}(d)^*\mathcal{E} \cong \mathcal{H}$) then β is coCartesian if and only if γ is Cartesian or, in other words, the natural exchange

   $$\widetilde{X}(b)^*\widetilde{X}(a)_* \to \widetilde{X}(c)_*\widetilde{X}(d)^*$$

   is an isomorphism on objects with support in $X(\pi_{13}(z))$.

2. *Let a and c be of type 2 and b and d of type 3. If $\mathcal{E}$ has support in $X(\pi_{13}(z))$ and α is strongly coCartesian ($\widetilde{X}(a)^*\mathcal{F} \cong \mathcal{E}$ inducing $\widetilde{X}(a)_!\mathcal{E} \cong \mathcal{F}$) and δ is coCartesian ($\widetilde{X}(d)^*\mathcal{E} \cong \mathcal{H}$) then β is coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange*

   $$\widetilde{X}(c)_!\widetilde{X}(d)^* \to \widetilde{X}(b)^*\widetilde{X}(a)_!$$

   is an isomorphism on objects with support in $X(\pi_{13}(z))$.

3. *Let a and c be of type 2 and b and d of type 1. If γ is strongly coCartesian ($\widetilde{X}(c)^*\mathcal{G} \cong \mathcal{H}$ inducing $\widetilde{X}(c)_!\mathcal{H} \cong \mathcal{G}$) and δ is Cartesian ($\mathcal{E} \cong \widetilde{X}(d)_*\mathcal{H}$) then β is Cartesian if and only if α is strongly coCartesian, or in other words, the natural exchange*

   $$\widetilde{X}(a)_!\widetilde{X}(d)_* \to \widetilde{X}(b)_*\widetilde{X}(c)_!$$

   is an isomorphism.

Proof. 1. follows from Lemma 7.8, and 2. from Lemma 7.9, using Lemma 7.5 in each case. 3. follows from Lemma 7.6, using Lemma 7.3. □

8 Preliminaries for the construction of the derivator six-functor-formalism (multi-case)

In this section the discussion in the previous section will be repeated making the necessary modifications to include to multi-case, needed later to include the monoidal structure into the derivator six-functor-formalism. It should be skipped on a first reading.
Let $\mathcal{S}$ be a category with compactifications, and $\mathbb{S}^{\mathrm{op}}$ the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ with its natural multicategory structure encoding the product. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a fibered multiderivator with domain Dirlf, satisfying axioms (F1)–(F6) and also (F4m)–(F5m).

8.1. Let $\mathcal{C}$ be a multicategory. For each pair of ordered set of objects $\mathcal{E} := (\mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{F} := (\mathcal{F}_1, \dots, \mathcal{F}_m)$ in $\mathcal{C}$ we define the set of **morphisms** from $\mathcal{E}$ to $\mathcal{F}$ to be a sequence of integers $0 \le n_1 \le \dots \le n_{m-1} \le n$ and multimorphisms

$$\mathcal{E}_1, \dots, \mathcal{E}_{n_1} \to \mathcal{F}_1;\ \mathcal{E}_{n_1+1}, \dots, \mathcal{E}_{n_2} \to \mathcal{F}_2;\ \dots;\ \mathcal{E}_{n_{m-1}+1}, \dots, \mathcal{E}_n \to \mathcal{F}_m$$

The integers n_i may be equal and also $n = 0$ is allowed. If $n = m = 0$ we understand there to be exactly one morphism.

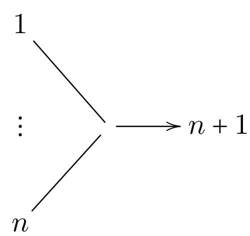
8.2. Let $\Xi \in \{\downarrow, \uparrow\}^l$ be sequence of arrow directions and let M be a multidiagram (i.e. a small multicategory like $\Delta_{1,n}$). If $\Xi_l = \downarrow$, we define a small multicategory ${}^{\Xi}M$ and if $\Xi_l = \uparrow$, we define a small opmulticategory ${}^{\Xi}M$. We concentrate on the case $\Xi_l = \downarrow$ for definiteness.
Objects are sequences

$$[S_{1,1}, \dots, S_{1,n_1}] \longrightarrow [S_{2,1}, \dots, S_{2,n_2}] \longrightarrow \cdots \longrightarrow [S_{l,1}]$$

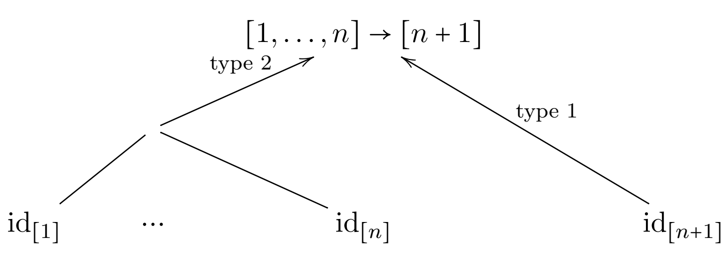
of l lists of objects (can be empty) and morphisms in the sense of 8.1 between them, where however the l-th list consist of exactly one object. Multimorphisms $S^{(1)}, \dots, S^{(n)} \to T$ are diagrams

$$\begin{tikzcd}
{[S^{(1)}_{1,1}, \dots, S^{(1)}_{1,n_1}, \dots]} \arrow[r] \arrow[d, leftrightarrow] & {[S^{(1)}_{2,1}, \dots, S^{(1)}_{2,n_2}, \dots]} \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & {[S^{(1)}_{l,1}, \dots, S^{(n)}_{l,1}]} \arrow[d] \\
{[T_{1,1}, \dots, T_{1,n_1}]} \arrow[r] & {[T_{2,1}, \dots, T_{2,n_2}]} \arrow[r] & \cdots \arrow[r] & {[T_{l,1}]}
\end{tikzcd}$$

where the arrow direction in the i-th column is determined by Ξ_i. Such a morphism is called of **type** i if all vertical morphisms except the i-th one are identities of lists. There are thus only n-ary morphisms for $n \neq 1$ of type l and not of any other type.
Example: For the tree $\Delta_{1,n}$



the multidiagram ${}^{\uparrow\downarrow}(\Delta_{1,n})$ is



8.3. Consider a tree Δ_T. The category ${}^{\Xi}\Delta_T$ can be generated by multimorphisms of type l

$$\begin{tikzcd}
{[S^{(1)}_{1,1}, \dots, S^{(1)}_{1,n_1}, \dots]} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & {[S^{(1)}_{i,1}, \dots, S^{(1)}_{i,n_i}, \dots]} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & {[S^{(1)}_{l}, \dots, S^{(n)}_{l}]} \arrow[d] \\
{[S^{(1)}_{i,1}, \dots, S^{(1)}_{i,n_i}, \dots]} \arrow[r] & \cdots \arrow[r] & {[S^{(1)}_{i,1}, \dots, S^{(1)}_{i,n_i}, \dots]} \arrow[r] & \cdots \arrow[r] & {[T_l]}
\end{tikzcd}$$

where the morphism $S_l^{(1)}, \dots, S_l^{(n)} \to T_l$ is a generating morphism of Δ_T and morphisms of type i

$$\begin{tikzcd} {[S_{1,1}, \dots, S_{1,n_1}]} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & {[S_{i,1}, \dots, S_{i,n_i}]} \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & {[S_l]} \arrow[d, equal] \\ {[S_{1,1}, \dots, S_{1,n_1}]} \arrow[r] & \cdots \arrow[r] & {[T_{i,1}, \dots, T_{i,n_i'}]} \arrow[r] & \cdots \arrow[r] & {[S_l]} \end{tikzcd}$$

in which the morphism of lists $[S_{i,1}, \dots, S_{i,n_i}] \leftrightarrow [T_{i,1}, \dots, T_{i,n_i'}]$ consists of *one* generating morphism of Δ_T and identities otherwise. These generators are subject to the relations requiring that squares

$$\begin{tikzcd} w_1, \dots, w_n \arrow[r] \arrow[d] & z \arrow[d] \\ x_1, \dots, x_n \arrow[r] & y, \end{tikzcd} \tag{10}$$

in which the vertical and horizontal morphisms are generators as above, are commutative. Necessarily also only one of the left vertical morpisms is not an identity. In the non-multi-case $\Delta_T = \Delta_n$ we do not have any non-trivial relation-squares in which the horizontal and vertical morphisms are of the same type. Otherwise this may happen for type $< l$.

8.4. Let Δ_T be a tree. We define the degree $d(a)$ of an element a in Δ_T to be its distance from the root (i.e. the final object). We say that a list $[a_0 \dots a_n]$ has degree equal to the sum of the degrees of its entries $\sum d(a_i)$.
This enables us to define a degree function on ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T$ as follow: An object

$$a_0 \xrightarrow{\nu_1} a_1 \xrightarrow{\nu_2} a_2 \xrightarrow{\nu_3} [a_3]$$

is mapped to $3d(a_0) - d(a_1) - d(a_2) - d(a_3)$. It has the following properties

1. The degree is always non-negative.
2. The objects of degree zero are those in which all ν_i are identities.
3. Every morphism of type 4 increases the degree, and morphisms of type 1, 2 and 3 decrease it.
4. In particular, the relation squares (10) have a determined maximal and a determined minimal corner.

8.5. Let $\mathcal{S}$ be an opmulticategory and M a multidiagram. A pseudo-functor of multicategories

$$M \to \mathcal{S}^{\mathrm{cor}}$$

can be seen as a functor of opmulticategories

$${}^{\downarrow\uparrow}M \to \mathcal{S}$$

which is **admissible** in the sense that every square

$$\begin{tikzcd} i \arrow[r] \arrow[d] & j_1, \dots, j_n \arrow[d] \\ i' \arrow[r] & j_1', \dots, j_n' \end{tikzcd}$$

in which the horizontal morphisms are of type 2 and the vertical ones of type 1 is mapped to a Cartesian square. As is the non-multi-case we say that the functor ${}^{\downarrow\uparrow}M \to \mathcal{S}$ is **weakly admissible** if the squares above are instead mapped to weakly Cartesian squares.
Similarly: Consider a diagram I (not multidiagram) and the 2-multicategory $\mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$. Let M be a multicategory. A pseudo-functor

$$M \to \mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$$

may be seen as a functor of usual 1-opmulticategories

$${}^{\downarrow\uparrow}M \to \mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})^{\mathrm{adm}}.$$

This functor has the property that each morphism of type 1 is mapped to a type 2 admissible morphism and every (multi)morphism of type 2 is mapped to a type 1 admissible (multi)morphism and each diagram

$$\begin{tikzcd}
(\mu', j) \arrow[r] \arrow[d] & (\mu, i_0), \cdots, (\mu, i_n) \arrow[d] \\
(\mu''', j) \arrow[r] & (\mu'', i_0), \cdots, (\mu'', i_n)
\end{tikzcd}$$

in which the horizontal morphisms are of type 2 and the vertical morphisms are of type 1 (necessarily 1-ary) is mapped to a Cartesian square.
Similarly pseudo-functors of 2-multicategories

$$M \to \mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor}})$$

resp.

$$M \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor}})$$

are the same as functors between 1-opmulticategories

$${}^{\downarrow\uparrow}M \to \mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})^{\mathrm{adm}}$$

in which every morphism of type 1 is mapped to a (weakly in the oplax case) type 2 admissible morphism and every multimorphism of type 2 is mapped to a multimorphism (weakly in the lax case) type 1 admissible multimorphism and in which still every diagram as above is mapped to a Cartesian square. It has obviously still the property, that the resulting functor

$${}^{\downarrow\uparrow}(I \times M) \to \mathcal{S}$$

is *weakly* admissible.

8.6. Let $\mathcal{S}$ now be a usual category with compactifications turned into an opmulticategory in the usual way. Let M be a locally finite multidiagram and let $X : {}^{\downarrow\uparrow}M \to \mathcal{S}$ be a functor of opmulticategories. Then it can be compactified as well yielding a point-wise dense embedding

$$X \hookrightarrow \overline{X}.$$

As for the plain case, $\overline{X}$ does not need to be admissible if X is. The reason is the particular opmulticategory structure on $\mathcal{S}$: For an opmulticategory M define a usual category M° replacing all multimorphisms in M in $\mathrm{Hom}(j; i_1, \dots, i_n)$ by a set of 1-ary morphisms $j \to i_1, \dots, j \to i_n$

[12], then a functor of opmulticategories $M \to \mathcal{S}$ is the same as a functor between usual categories $M^\circ \to \mathcal{S}$. This also shows that we get an interior compactification

$$\widetilde{X} : {}^{\downarrow\downarrow\uparrow}M \to \mathcal{S}$$

for any (weakly) admissible

$$X : {}^{\downarrow\uparrow}M \to \mathcal{S}$$

(We leave it to the reader to construct a functor $({}^{\downarrow\downarrow\uparrow}M)^\circ \to {}^{\downarrow\downarrow}(({}^{\downarrow\uparrow}M)^\circ)$ analogously to 7.1.)

For an interior compactification

$$\widetilde{X} : {}^{\downarrow\downarrow\uparrow}M \to \mathcal{S}$$

we denote by $\widetilde{X}(i \to j \to k)$ where i, j and k are *lists of objects* in M the following. Let $k = [k_1, \dots, k_n]$. Then i and j break up into sublists $i_1, \dots, i_n$ and $j_1, \dots, j_n$ with morphisms $i_1 \to j_1 \to [k_1]$, etc. Then define

$$\widetilde{X}(i \to j \to k) := \prod_\nu \widetilde{X}(i_\nu \to j_\nu \to [k_\nu]).$$

For any (multi)morphism $(i \to j \to k) \to (i' \to j' \to k')$ in the obvious sense, we get a corresponding morphism $\widetilde{X}(i \to j \to k) \to \widetilde{X}(i' \to j' \to k')$.

With this definition the same Lemmas as in the previous section hold mutatis mutandis, namely:

Lemma 8.7. *Let M be a multidiagram and $X : {}^{\downarrow\uparrow}M \to \mathcal{S}$ be weakly admissible and let $\widetilde{X}$ be any interior compactification of it. Any square of the form*

$$\begin{tikzcd}
\widetilde{X}(i \to j \to [k]) \arrow[r, two heads] \arrow[d, hook] & \widetilde{X}(i' \to i \to [k]) \arrow[d, hook] \\
\widetilde{X}(i \to j' \to [k]) \arrow[r, two heads] & \widetilde{X}(i' \to j' \to [k])
\end{tikzcd}$$

is Cartesian for all i, i', j, j' lists of objects of M, and all objects $k \in M$.

Proof. Cf. Lemma 2.5. □

Lemma 8.8. *Let M be a multidiagram and $X : {}^{\downarrow\uparrow}M \to \mathcal{S}$ be weakly admissible and let $\widetilde{X}$ be any interior compactification of it. Any square of the form*

$$\begin{tikzcd}
\widetilde{X}(i = i \to [k]) \arrow[r] \arrow[d, hook] & \widetilde{X}(i = i \to [k_1]), \dots, \widetilde{X}(i = i \to [k_n]) \arrow[d, hook] \\
\widetilde{X}(i \to j \to [k]) \arrow[r] & \widetilde{X}(i \to j \to [k_1]), \dots, \widetilde{X}(i \to j \to [k_n])
\end{tikzcd}$$

is Cartesian for all i, j lists of objects of M, and all objects k and $k_1, \dots, k_n$ of M.

Proof. Writing $k' := [k_1, \dots, k_n]$ we have to check that the top square in the following diagram is Cartesian:

$$\begin{tikzcd}
\widetilde{X}(i = i \to [k]) \arrow[r] \arrow[d, hook] & \widetilde{X}(i = i \to k') \arrow[d, hook] \\
\widetilde{X}(i \to j \to [k]) \arrow[r] \arrow[d, two heads] & \widetilde{X}(i \to j \to k') \arrow[d, two heads] \\
\widetilde{X}(j = j \to [k]) \arrow[r] & \widetilde{X}(j = j \to k')
\end{tikzcd}$$

[12]that means, in particular, forgetting all 0-ary morphisms

in which the outer square is weakly Cartesian, because X is weakly admissible. The statement follows therefore from Lemma 2.7. □

Lemma 8.9. *Let M be a multidiagram and $X : {}^{\downarrow\uparrow}M \to \mathcal{S}$ be weakly admissible and let $\widetilde{X}$ be any interior compactification. Then the cubes*

$$\begin{tikzcd}
& \widetilde{X}(i=i\to[k]) \arrow[rr] \arrow[dl, equal] \arrow[dd, hook] & & \widetilde{X}(i=i\to[k_1]),\dots,\widetilde{X}(i=i\to[k_n]) \arrow[dl, equal] \arrow[dd, hook] \\
\widetilde{X}(i=i\to[k]) \arrow[rr, crossing over] \arrow[dd, hook] & & \widetilde{X}(i=i\to[k_1]),\dots,\widetilde{X}(i=i\to[k_n]) & \\
& \widetilde{X}(i\to j\to[k]) \arrow[rr, "\overline{G}"] \arrow[dl, hook] & & \widetilde{X}(i\to j\to[k_1]),\dots,\widetilde{X}(i\to j\to[k_n]) \arrow[dl, hook] \\
\widetilde{X}(i\to j'\to[k]) \arrow[rr, "\overline{g}"'] & & \widetilde{X}(i\to j'\to[k_1]),\dots,\widetilde{X}(i\to j'\to[k_n]) \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

is a weak compactifications of the top (trivially) Cartesian square, If X is admissible, then the cube

$$\begin{tikzcd}
& \widetilde{X}(i=i\to[k]) \arrow[rr] \arrow[dl] \arrow[dd, hook] & & \widetilde{X}(i=i\to[k_1]),\dots,\widetilde{X}(i=i\to[k_n]) \arrow[dl] \arrow[dd, hook] \\
\widetilde{X}(i'=i'\to[k]) \arrow[rr, crossing over] \arrow[dd, hook] & & \widetilde{X}(i'=i'\to[k_1]),\dots,\widetilde{X}(i'=i'\to[k_n]) & \\
& \widetilde{X}(i\to j\to[k]) \arrow[rr, "\overline{G}"] \arrow[dl] & & \widetilde{X}(i\to j\to[k_1]),\dots,\widetilde{X}(i\to j\to[k_n]) \arrow[dl] \\
\widetilde{X}(i'\to j\to[k]) \arrow[rr, "\overline{g}"'] & & \widetilde{X}(i'\to j\to[k_1]),\dots,\widetilde{X}(i'\to j\to[k_n]) \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

is a weak compactifications of the top Cartesian square.

Proof. We need to show that in each case the front and back squares are Cartesian squares. This is a consequence of Lemma 8.8. □

Lemma 8.10. *Consider a weak compactification of a Cartesian square.*

$$\begin{tikzcd}
& W \arrow[rr] \arrow[dl] \arrow[dd, hook] & & Z_1,\dots,Z_n \arrow[dl] \arrow[dd] \\
Y \arrow[rr, crossing over] \arrow[dd, hook] & & X_1,\dots,X_n & \\
& \overline{W} \arrow[rr, "\overline{G}"] \arrow[dl, "\overline{F}"'] & & \overline{Z}_1,\dots,\overline{Z}_n \arrow[dl, "\overline{f}_1,\dots,\overline{f}_n"] \\
\overline{Y} \arrow[rr, "\overline{g}"'] & & \overline{X}_1,\dots,\overline{X}_n \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

in which the $\overline{f}_i$ and $\overline{F}$ are proper. Then for a square

$$\begin{tikzcd}
\mathcal{H} & \mathcal{E}_1,\dots,\mathcal{E}_n \arrow[l, "\delta"'] \\
\mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1,\dots,\mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"']
\end{tikzcd}$$

in $\mathbb{D}(\cdot)$ above the bottom square the following holds: If $\mathcal{E}_i$ has support in Z_i for all i, and the α_i are Cartesian ($\mathcal{F}_i \cong \overline{f}_{i,*}\mathcal{E}_i$) and δ is coCartesian ($\overline{G}^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is Cartesian or, in other words, the natural exchange

$$\overline{g}^*(\overline{f}_{1,*}-, \dots, \overline{f}_{n,*}-) \to \overline{F}_* \overline{G}^*(-, \dots, -)$$

is an isomorphism on tupels of objects with support in $Z_1, \dots, Z_n$.

Proof. We look at the following diagram

$$\begin{tikzcd}
W \arrow[r, equal] \arrow[d, hook, "\iota_W"'] \arrow[rd, phantom, "1"] & W \arrow[r, "G"] \arrow[d, hook, "\iota"] \arrow[rd, phantom, "2"] & Z_1, \dots, Z_n \arrow[d, hook, "\iota_Z"] \\
\overline{W} \arrow[r, two heads, "\overline{f}''"'] \arrow[d, "\overline{F}"'] & \square \arrow[r, "G'"] \arrow[d, "\overline{f}'"] \arrow[rd, phantom, "3"] & \overline{Z}_1, \dots, \overline{Z}_n \arrow[d, "\overline{f}"] \\
\overline{Y} \arrow[r, equal] & \overline{Y} \arrow[r, "\overline{g}"'] & \overline{X}_1, \dots, \overline{X}_n
\end{tikzcd}$$

in which squares 2 and 3 are Cartesian. The middle left horizontal morphism is proper because of (S2). Because of the support condition, we have to show that the natural exchange

$$\overline{g}^*(\overline{f}_{1,*}\iota_{Z_1,!}-, \dots, \overline{f}_{n,*}\iota_{Z_n,!}-) \to \overline{F}_* \overline{G}^*(\iota_{Z_1,!}-, \dots, \iota_{Z_n,!}-)$$

is an isomorphism. Elementary properties of exchange morphisms imply that the morphism is the composition of the following exchange morphisms which are all isomorphisms because of the indicated reason.

$$\begin{array}{lll}
\overline{g}^*(\overline{f}_{1,*}\iota_{Z_1,!}-, \dots, \overline{f}_{n,*}\iota_{Z_n,!}-) \stackrel{\sim}{\longrightarrow} & \overline{f}'_*(G')^*(\iota_{Z_1,!}-, \dots, \iota_{Z_n,!}-) & \text{because 3 is Cartesian and (F4) in the form of Lemma 6.5} \\
\stackrel{\sim}{\longleftarrow} & \overline{f}'_*\iota_! G^*(-, \dots, -) & \text{because 2 is Cartesian and (F5m) in the form of Lemma 6.6} \\
\stackrel{\sim}{\longrightarrow} & \overline{f}'_*\overline{f}''_*\iota_{W,!}G^*(-, \dots, -) & \text{applying Lemma 7.6 for 1} \\
\stackrel{\sim}{\longrightarrow} & \overline{F}_*\iota_{W,!}G^*(-, \dots, -) & \\
\stackrel{\sim}{\longrightarrow} & \overline{F}_*\overline{G}^*(\iota_{Z_1,!}-, \dots, \iota_{Z_n,!}-) & \text{because the composite of 1 and 2 is Cartesian and (F5m) in the form of Lemma 6.6.}
\end{array}$$

□

Lemma 8.11. *Consider a weak compactification of a Cartesian square.*

$$\begin{tikzcd}
& W \arrow[rr] \arrow[ld, hook, "I"'] \arrow[dd, hook] & & Z_1, \dots, Z_n \arrow[ld, hook, "\iota_1, \dots, \iota_n"'] \arrow[dd, hook] \\
Y \arrow[rr, crossing over] \arrow[dd, hook] & & X_1, \dots, X_n & \\
& \overline{W} \arrow[rr, "\overline{G}"] \arrow[ld, hook, "\overline{I}"'] & & \overline{Z}_1, \dots, \overline{Z}_n \arrow[ld, hook, "\overline{\iota}_1, \dots, \overline{\iota}_n"] \\
\overline{Y} \arrow[rr, "\overline{g}"'] & & \overline{X}_1, \dots, \overline{X}_n \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

in which all $\bar{\iota}_i$, all ι_i, I, and $\overline{I}$ are embeddings. Then for a commutative square

$$\begin{tikzcd} \mathcal{H} & \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"'] \end{tikzcd}$$

in $\mathbb{D}(\cdot)$ above the bottom square the following holds: If $\underline{\mathcal{E}_i \text{ has support in } Z_i}$ for all I, and the α_i are strongly coCartesian ($\bar{\iota}_i^* \mathcal{F}_i \cong \mathcal{E}_i$ inducing $\bar{\iota}_{i,!}\mathcal{E}_i \cong \mathcal{F}_i$) and δ is coCartesian ($\overline{G}^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange

$$\overline{I}_! \overline{G}^*(-, \dots, -) \to \overline{g}^*(\bar{\iota}_{1,!}-, \dots, \bar{\iota}_{n,!}-)$$

is an isomorphism on tupels of objects with support in $Z_1, \dots, Z_n$.

Proof. Because of the support condition it suffices to see that the natural exchange

$$\overline{I}_! \overline{G}^*(\iota_{Z_1,!}-, \dots, \iota_{Z_n,!}-) \to \overline{g}^*(\bar{\iota}_{1,!}\iota_{Z_1,!}-, \dots, \bar{\iota}_{n,!}\iota_{Z_n,!}-)$$

is an isomorphism. Elementary properties of exchange morphisms imply that the morphism is the composition of the following exchange morphisms which are all isomorphisms because of pseudo-functoriality and because of the indicated reason:

$\overline{I}_! \overline{G}^*(\iota_{Z_1,!}-, \dots, \iota_{Z_n,!}-)$	$\overset{\sim}{\longleftarrow}$	$\overline{I}_! \iota_{W,!} G^*(-, \dots, -)$	because the back square in Cartesian and (F5m) in the form of Lemma 6.6
	$\overset{\sim}{\longrightarrow}$	$\iota_{Y,!} I_! G^*(-, \dots, -)$	
	$\overset{\sim}{\longrightarrow}$	$\iota_{Y,!} g^*(\iota_{1,!}-, \dots, \iota_{n,!}-)$	because the top square is Cartesian and and (F5m) in the form of Lemma 6.6
	$\overset{\sim}{\longrightarrow}$	$\overline{g}^*(\iota_{X_1,!}\iota_{1,!}-, \dots, \iota_{X_n,!}\iota_{n,!}-)$	because the front square is Cartesian and (F5m) in the form of Lemma 6.6
	$\overset{\sim}{\longrightarrow}$	$\overline{g}^*(\bar{\iota}_{1,!}\iota_{Z_1,!}-, \dots, \bar{\iota}_{n,!}\iota_{Z_n,!}-)$	

□

We summarize the discussion in the following Proposition:

Proposition 8.12. *Let M be a multidiagram, $X : {}^{\downarrow\uparrow}M \to \mathcal{S}$ be weakly admissible, and let $\widetilde{X} : {}^{\downarrow\downarrow\uparrow}M \to \mathcal{S}$ be any interior compactification of it. Consider a diagram of the form*

$$\begin{tikzcd} w \arrow[r, "d"] \arrow[d, "c"'] & z_1, \dots, z_n \arrow[d, "a_1,\dots,a_n"] \\ y \arrow[r, "b"'] & x_1, \dots, x_n \end{tikzcd}$$

in ${}^{\downarrow\downarrow\uparrow}M$ where the b and d are multimorphisms of type 3. Let

$$\begin{tikzcd} \mathcal{H} & \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"'] \end{tikzcd}$$

be a diagram in $\mathbb{D}(\cdot)$ above $\widetilde{X}^{\mathrm{op}}$ applied to the top square. *Then the following holds:*

1. *If X is admissible, let the a_i and c be of type 1. If $\underline{\mathcal{E}_i \text{ has support in } X(\pi_{13}(z_i))}$ for all i, and all α_i are Cartesian ($\mathcal{F}_i \cong \widetilde{X}(a_i)_*\mathcal{E}_i$) and δ is coCartesian ($\widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is Cartesian or, in other words, the natural exchange*

$$\widetilde{X}(b)^*(\widetilde{X}(a_1)_*-, \dots, \widetilde{X}(a_n)_*-) \to \widetilde{X}(c)_*\widetilde{X}(d)^*(-, \dots, -)$$

is an isomorphism on n-tupels of objects with support in $X(\pi_{13}(z_1)), \dots, X(\pi_{13}(z_n))$.

2. *Let the a_i and c be of type 2. If $\underline{\mathcal{E}_i \text{ has support in } X(\pi_{13}(z_i))}$ for all i, and all α_i are strongly coCartesian ($\widetilde{X}(a_i)^*\mathcal{F}_i \cong \mathcal{E}_i$ inducing $\widetilde{X}(a_i)_!\mathcal{E}_i \cong \mathcal{F}_i$) and δ is coCartesian ($\widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange*

$$\widetilde{X}(c)_!\widetilde{X}(d)^*(-, \dots, -) \to \widetilde{X}(b)^*(\widetilde{X}(a_1)_!-, \dots, \widetilde{X}(a_n)_!-)$$

is an isomorphism on n-tupels of objects with support in $X(\pi_{13}(z_1)), \dots, X(\pi_{13}(z_n))$.

3. *Let a and c be of type 2 and b and d of type 1. If γ is strongly coCartesian ($\widetilde{X}(c)^*\mathcal{G} \cong \mathcal{H}$ inducing $\widetilde{X}(c)_!\mathcal{H} \cong \mathcal{G}$) and δ is Cartesian ($\mathcal{E} \cong \widetilde{X}(d)_*\mathcal{H}$) then β is Cartesian if and only if α is strongly coCartesian, or in other words, the natural exchange*

$$\widetilde{X}(a)_!\widetilde{X}(d)_* \to \widetilde{X}(b)_*\widetilde{X}(c)_!$$

is an isomorphism.

Proof. 1. follows from Lemma 8.10, and 2. from Lemma 8.11, using Lemma 8.9 in each case. 3. is proven exactly as in Proposition 7.10. □

9 Fibered multiderivators over 2-categorical bases

In this section we recall from [13] the notion of 2-pre-multiderivator and fibered multiderivator (with 2-categorical bases).

Definition 9.1 ([13, Definition 2.1])**.** *A* **2-pre-multiderivator** *is a functor $\mathbb{S} : \mathrm{Dia}^{1-\mathrm{op}} \to$ 2-$\mathcal{MCAT}$ which is strict in 1-morphisms (functors) and pseudo-functorial in 2-morphisms (natural transformations). More precisely, it associates with a diagram I a 2-multicategory $\mathbb{S}(I)$, with a functor $\alpha : I \to J$ a strict functor*

$$\mathbb{S}(\alpha) : \mathbb{S}(J) \to \mathbb{S}(I)$$

denoted also α^* if $\mathbb{S}$ is understood, and with a natural transformation $\mu : \alpha \Rightarrow \alpha'$ a pseudo-natural transformation

$$\mathbb{S}(\eta) : \alpha^* \Rightarrow (\alpha')^*$$

such that the following holds:

1. *The association*

$$\mathrm{Fun}(I, J) \to \mathrm{Fun}(\mathbb{S}(J), \mathbb{S}(I))$$

given by $\alpha \mapsto \alpha^*$, resp. $\mu \mapsto \mathbb{S}(\mu)$, is a pseudo-functor (this involves, of course, the choice of further data). Here $\mathrm{Fun}(\mathbb{S}(J), \mathbb{S}(I))$ is the 2-category of strict 2-functors, pseudo-natural transformations, and modifications.

2. *(Strict functoriality w.r.t. compositons of 1-morphisms) For functors $\alpha : I \to J$ and $\beta : J \to K$, we have an* equality *of pseudo-functors* $\mathrm{Fun}(I,J) \to \mathrm{Fun}(\mathbb{S}(I), \mathbb{S}(K))$

$$\beta^* \circ \mathbb{S}(-) = \mathbb{S}(\beta \circ -).$$

A **symmetric, resp. braided 2-pre-multiderivator** *is given by the structure of strictly symmetric (resp. braided) 2-multicategory on $\mathbb{S}(I)$ such that the strict functors α^* are equivariant w.r.t. the action of the symmetric groups (resp. braid groups).*
Similarly we define a **lax, resp. oplax, 2-pre-multiderivator** *where the same as before holds but where the*

$$\mathbb{S}(\eta) : \alpha^* \Rightarrow (\alpha')^*$$

are lax (resp. oplax) natural transformations and in 1. "pseudo-natural transformations" is replaced by "lax (resp. oplax) natural transformations".

Definition 9.2 ([13, Definition 2.2])**.** *A strict morphism $p : \mathbb{D} \to \mathbb{S}$ of 2-pre-multiderivators (resp. lax/oplax 2-pre-multiderivators) is given by a collection of strict 2-functors*

$$p(I) : \mathbb{D}(I) \to \mathbb{S}(I)$$

for each $I \in \mathrm{Dia}$ such that we have $\mathbb{S}(\alpha) \circ p(J) = p(I) \circ \mathbb{D}(\alpha)$ and $\mathbb{S}(\mu) * p(J) = p(I) * \mathbb{D}(\mu)$ for all functors $\alpha : I \to J$, $\alpha' : I \to J$ and natural transformations $\mu : \alpha \Rightarrow \alpha'$ as illustrated by the following diagram:

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "p(J)"] \arrow[d, bend right, "\mathbb{D}(\alpha)"'] \arrow[d, bend left, "\mathbb{D}(\alpha')"] \arrow[d, phantom, "\overset{\mathbb{D}(\mu)}{\Rightarrow}"] & \mathbb{S}(J) \arrow[d, bend right, "\mathbb{S}(\alpha)"'] \arrow[d, bend left, "\mathbb{S}(\alpha')"] \arrow[d, phantom, "\overset{\mathbb{S}(\mu)}{\Rightarrow}"] \\
\mathbb{D}(I) \arrow[r, "p(I)"] & \mathbb{S}(I)
\end{tikzcd}$$

9.3. As with usual pre-multiderivators we consider the following axioms:

(Der1) For $I, J \in \mathrm{Dia}$, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence of 2-multicategories. Moreover $\mathbb{D}(\varnothing)$ is not empty.

(Der2) For $I \in \mathrm{Dia}$ the 'underlying diagram' functor

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Fun}(I, \mathbb{D}(\cdot)) \quad \text{resp. } \mathrm{Fun}^{\mathrm{lax}}(I, \mathbb{D}(\cdot)) \quad \text{resp. } \mathrm{Fun}^{\mathrm{oplax}}(I, \mathbb{D}(\cdot))$$

is 2-conservative (this means that it is conservative on 2-morphisms and that a 1-morphism α is an equivalence if $\mathrm{dia}(\alpha)$ is an equivalence).

And we consider the following axioms on a strict morphism $p : \mathbb{D} \to \mathbb{S}$ of 2-pre-multiderivators (where (FDer0 left) is assumed for (FDer3–5 left) and similarly for the right case):

(FDer0 left) For each I in Dia the morphism p specializes to an 1-opfibered 2-multicategory with 1-categorical fibers. It is, in addition, 2-fibered in the lax case and 2-opfibered in the oplax case. Moreover any *fibration* $\alpha : I \to J$ in Dia induces a diagram

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\
\mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I)
\end{tikzcd}$$

of 1-opfibered and 2-(op)fibered 2-multicategories, i.e. the top horizontal functor maps coCartesian 1-morphisms to coCartesian 1-morphisms and (co)Cartesian 2-morphisms to (co)Cartesian 2-morphisms.

We assume that corresponding push-forward functors between the fibers have been chosen and those will be denoted by $(-)_\bullet$.

(FDer3 left) For each functor $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 left))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left adjoint $\alpha_!^{(S)}$.

(FDer4 left) For each functor $\alpha : I \to J$ in Dia, and for any object $j \in J$, and for the 2-commutative square

$$\begin{tikzcd} I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors

$$\alpha_{j,!}^{(j^*S)} \mathbb{S}(\mu)(S)_\bullet \iota^* \to j^* \alpha_!^{(S)}$$

is an isomorphism for all $S \in \mathbb{S}(J)$.

(FDer5 left) For any *opfibration* $\alpha : I \to J$ in Dia, and for any 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(J)$ for some $n \geq 1$, the natural transformations of functors

$$\alpha_!(\alpha^*\xi)_\bullet(\alpha^*-, \cdots, \alpha^*-, \underbrace{-}_{\text{at } i}, \alpha^*-, \cdots, \alpha^*-) \cong \xi_\bullet(-, \cdots, -, \underbrace{\alpha_!-}_{\text{at } i}, -, \cdots, -)$$

are isomorphisms for all $i = 1, \dots, n$.

Dually, we consider the following axioms:

(FDer0 right) For each I in Dia the morphism p specializes to a 1-fibered multicategory with 1-categorical fibers. It is, in addition, 2-opfibered in the lax case, and 2-fibered in the oplax case. Furthermore, any *opfibration* $\alpha : I \to J$ in Dia induces a diagram

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of 1-fibered and 2-(op)fibered multicategories, i.e. the top horizontal functor maps Cartesian 1-morphisms w.r.t. the i-th slot to Cartesian 1-morphisms w.r.t. the i-th slot for any i and maps (co)Cartesian 2-morphisms to (co)Cartesian 2-morphisms.

We assume that corresponding pull-back functors between the fibers have been chosen and those will be denoted by $(-)^{\bullet,i}$.

(FDer3 right) For each functor $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 right))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a right adjoint $\alpha_*^{(S)}$.

(FDer4 right) For each morphism $\alpha : I \to J$ in Dia, and for any object $j \in J$, and for the 2-commutative square

$$\begin{tikzcd} j \times_{/J} I \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors

$$j^* \alpha_*^{(S)} \to \alpha_{j,*}^{(j^* S)} \mathbb{S}(\mu)(S)^\bullet \iota^*$$

is an isomorphism for all $S \in \mathbb{S}(J)$.

(FDer5 right) For any *fibration* $\alpha : I \to J$ in Dia, and for any 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(J)$ for some $n \geq 1$, the natural transformations of functors

$$\alpha_* (\alpha^* \xi)^{\bullet, i} (\alpha^* -, \overset{\widehat{i}}{\dots}, \alpha^* - \ ; \ -) \cong \xi^{\bullet, i} (-, \overset{\widehat{i}}{\dots}, - \ ; \ \alpha_* -)$$

are isomorphisms for all $i = 1, \dots, n$.

Definition 9.4. *A strict morphism $p : \mathbb{D} \to \mathbb{S}$ of (op)lax 2-pre-multiderivators is called a* **(op)lax left (resp. right) fibered multiderivator** *if $\mathbb{D}$ and $\mathbb{S}$ both satisfy (Der1) and (Der2) if (FDer0 left/right) and (FDer3–5 left/right) hold true.*

Remark 9.5. *One can show that the axioms imply, in the plain case, that the second part of (FDer0 left) and (FDer5 right) — which are then adjoint to each other — hold true for any functor $\alpha : I \to J$. Similarly* for 1-ary morphisms *the second part of (FDer0 right) and (FDer5 left) hold true for any functor $\alpha : I \to J$.*
In the oplax case the second part of (FDer0 left) and (FDer5 right) should be claimed to hold for any functor $\alpha : I \to J$, and in the lax case, and for 1-ary morphisms, *the second part of (FDer0 right) and (FDer5 left) should be claimed to hold for any functor $\alpha : I \to J$. It seems that this does not follow from the other axioms as stated. For the oplax left and lax right fibered multiderivators constructed in section 13 we will show explicitly that these stronger statements hold true.*

If in $\mathbb{S}$ all 2-morphisms are invertible then there is no difference between lax and oplax and we just say left (resp. right) fibered multiderivator.

Definition 9.6. *For (op)lax fibered multiderivators over an (op)lax 2-pre-multiderivator $p : \mathbb{D} \to \mathbb{S}$ and an object $S \in \mathbb{S}(I)$ we have that*

$$\mathbb{D}_{I,S} : J \mapsto \mathbb{D}(I \times J)_{\mathrm{pr}_2^* S}$$

is a usual derivator. We say that p has **stable fibers** *if $\mathbb{D}_{I,S}$ is stable for all $S \in \mathbb{S}(I)$ and for all I. In fact, it suffices to require this for $I = \cdot$.*

In [13] we defined a fibered multiderivator as below. Above we gave the equivalent patchwork definition because the axioms are anyway the ones to be checked.
A strict morphism $\mathbb{D} \to \mathbb{S}$ of (op)lax 2-pre-multiderivators (Definition 9.2) such that $\mathbb{D}$ and $\mathbb{S}$ each satisfy (Der1) and (Der2) (cf. 9.3) is a

1. lax left (resp. oplax right) fibered multiderivator if and only if the corresponding strict functor of 2-multicategories
$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$$
(cf. [13, Definition 3.6]) is a 1-opfibration (resp. 1-fibration) and 2-fibration with 1-categorical fibers.

2. oplax left (resp. lax right) fibered multiderivator if and only if the corresponding strict functor of 2-multicategories
$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$
(cf. [13, Definition 3.6]) is a 1-opfibration (resp. 1-fibration) and 2-fibration with 1-categorical fibers.

10 The construction of derivator six-functor-formalisms

10.1. Let $\mathcal{S}$ be a category with compactifications, and $\mathbb{S}^{\mathrm{op}}$ the symmetric pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ with domain Dirlf, where $\mathcal{S}^{\mathrm{op}}$ carries the natural symmetric multicategory structure encoding the product. Recall from [15, Section 3] the definition of the symmetric 2-multicategory $\mathcal{S}^{cor}$ (resp. $\mathcal{S}^{cor,0}$ formed w.r.t. the given class of proper morphisms). Denote its associated represented symmetric 2-pre-multiderivator with domain Cat by $\mathbb{S}^{\mathrm{cor}}$, $\mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$, and $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$, respectively. Recall:

Definition 10.2 ([13, Definition 6.1]). *1. A* **(symmetric) derivator six-functor-formalism** *is a left and right fibered (symmetric) multiderivator with domain* Cat
$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}.$$

2. A **(symmetric) proper derivator six-functor-formalism** *is as before with an extension as oplax left fibered (symmetric) multiderivator with domain* Cat
$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}},$$
and an extension as lax right fibered (symmetric) multiderivator with domain Cat
$$\mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}.$$

There is a dual notion of etale derivator six-functor-formalism which will not play any role in this article. The word "symmetric" in brackets indicates that there is a symmetric and a non-symmetric variant of the definition. In the symmetric variant all functors of multicategories occuring the various definitions have to be compatible with the actions of the symmetric groups.

10.3. Let $\mathbb{S}^{\mathrm{op}}$ be the symmetric pre-multiderivator as in 10.1. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a (symmetric) fibered multiderivator with domain Dirlf satisfying axioms (F1)–(F6) and (F4m)–(F5m) of 6.1. Assume that $\mathbb{D}$ is infinite (i.e. satisfies (Der1) also for infinite coproducts).

The goal is to construct a natural (symmetric) derivator six-functor-formalism $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ (and finally a (symmetric) *proper* derivator six-functor-formalism) whose restriction to $\mathbb{S}^{\mathrm{op}}$ is equivalent to $\mathbb{D}$. We will first construct a left fibered multiderivator $\mathbb{E} \to \mathbb{S}^{\mathrm{cor,comp}}$ with domain Catlf such that

$$\mathbb{E}(I)_{X \hookrightarrow \overline{X}} = \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{X}^{\mathrm{op}}}$$

for all compactified correspondences $X \hookrightarrow \overline{X}$ in $\mathbb{S}^{\mathrm{cor,comp}}(I)$, and where $\widetilde{X}$ denotes the corresponding interior compactification. The superscripts mean that we consider the full subcategory where the underlying morphism in $\mathbb{D}(\cdot)$ for any morphism of type 4 in ${}^{\downarrow\uparrow\uparrow\downarrow}I$ is coCartesian, strongly coCartesian for any morphism of type 3, and Cartesian for any morphism of type 2. We will also say that the objects are 4-coCartesian, strongly 3-coCartesian, and 2-Cartesian, respectively.

Example 10.4. *If $I = \Delta_1$, and $X : I \to \mathcal{S}^{\mathrm{cor}}$ is a correspondence*

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ S & & T \end{tikzcd}$$

with compactification $X \hookrightarrow \overline{X}$ in $\mathcal{S}^{\mathrm{cor,comp}}$, inducing the interior compactification $\widetilde{X}$

$$\begin{tikzcd} & A \arrow[ddl, "g"'] \arrow[dr, hook, "\iota"] & & \\ & & \overline{A} \arrow[dr, "\overline{f}"] & \\ S & & & T \end{tikzcd}$$

then the underlying diagram *of an object in $\mathcal{E} \in \mathbb{E}(I)_{X \hookrightarrow \overline{X}}$ is determined (up to isomorphism) by an object $\mathcal{E}_0$ in $\mathbb{D}(\cdot)_S$ and $\mathcal{E}_1$ in $\mathbb{D}(\cdot)_T$ together with a morphism*

$$\overline{f}_* \iota_! g^* \mathcal{E}_0 \to \mathcal{E}_1.$$

This is, of course, what is intended. It is therefore reasonable to believe that the above definition gives the right derivator enhancement of this situation.

We begin with a couple of Lemmas that are used to construct a 1-opfibration and 2-fibration $\mathbb{E}(I) \to \mathbb{S}^{\mathrm{cor,comp}}(I)$. The first crucial step is to understand how the given pull-back functors, (even multivalued) push-forward functors, and the additional $\iota_!$, for $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I) \to \mathbb{S}^{\mathrm{op}}({}^{\downarrow\uparrow\uparrow\downarrow}I)$ behave w.r.t. to the conditions of being 4-coCartesian, strongly 3-coCartesian, and 2-Cartesian, respectively.

10.5. Let I be in Catlf, let Δ_T be a tree, and let

$$\xi : \Delta_T \to \mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$$

be a functor of 2-multicategories. We can associate with it a functor of opmulticategories

$$X : {}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$$

which is admissible. By the construction in 8.6 it has an interior compactification

$$\widetilde{X} : {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$$

to which we may apply the results of the previous section.
Let o be a multimorphism in ${}^{\downarrow\downarrow\uparrow}\Delta_T$. If o is of type 3 (resp. type 2, resp. type 1) denote by

$$\begin{array}{rccl} \widetilde{g}: & \widetilde{A} & \to & \widetilde{S}_1, \dots, \widetilde{S}_n \\ \widetilde{\iota}: & \widetilde{A} & \to & \widetilde{A}' \\ \widetilde{f}: & \widetilde{A}' & \to & \widetilde{T} \end{array}$$

their images in

$$\mathrm{Fun}({}^{\downarrow\downarrow\uparrow}I, \mathcal{S}).$$

The notation is borrowed from the following example. Keep in mind, however, that not every multimorphism o occurs as the ones considered there. In particular the $\widetilde{A}, \widetilde{S}_i, \widetilde{T}$ do not need to be — unlike in the example — compactifications of admissible diagrams themselves.
For each object x in ${}^{\downarrow\downarrow\uparrow}\Delta_T$ there is a canonical object y in the image of ${}^{\downarrow\uparrow}\Delta_T$ with a morphism

$$o : y \to x$$

of type 2. Let $\widetilde{\iota}: \widetilde{S}' \to \widetilde{S}$ be the corresponding morphism. We say that an object in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{S}^{\mathrm{op}}}$ is **well-supported**, if it is strongly 3-coCartesian and the underlying diagram lies *point-wise* at all $i \in {}^{\downarrow\uparrow\uparrow\downarrow}I$ in the essential image of $\widetilde{\iota}(\pi_{234}(i))_!$. The corresponding full subcategory will be denoted by $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{\mathrm{ws}}_{\pi_{234}^* \widetilde{S}^{\mathrm{op}}}$.

Example 10.6 ($\Delta_T = \Delta_{1,n}$). *In this case $\xi : S_1, \dots, S_n \to T$ is a multimorphism in* $\mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$ *inducing the diagram in* $\mathrm{Fun}({}^{\downarrow\uparrow}I, \mathcal{S})$

$$\begin{tikzcd} & A \arrow[dl, "g_1"'] \arrow[dr, "g_n"] \arrow[drrr, "f"] & & & \\ S_1 & \dots & S_n & ; & T \end{tikzcd}$$

(hence with $g = (g_1, \dots, g_n)$ type 1 admissible as multimorphism, and f type 2 admissible) with exterior compactification

$$\begin{tikzcd} & \overline{A} \arrow[dl, "\overline{g}_1"'] \arrow[dr, "\overline{g}_n"] \arrow[drrr, "\overline{f}"] & & & \\ \overline{S}_1 & \dots & \overline{S}_n & ; & \overline{T} \end{tikzcd}$$

(Note that the diagrams $\overline{S}_1, \dots, \overline{S}_n, \overline{A}, \overline{T}$ have no admissibility properties and neither do the morphisms.)
It induces an interior compactification, i.e. a diagram of shape $({}^{\downarrow\downarrow\uparrow}\Delta_{1,n})^\circ$ in $\mathrm{Fun}({}^{\downarrow\downarrow\uparrow}I \to \mathcal{S})$*:*

$$\begin{tikzcd} & \widetilde{A} \arrow[ddl, "\widetilde{g}_1"'] \arrow[ddr, "\widetilde{g}_n"] \arrow[dr, hook, "\widetilde{\iota}"] & & & \\ & & \widetilde{A}' \arrow[drr, "\widetilde{f}"] & & \\ \widetilde{S}_1 & \dots & \widetilde{S}_n & ; & \widetilde{T} \end{tikzcd}$$

Lemma 10.7. *With the notation as in 10.3 and 10.5.*

1. *If ξ is a functor $\Delta_T \to \mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$, for any o of type 3, the multivalued functor*

$$(\pi_{234}^* \widetilde{g})^* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{S}_1^{\mathrm{op}}} \times \cdots \times \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{S}_n^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{A}^{\mathrm{op}}}.$$

is computed point-wise (in ${}^{\downarrow\uparrow\uparrow\downarrow}I$) and on well-supported objects preserves the condition of being 4-coCartesian, well-supported, and 2-Cartesian.

2. *For any o of type 2, the functor*

$$(\pi_{234}^* \widetilde{\iota})_! : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{A}^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* (\widetilde{A}')^{\mathrm{op}}}$$

i.e. the left adjoint of $(\pi_{234}^* \widetilde{\iota})^*$, which exists by (F1), is computed point-wise (in ${}^{\downarrow\uparrow\uparrow\downarrow}I$) on 4-coCartesian and well-supported objects, and on such it preserves the conditions of being 4-coCartesian, well-supported, and 2-Cartesian.

3. *If ξ is a functor $\Delta_T \to \mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$, for any o of type 1, the functor*

$$(\pi_{234}^* \widetilde{f})_* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* (\widetilde{A}')^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{T}^{\mathrm{op}}}.$$

is computed point-wise (in ${}^{\downarrow\uparrow\uparrow\downarrow}I$) (cf. Definition 6.10) and on well-supported objects it preserves the condition of being 4-coCartesian, well-supported, and 2-Cartesian.

Proof. 1. $(\pi_{234}^* \widetilde{g})^*$ is computed point-wise by axiom (FDer0 left) even for n-ary $\widetilde{g}$. Therefore the preservation of the coCartesianity is clear, and the preservation of the conditions of being well-supported and 2-Cartesian follows from Proposition 8.12, 1.–2. The fact that the inputs are well-supported is needed for the support condition of the Proposition.
2. The functor $(\pi_{234}^* \widetilde{\iota})_!$, which exists by Axiom (F1), it is not automatically computed point-wise. We will show below in two steps that it is actually computed point-wise *on the specified full subcategory*. Therefore again, the preservation of good support is clear, and the preservation of Cartesianity and coCartesianity conditions follows from Proposition 8.12, 2.–3.
STEP 1: Note that $\pi_{12} : {}^{\downarrow\uparrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow}I$ is an opfibration. For $\alpha \in {}^{\downarrow\uparrow}I$ denote $e_\alpha : ({}^{\downarrow\uparrow\uparrow\downarrow}I)_\alpha \hookrightarrow {}^{\downarrow\uparrow\uparrow\downarrow}I$ the inclusion of the fiber. We claim that the functor $\widetilde{\iota}_!$ commutes with e_α^* for $e_\alpha : ({}^{\downarrow\uparrow\uparrow\downarrow}I)_\alpha \hookrightarrow {}^{\downarrow\uparrow\uparrow\downarrow}I$ denoting the inclusion of the fiber. By Lemma 6.11, 1. we only have to show that for each coCartesian $\rho : \mu \to \tau_\bullet \mu$ the square

$$\begin{tikzcd}
(\widetilde{A}')^{\mathrm{op}}(\pi_{234}(\tau_\bullet \mu)) \arrow[r, two heads, "\widetilde{S}^{\mathrm{op}}(\pi_{234}\rho)"] \arrow[d, hook] & (\widetilde{A}')^{\mathrm{op}}(\pi_{234}(\mu)) \arrow[d, hook] \\
\widetilde{T}^{\mathrm{op}}(\pi_{234}(\tau_\bullet \mu)) \arrow[r, two heads, "\widetilde{T}^{\mathrm{op}}(\pi_{234}\rho)"'] & \widetilde{T}^{\mathrm{op}}(\pi_{234}(\mu))
\end{tikzcd}$$

is Cartesian. However, $\pi_{234}(\rho)$ is of type 1 and hence the horizontal morphisms are proper and the vertical morphisms come from morphisms of type 2 and are hence dense embeddings. The square is therefore Cartesian by Lemma 7.3.
STEP 2: We are thus reduced to show that on $\mathbb{D}(({}^{\downarrow\uparrow\uparrow\downarrow}I)_\alpha)_{e_\alpha^* \pi_{234}^* \widetilde{S}^{\mathrm{op}}}$ the functor $\iota_!$ is computed point-wise when restricted to 4-coCartesian, and well-supported objects. Note however, that all morphisms in the fiber are compositions of ones of type 3 and type 4 and hence the objects are thus (absolutely) coCartesian. Using Lemma 6.11, 3., we are reduced to show that for each morphism $\mu : x \to z$ in $({}^{\downarrow\uparrow\uparrow\downarrow}I)_\alpha$ denoting $H := e_\alpha^* \pi_{234}^* (\widetilde{A}')^{\mathrm{op}}(\mu)$ and $h := e_\alpha^* \pi_{234}^* \widetilde{T}^{\mathrm{op}}(\mu)$ the morphism

$$\widetilde{\iota}(y)_! H^* \mathcal{E}_x \to h^* \widetilde{\iota}(x)_! \mathcal{E}_x$$

is an isomorphism for 4-coCartesian, and strongly 3-coCartesian objects $\mathcal{E}$. Here, we wrote $\mathcal{E}_x$ for $x^*\mathcal{E}$. Writing $\mu = \mu_3 \circ \mu_4 : x \to y \to z$ where μ_3 is of type 3, and μ_4 is of type 4, and denoting $h = h_4 \circ h_3$, resp. $H = H_4 \circ H_3$ the corresponding factorization of h, resp. H, we have thus

$$\begin{array}{rcll} \widetilde{\iota}(z)_! H^* \mathcal{E}_x & \cong & \widetilde{\iota}(z)_! H_3^* H_4^* \mathcal{E}_x & \\ & \cong & \widetilde{\iota}(z)_! H_3^* \mathcal{E}_y & \text{coCartesianity} \\ & \cong & h_3^* \widetilde{\iota}(y)_! \mathcal{E}_y & \text{because } \mathcal{E}_y \text{ has support in } (\widetilde{A}')^{\mathrm{op}}(\pi_{234} z) \\ & \cong & h_3^* \widetilde{\iota}(y)_! H_4^* \mathcal{E}_x & \text{coCartesianity} \\ & \cong & h_3^* h_4^* \widetilde{\iota}(x)_! \mathcal{E}_x & \text{Proposition 8.12, 2.} \\ & \cong & h^* \widetilde{\iota}(x)_! \mathcal{E}_x & \end{array}$$

It is not true that, in general, $\widetilde{\iota}_!$ commutes with h^* here! The support conditions are essential.
3. The functors $(\pi_{234}^* \widetilde{f})_*$, which exist by (FDer0 right) are also computed point-wise *for 1-ary* $\widetilde{f}$. Therefore the preservation of Cartesianity is clear, and the preservation of good support and coCartesianity conditions follows from from Proposition 8.12, 1. and 3. The fact that the input is well-supported is needed for the support condition in 1. of the Proposition. □

Proposition 10.8. *Let Δ_T be a tree, $\xi : \Delta_T \to \mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$ be a functor, and let $\widetilde{X} : {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ be any interior compactification of the corresponding $X : {}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$. Consider a diagram of the form*

$$\begin{tikzcd} w \arrow[r, "d"] \arrow[d, "c"'] & z_1, \dots, z_n \arrow[d, "a_1,\dots,a_n"] \\ y \arrow[r, "b"'] & x_1, \dots, x_n \end{tikzcd}$$

in ${}^{\downarrow\downarrow\uparrow}\Delta_T$. Let

$$\begin{tikzcd} \mathcal{H} & \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"'] \end{tikzcd}$$

be a diagram in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)$ above π_{234}^* of $\widetilde{X}^{\mathrm{op}}$ applied to the top square.
Then the following holds:

1. *Let the a_i and c be of type 1 and let b and d be multimorphisms of type 3.*

   Assume that the $\mathcal{E}_i$ are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   If all α_i are Cartesian ($\mathcal{F}_i \cong \widetilde{X}(a_i)_* \mathcal{E}_i$) and δ is coCartesian ($\widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is Cartesian or, in other words, the natural exchange

$$\widetilde{X}(b)^*(\widetilde{X}(a_1)_* -, \dots, \widetilde{X}(a_n)_* -) \to \widetilde{X}(c)_* \widetilde{X}(d)^*(-, \dots, -) \tag{11}$$

   is an isomorphism on n-tupels of objects in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   If this is the case then all other objects are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

2. *Let the a_i and c be of type 2 and let b and d multimorphisms of type 3.*

   Assume that the $\mathcal{E}_i$ are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

If all α_i are strongly coCartesian ($\widetilde{X}(a_i)^*\mathcal{F}_i \cong \mathcal{E}_i$ inducing $\widetilde{X}(a_i)_!\mathcal{E}_i \cong \mathcal{F}_i$) and δ is coCartesian ($\widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange

$$\widetilde{X}(c)_!\widetilde{X}(d)^*(-,\dots,-) \to \widetilde{X}(b)^*(\widetilde{X}(a_1)_!-,\dots,\widetilde{X}(a_n)_!-)$$

is an isomorphism on n-tupels of well-supported objects in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},\mathrm{ws},2-\mathrm{cart}}$.

If this is the case then all other objects in the diagram are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},\mathrm{ws},2-\mathrm{cart}}$.

3. *Let a and c be of type 2 and b and d of type 1 (now necessarily 1-ary).*

   If γ is strongly coCartesian ($\widetilde{X}(c)^*\mathcal{G} \cong \mathcal{H}$ inducing $\widetilde{X}(c)_!\mathcal{H} \cong \mathcal{G}$) and δ is Cartesian ($\mathcal{E} \cong \widetilde{X}(d)_*\mathcal{H}$) then β is Cartesian if and only if α is strongly coCartesian, or in other words, the natural exchange

$$\widetilde{X}(a)_!\widetilde{X}(d)_* \to \widetilde{X}(b)_*\widetilde{X}(c)_!$$

   is an isomorphism.

   If this is the case, and $\mathcal{H}$ is in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},\mathrm{ws},2-\mathrm{cart}}$, then all other objects in the diagram are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},\mathrm{ws},2-\mathrm{cart}}$.

Proof. This follows from Proposition 8.12 taking into account (cf. Lemma 10.7) that strongly (co)Cartesian can be checked point-wise. □

Fundamental Lemma 10.9. *With the notation as in 10.3.*

1. *Let Δ_T be a tree, and let $\xi : \Delta_T \to \mathrm{Fun}(I, \mathcal{S}^{\mathrm{cor}})$ be a functor of multicategories. Let $\widetilde{X} : {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ be an interior compactification (8.6) of the corresponding admissible $X : {}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$.*

   Let $(\mathcal{E}_o)_{o \in \Delta_T}$ be a collection of objects with $\mathcal{E}_o \in \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{X}_o^{\mathrm{op}}}$, where $\widetilde{X}_o$ is the value of $\widetilde{X}$ at ${}^{\downarrow\downarrow\uparrow}o$. Then the category[13]

$$M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \coloneqq \{\mathcal{F} \in \mathrm{Fun}({}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T, \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I))^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{X}^{\mathrm{op}}}, ({}^{\downarrow\uparrow\uparrow\downarrow}o)^*\mathcal{F} \cong \mathcal{E}_o\}$$

   is equivalent to a set.

2. *For $\Delta_T = \Delta_{T_1} \circ_i \Delta_{T_2}$, where i is a source object of Δ_{T_1} (which we identify with the final object of Δ_{T_2}), the square*

$$\begin{tikzcd} M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \arrow[r] \arrow[d] & M_{\widetilde{X}_1}((\mathcal{E}_o)_{o\in\Delta_{T_1}}) \arrow[d] \\ M_{\widetilde{X}_2}((\mathcal{E}_o)_{o\in\Delta_{T_2}}) \arrow[r] & \{\ \textit{Iso.class of } \mathcal{E}_i\ \} \end{tikzcd}$$

   is 2-Cartesian. Hence if we consider the M as sets, we have

$$M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \cong M_{\widetilde{X}_1}((\mathcal{E}_o)_{o\in\Delta_{T_1}}) \times M_{\widetilde{X}_2}((\mathcal{E}_o)_{o\in\Delta_{T_2}}).$$

[13] The supscript "4−cocart" means here *equivalently* 1. that the functor $\mathcal{F}$ maps morphisms of type 4 to coCartesian morphisms for the bifibration $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I) \to \mathbb{S}({}^{\downarrow\uparrow\uparrow\downarrow}I)$, or 2. that the total underlying diagram in $\mathrm{Fun}({}^{\downarrow\uparrow\uparrow\downarrow}(\Delta_T \times I), \mathbb{D}(\cdot))$ maps morphisms of type 4 to coCartesian morphisms for the bifibration $\mathbb{D}(\cdot) \to \mathcal{S}^{\mathrm{op}}$. Similarly for the other supscripts.

3. *For $\Delta_T = \Delta_{1,n}$, we have canonically an isomorphism of sets*

$$M_{\widetilde{X}}(\mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{E}_{n+1}) \cong \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{T}^{\mathrm{op}}}}(\widetilde{f}_* \widetilde{\iota}_! \widetilde{g}^* (\mathcal{E}_1, \dots, \mathcal{E}_n); \mathcal{E}_{n+1})$$

for any choice of pull-back $\widetilde{g}^*$, push-forward $\widetilde{f}_*$, and adjoint $\widetilde{\iota}_!$ to the pull-back $\widetilde{\iota}^*$. Here $\widetilde{T}$ is the restriction of $\widetilde{X}$ to ${}^{\downarrow\downarrow\uparrow}(n+1)$, and $\widetilde{g}$, $\widetilde{\iota}$, and $\widetilde{f}$ are the components of $\widetilde{X}$ as in 10.6.

An object $\mathcal{F}$ on the left hand side is mapped to an isomorphism if and only if it is also Cartesian (or equivalently coCartesian) w.r.t. the projection

$$\pi_{234} \times \mathrm{id} : {}^{\downarrow\uparrow\uparrow\downarrow}(\Delta_{1,n} \times I) \to {}^{\uparrow\uparrow\downarrow}\Delta_{1,n} \times {}^{\downarrow\uparrow\uparrow\downarrow}I.$$

3. *For each interior compactification $\widetilde{f} : \Delta_1 \times {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ which comes from a refinement of exterior compactifications $\overline{f} : \overline{X}_1 \to \overline{X}_2$ of X, the functors $(\pi_{234}^* \widetilde{f})^*, (\pi_{234}^* \widetilde{f})_*$ are mutually inverse bijections between*

$$\begin{tikzcd} M_{e_0^* \widetilde{f}}((\mathcal{E}_o)_{o \in \Delta_T}) \arrow[r, shift left, "(\pi_{234}\widetilde{f})_*"] & M_{e_1^* \widetilde{f}}((\mathcal{E}_o)_{o \in \Delta_T}) \arrow[l, shift left, "(\pi_{234}^*\widetilde{f})^*"] \end{tikzcd}$$

Proof. 1. Every object x of ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T$ is connected by a sequence of morphisms

$$x \to \cdots \leftarrow \cdots \to {}^{\downarrow\uparrow\uparrow\downarrow}o$$

of type 2,3 and 4 to an object of the form ${}^{\downarrow\uparrow\uparrow\downarrow}o$ for $o \in \Delta_T$. Morphisms of type 4 go to the right and morphisms of type 2 and 3 go to the left, i.e. the degree (cf. 8.4) is strictly decreasing from left to right. Therefore the statement follows from the uniqueness (up to unique isomorphism) of the source (resp. target) of a Cartesian (resp. strongly coCartesian, resp. coCartesian) morphism.
2. Given restrictions of $\mathcal{F}$ to the union of ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_{T_1}$ and ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_{T_2}$, we construct an extension to the whole diagram ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T$ (here $\Delta_T := \Delta_{T_2} \circ_i \Delta_{T_1}$ is the concatenation) by induction on the degree (cf. 8.4).
Note that every object of the form ${}^{\downarrow\uparrow\uparrow\downarrow}o$ lies already in ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_{T_1}$ or ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_{T_2}$. Therefore, using Lemma 10.7, and the observation of 1., we can define an extension on objects. It has the property that all values are again in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{S}_o^{\mathrm{op}}}$ and well-supported.
To define the extension on morphisms observe that by 8.3 the multidiagram ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T$ is generated by its multimorphisms of type 1,2,3 or 4 and "length 1" and with relations being commutative squares

$$\begin{tikzcd} w_1, \dots, w_n \arrow[r] \arrow[d] & z \arrow[d] \\ x_1, \dots, x_n \arrow[r] & y \end{tikzcd} \tag{12}$$

We say that (12) is of type (a, b) if the horizontal morphism is of type a and the vertical morphism is of type b. By symmetry, it suffices to consider the cases $(4,3), (4,2), (4,1), (3,3), (3,2), (3,1), (2,2), (2,1)$, and $(1,1)$. The case $(4,4)$ does not appear.
Consider a functor

$$\mathcal{F} : {}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}.$$

with the conditions defining $M_{\widetilde{S}}$.

Let (12) be mapped by $\mathcal{F}$ to a square

$$\begin{tikzcd} \mathcal{F}_{w_1},\dots,\mathcal{F}_{w_n} \arrow[r] \arrow[d] & \mathcal{F}_z \arrow[d] \\ \mathcal{F}_{x_1},\dots,\mathcal{F}_{x_n} \arrow[r] & \mathcal{F}_y \end{tikzcd}$$

By Proposition 10.8, in cases $(3,3)$, $(2,2)$, $(3,2)$, $(4,2)$, and $(4,3)$, when there are conditions on the horizontal *and* vertical morphisms, a square with a part of bounded degree (cf. 8.4) filled in, can be always completed (uniquely up to unique isomorphism) to a full square.
In the cases $(2,1)$, $(3,1)$, $(4,1)$, when there is a condition only on the horizontal morphisms, a square with a part of bounded degree (cf. 8.4) filled in, can be always completed (uniquely up to unique isomorphism) to a full square, provided that the square contains the whole morphism lying over the one of type 1. Therefore, by induction, an extension can be defined on morphisms, such that all relations in ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T$ are respected, and such that the conditions defining $M_{\widetilde{S}}$ are fulfilled.
In the case $(1,1)$ all morphisms are already determined uniquely by other relation squares. Because they are unique the square must commute.
3. is clear.
4. Note that the functors $(\pi_{234}^*\widetilde{f})^*$ and $(\pi_{234}^*\widetilde{f})_*$ are both computed point-wise. By Lemma 10.7, 3. the functor $(\pi_{234}^*\widetilde{f})_*$ preserves the conditions of being 4-coCartesian, strongly 3-coCartesian, and 2-Cartesian. For each element $i_1 \to i_2 \to i_3 \to i_4 \in {}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T$ where $i_1,\dots,i_4$ are lists of objects in Δ_T, with i_4 containing one object, consider

$$\begin{tikzcd} \widetilde{X}_1(i_2 = i_2 \to i_4) \arrow[r, hook] \arrow[d, equal] & \widetilde{X}_1(i_2 \to i_3 \to i_4) \arrow[r, two heads] \arrow[d, "\widetilde{f}(i_2\to i_3\to i_4)"] & \widetilde{X}_1(i_3 \to i_3 \to i_4) \arrow[d, equal] \\ \widetilde{X}_2(i_2 = i_2 \to i_4) \arrow[r, hook] & \widetilde{X}_2(i_2 \to i_3 \to i_4) \arrow[r, two heads] & \widetilde{X}_2(i_3 \to i_3 \to i_4) \end{tikzcd}$$

By Lemma 2.5 the left square is (point-wise) Cartesian. Therefore by (F5) and (F6) $\widetilde{f}_*$ and $\widetilde{f}^*$ are inverse (up to isomorphism) to each other on strongly 3-coCartesian objects. By Lemma 10.7, $\widetilde{f}_*$ preserves the conditions of being 2-Cartesian and 4-coCartesian, and $\widetilde{f}^*$ preserves the condition of being 4-coCartesian. We claim that the latter also preserves the condition of being 2-Cartesian. Indeed for strongly 3-coCartesian objects it suffices to check 2-Cartesianity over morphisms of the form

$$(i_1 \to i_2 \to i_3 \to i_4) \twoheadrightarrow (i_1 \to i_3 = i_3 \to i_4)$$

where, however, $\widetilde{f}_*$ reflects 2-Cartesianity by the diagram above.
Therefore both functors preserve all conditions of being (strongly) (co)Cartesian. Since they are adjoints (between groupoids) they become mutually inverse when we pass to the sets of isomorphism classes. □

From now on, we will use $M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T})$ for the corresponding *set*.

Lemma 10.10. *Assume $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is symmetric.*
For $\Delta_T = \Delta_{1,n}$ and $\sigma \in S_n$ there is a natural isomorphism

$$M_{\widetilde{X}}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{E}_{n+1}) \to M_{\sigma(\widetilde{X})}(\mathcal{E}_{\sigma(1)},\dots,\mathcal{E}_{\sigma(n)};\mathcal{E}_{n+1})$$

These form an action of the symmetric group. Here $\sigma(\widetilde{X})$ is the interior compactification induced by $\sigma(\overline{X}) \to \sigma(X)$.

Proof. The action is constructed in the obvious way using the symmetric structure on the multicategory $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)$ taking into account the special shape of ${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_{1,n}$. $\square$

Definition 10.11. *With the notation as in 10.3, let I be a diagram in* Catlf. *We define a 2-multicategory together with a strict functor between 2-multicategories*

$$\mathbb{E}(I) \quad \to \quad \mathbb{S}^{\mathrm{cor,comp}}(I)$$

as follows:

1. *Objects in $\mathbb{E}(I)$ are pairs $(X \hookrightarrow \overline{X}, \mathcal{E})$ of an exterior compactification $X \hookrightarrow \overline{X}$ in $\mathbb{S}^{\mathrm{cor,comp}}(I)$ and an object*
$$\mathcal{E} \in \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{X}^{\mathrm{op}}}$$
where $\widetilde{X}$ is the induced interior compactification.

2. *1-morphisms from $(X_1 \hookrightarrow \overline{X}_1, \mathcal{E}_1), \dots, (X_n \hookrightarrow \overline{X}_n, \mathcal{E}_n)$ to $(Y \hookrightarrow \overline{Y}, \mathcal{F})$ are pairs $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m, x)$ of a 1-morphism $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$ in $\mathbb{S}^{\mathrm{cor,comp}}(I)$ with the given sources and destination, and an element $x \in M_{\widetilde{\Xi}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$. Recall that for the 1-morphism $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$, we fixed a compactification $\Xi \hookrightarrow \overline{\Xi}$ of the composition in $\mathbb{S}^{\mathrm{cor}}(I)$. (The compactification $\Xi \hookrightarrow \overline{\Xi}$ is the pullback along $\Delta_{1,n} \to \Delta_T$ of the compactification constructed in Lemma 5.11). The $\widetilde{\Xi}$ appearing is the induced interior compactification.*

3. *Any 2-morphism $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m) \Rightarrow (\Delta_{T'}, (\overline{S}'_o)_o, (\overline{\xi}'_m)_m)$ in the 2-multicategory $\mathbb{S}^{\mathrm{cor,comp}}(I)$, which can be represented by a morphism of compactifications $\overline{\Xi} \to \overline{\Xi}'$, gives an isomorphism $M_{\widetilde{\Xi}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to M_{\widetilde{\Xi}'}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ and we declare a 2-morphism from elements in source to the corresponding element in the destination. An arbitrary 2-morphism in $\mathbb{S}^{\mathrm{cor}}(I)$ is a morphism in the localization. Since the inverted morphisms are mapped to isomorphisms (cf. Lemma 10.9, 4.) and the association is functorial, the construction uniquely extends to the localization.*

4. *The composition of 1-morphisms is defined as follows. For each pair of composable 1-morphisms $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$ and $(\Delta_{T'}, (\overline{S}'_o)_o, (\overline{\xi}'_m)_m)$ we may extract a compactification $\Phi \hookrightarrow \overline{\Phi}$ of the corresponding morphism ${}^{\downarrow\uparrow}((\Delta_{1,n} \circ_i \Delta_{1,n'}) \times I) \to \mathcal{S}$ from the one constructed in Lemma 5.11 (Pullback along the corresponding functor $\Delta_{1,n} \circ_i \Delta_{1,n'} \to \Delta_T \circ_i \Delta'_T$). The compactifications[14] $({}^{\downarrow\uparrow}e_{01})^*\overline{\Phi}$, $({}^{\downarrow\uparrow}e_{12})^*\overline{\Phi}$, and $({}^{\downarrow\uparrow}e_{02})^*\overline{\Phi}$ are equal to the three fixed ones used to define the sets of 1-morphisms in $\mathbb{E}(I)$. Therefore we need to define a morphism*
$$\begin{aligned} &M_{({}^{\downarrow\downarrow\uparrow}e_{01})^*\widetilde{\Phi}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}_i) \times M_{({}^{\downarrow\downarrow\uparrow}e_{12})^*\widetilde{\Phi}}(\mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}) \\ &\qquad \to M_{({}^{\downarrow\downarrow\uparrow}e_{02})^*\widetilde{\Phi}}(\mathcal{E}_1, \dots, \mathcal{F}_1, \dots, \mathcal{F}_m, \dots, \mathcal{E}_n; \mathcal{G}). \end{aligned} \tag{13}$$
However, by Lemma 10.9, there are morphisms from the set
$$M_{\widetilde{\Phi}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}_1, \dots, \mathcal{F}_m; \mathcal{G}) \tag{14}$$
to the three sets in (13) and Fundamental Lemma 10.9, 3. implies that the corresponding relation gives a well-defined composition. The analogous reasoning for a composition of 3 morphisms implies that this composition is associative.

[14] The functors e_{01}, e_{12}, e_{02} denote respectively the inclusion of the first multimorphism $\Delta_{1,n'}$, second multimorphism $\Delta_{1,n}$ and composite multimorphism $\Delta_{1,n+n'-1}$ into $\Delta_{1,n} \circ_i \Delta_{1,n'}$.

Lemma 10.12. *The functor of 2-multicategories*

$$\mathbb{E}(I) \quad \to \quad \mathbb{S}^{\mathrm{cor,comp}}(I)$$

constructed in Definition 10.11 is a 1-opfibration and 2-bifibration with 1-categorical fibers.

Proof. The functors have 1-categorical fibers and are 2-bifibered by Definition 10.11, 3. By Lemma 10.9, 3., weakly coCartesian morphisms are exactly those such that the corresponding object $\mathcal{F}$ is also $\pi_{234} \times$ id-(co)Cartesian. In the construction (Definition 10.11, 4.) two $\pi_{234} \times$ id-(co)Cartesian objects can only come from a $\pi_{234} \times$ id-(co)Cartesian object in (14). Therefore the composition of weakly coCartesian morphisms are weakly coCartesian and hence the functor is a 1-opfibration [15, Proposition 2.7]. □

Definition 10.13. *With the notation as in 10.3, we will now construct strict morphisms of pre-2-multiderivators of domain* Catlf

$$\mathbb{E} \quad \to \quad \mathbb{S}^{\mathrm{cor,comp}}.$$

The values $\mathbb{E}(I)$ are as defined already in Definition 10.11. For a functor $\alpha : I \to J$ in Catlf *we define the pullback $\alpha^* = \mathbb{E}(\alpha)$ to be $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)$. Note that α induces a functor ${}^{\downarrow\uparrow\uparrow\downarrow}\alpha : {}^{\downarrow\uparrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow\uparrow\downarrow}J$ and that $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)$ preserves the conditions of being (strongly) (co)Cartesian. The pre-2-multiderivator $\mathbb{E}$ is defined on natural transformations as follows. A natural transformation $\alpha \Rightarrow \beta$ can be seen as a morphism $\Delta_1 \times I \to J$. Pullback of a diagram in $\mathcal{E} \in \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)$ and taking partial underlying diagram gives a functor in* $\mathrm{Fun}({}^{\downarrow\uparrow\uparrow\downarrow}\Delta_1, \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I))$ *which has the right strong (co)Cartesianity conditions. It is, by definition, a morphism*

$$\alpha^*\mathcal{E} = e_0^*\mathcal{F} \to \beta^*\mathcal{E} = e_1^*\mathcal{F}$$

in $\mathbb{E}(I)$ which we define to be the natural transformation $\mathbb{E}(\mu)$ at $\mathcal{E}$.

The next goal is to establish that the morphism of pre-2-multiderivators

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor,comp}}$$

is a left fibered multiderivator with domain Catlf. The only missing step is the construction of relative Kan extensions.
In the stable case, using the equivalence of 2-pre-multiderivators $\mathbb{S}^{\mathrm{cor,comp}} \cong \mathbb{S}^{\mathrm{cor}}$, this will allow (using Brown representability) to construct the desired derivator six-functor-formalism, i.e. a left *and right* fibered multiderivator

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor}}.$$

11 Relative Kan extensions

Lemma 11.1. *Let $\alpha : I \to J$ be an opfibration in* Catlf*, and consider the sequence of functors:*

$${}^{\downarrow\uparrow\uparrow\downarrow}I \xrightarrow{q_1 = ({}^{\downarrow\uparrow\uparrow\downarrow}\alpha, \pi_{123})} {}^{\downarrow\uparrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow\uparrow}J)} {}^{\downarrow\uparrow\uparrow}I \xrightarrow{q_2 = \mathrm{id} \times \pi_1} {}^{\downarrow\uparrow\uparrow\downarrow}J \times_J I.$$

1. *The functor q_1 is an opfibration. The fiber of q_1 over a pair $j_1 \to j_2 \to j_3 \to j_4$ and $i_1 \to i_2 \to i_3$ is*

$$i_4 \times_{/I_{j_4}} I_{j_4}$$

where i_4 is the target of a coCartesian arrow over $j_3 \to j_4$ with source i_3.

2. *The functor q_2 is a fibration. The fiber of q_2 over a pair $j_1 \to j_2 \to j_3 \to j_4$ and i_1 (lying over j_1) is*
$$(i_2 \times_{/I_{j_2}} I_{j_2} \times_{/I_{j_3}} I_{j_3})^{\mathrm{op}}$$
where i_2 is the target of a coCartesian arrow over $j_1 \to j_2$ with source i_1 and the second comma category is constructed via the functor $I_{j_2} \to I_{j_3}$ being the coCartesian push-forward along $j_2 \to j_3$.

Proof. Straightforward. □

Lemma 11.2. *Under the assumptions of 10.3, if $\alpha : I \to J$ is an opfibration in* Catlf *then the functors*

$$\begin{array}{c} \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J \times_J I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \\ \Big\downarrow q_2^* \\ \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow\uparrow}J)} {}^{\downarrow\uparrow\uparrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \\ \Big\downarrow q_1^* \\ \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \end{array}$$

are equivalences. In particular (applying this to $J = \cdot$ and variable I) we have an equivalence of fibers:
$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Here $\mathbb{E}$ is the pre-2-multiderivator constructed in Definiton 10.13. Note that $\mathbb{E}_S$ is a usual pre-derivator, though.

Proof. We first treat the case of q_1^*. We know by Lemma 11.1 that q_1 is an opfibration with fibers of the form $i_4 \times_{/I_{j_4}} I_{j_4}$. Neglecting the conditions of being (co)Cartesian, we know that q_1^* has a left adjoint:
$$q_{1,!} : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J \times_{({}^{\uparrow\uparrow\downarrow}J)} {}^{\uparrow\uparrow\downarrow}I)_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})}$$
We will show that the unit and counit
$$\mathrm{id} \Rightarrow q_1^* q_{1,!} \qquad q_{1,!} q_1^* \Rightarrow \mathrm{id}$$
are isomorphisms *when restricted to the subcategory of* 4*-coCartesian objects*. Since the conditions of being 2-Cartesian and strongly 3-coCartesian objects match under q_1^* this shows the first assertion. Since q_1 is an opfibration this is the same as to show that for any object in ${}^{\downarrow\uparrow\uparrow\downarrow}J \times_{({}^{\uparrow\uparrow\downarrow}J)} {}^{\uparrow\uparrow\downarrow}I$ with fiber $F = i_4 \times_{/I_{j_4}} I_{j_4}$ the unit and counit
$$\mathrm{id} \Rightarrow p_F^* p_{F,!} \qquad p_{F,!} p_F^* \Rightarrow \mathrm{id} \tag{15}$$
are isomorphisms when restricted to the subcategory of 4-coCartesian objects. Since all morphisms in the fiber F are of type 4, we have to show that the morphisms in (15) are isomorphisms when restricted to (absolutely) (co)Cartesian objects. This follows from the fact that F has an initial object [13, Lemma 7.21 and Corollary 7.22].

We now treat the case of q_2^*. We know by Lemma 11.1 that q_2 is a fibration with fibers of the form $(i_2 \times_{/I_{j_2}} I_{j_2} \times_{/I_{j_3}} I_{j_3})^{\mathrm{op}}$. Neglecting the conditions of being (co)Cartesian, we know that q_1^* has a right adjoint:

$$q_{2,*} : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J \times_{({}^{\uparrow\uparrow\downarrow}J)} {}^{\uparrow\uparrow\downarrow}I)_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J \times_J I)_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})}$$

We will show that the unit and counit

$$\mathrm{id} \Rightarrow q_{2,*}q_2^* \qquad q_2^* q_{2,*} \Rightarrow \mathrm{id}$$

are isomorphisms *when restricted to the subcategory of* 2*-Cartesian and strongly* 3*-coCartesian objects.* Since the conditions of being 4-coCartesian match under q_2^* this shows the second assertion. Since q_2 is a fibration this is the same as to show that for any object in ${}^{\downarrow\uparrow\uparrow\downarrow}J \times_J I$ with fiber $F = (i_2 \times_{/I_{j_2}} I_{j_2} \times_{/I_{j_3}} I_{j_3})^{\mathrm{op}}$ the the unit and counit

$$\mathrm{id} \Rightarrow p_{F,*}p_F^* \qquad p_F^* p_{F,*} \Rightarrow \mathrm{id} \tag{16}$$

are isomorphisms when restricted to the subcategory of 2-Cartesian and strongly 3-coCartesian objects.

Since every morphism in the fiber $(i_2 \times_{/I_{j_2}} I_{j_2} \times_{/I_{j_3}} I_{j_3})^{\mathrm{op}}$ is a composition of morphisms of type 2 and 3, this means that it suffices to show that (16) are isomorphisms when restricted to (absolutely) (co)Cartesian objects. This follows from the fact that $(i_2 \times_{/I_{j_2}} I_{j_2} \times_{/I_{j_3}} I_{j_3})^{\mathrm{op}}$ has a final object [13, Lemma 7.21 and Corollary 7.22]. □

Lemma 11.3. *Let the situation be as in 10.3 and let $p' : \mathbb{E} \to \mathbb{S}^{\mathrm{cor,comp}}$ be the morphism of 2-pre-multiderivators defined in Definition 10.13. Let $\alpha : I \to J$ be an opfibration in* Catlf *and $X \hookrightarrow \overline{X}$ an element of $\mathbb{S}^{\mathrm{cor,comp}}(J)$. Then $\alpha^* : \mathbb{E}(J)_{X \hookrightarrow \overline{X}} \to \mathbb{E}(I)_{\alpha^* X \hookrightarrow \alpha^* \overline{X}}$ has a left adjoint $\alpha_!^{(X \hookrightarrow \overline{X})}$.*

Proof. Let $\widetilde{X} \in \mathrm{Fun}({}^{\downarrow\downarrow\uparrow}J, \mathcal{S})$ be the corresponding interior compactification. We have to show that

$$({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{4\text{–cocart},3\text{–cocart}^*,2\text{–cart}}_{\pi_{234}^*(\widetilde{X}^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4\text{–cocart},3\text{–cocart}^*,2\text{–cart}}_{\pi_{234}^*({}^{\downarrow\downarrow\uparrow}\alpha^*\widetilde{X})^{\mathrm{op}}}$$

has a left adjoint. The right hand side category is by Lemma 11.2 equivalent to

$$\mathbb{D}(({}^{\downarrow\uparrow\uparrow\downarrow}J) \times_J I)^{4\text{–cocart},3\text{–cocart}^*,2\text{–cart}}_{\pi_{234}^*(\widetilde{X}^{\mathrm{op}})},$$

hence we have to show that

$$\mathrm{pr}_1^* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{4\text{–cocart},3\text{–cocart}^*,2\text{–cart}}_{\pi_{234}^*(\widetilde{X}^{\mathrm{op}})} \to \mathbb{D}(({}^{\downarrow\uparrow\uparrow\downarrow}J) \times_J I)^{4\text{–cocart},3\text{–cocart}^*,2\text{–cart}}_{\pi_{234}^*(\widetilde{X}^{\mathrm{op}})}$$

has a left adjoint. By assumption the functor

$$\mathrm{pr}_1^* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)_{\pi_{234}^*(\widetilde{X}^{\mathrm{op}})} \to \mathbb{D}(({}^{\downarrow\uparrow\uparrow\downarrow}J) \times_J I)_{\pi_{234}^*(\widetilde{X}^{\mathrm{op}})}$$

has a left adjoint $\mathrm{pr}_{1,!}$. We claim that it preserves the conditions of being 4-coCartesian, 2-Cartesian, and strongly 3-coCartesian, respectively. The statement then follows.

Strongly 3*-coCartesian:* Let $\kappa : j \to j'$

$$\begin{tikzcd}
j = (j_1 \arrow[r] \arrow[d, equal] & j_2 \arrow[r] \arrow[d, equal] & j_3 \arrow[r] & j_4) \arrow[d, equal] \\
j' = (j_1 \arrow[r] & j_2 \arrow[r] & j_3' \arrow[r] \arrow[u] & j_4)
\end{tikzcd}$$

be a morphism of type 3 in ${}^{\downarrow\uparrow\uparrow\downarrow}J$. Denote

$$\iota := \widetilde{X}(\pi_{234}(\kappa)) : \widetilde{X}(\pi_{234}(j)) \to \widetilde{X}(\pi_{234}(j'))$$

the corresponding morphism in $\mathcal{S}$. Note that ι is an open embedding by the properties of induced interior compactifications.
We have to show that the induced map

$$\iota^* j^* \operatorname{pr}_{1,!} \to (j')^* \operatorname{pr}_{1,!}$$

is an isomorphism on strongly 3-coCartesian objects and its inverse induces an isomorphism

$$\iota_!(j')^* \operatorname{pr}_{1,!} \to j^* \operatorname{pr}_{1,!}.$$

Since pr_1 is an opfibration, the first step is the same as to show that the natural morphism

$$\iota^* \operatorname*{hocolim}_{I_{j_1}} e_j^* \to \operatorname*{hocolim}_{I_{j_1}} e_{j'}^*$$

is a isomorphism on 2-Cartesian objects where e_j, resp. $e_{j'}$ denotes the inclusion of the respective fiber. Since ι^* commutes with homotopy colimits, this is to say that

$$\operatorname*{hocolim}_{I_{j_1}} \iota^* e_j^* \to \operatorname*{hocolim}_{I_{j_1}} e_{j'}^*$$

is an isomorphism. However the fibers over j and j' in $({}^{\downarrow\uparrow\uparrow\downarrow}J) \times_J I$ are both isomorphic to I_{j_1} and the natural morphism

$$\iota^* e_j^* \to e_{j'}^*$$

is already an isomorphism on 3-coCartesian objects by definition. Similarly its inverse induces an isomorphism

$$\iota_! e_{j'}^* \to e_j^*$$

and the same reasoning using 1. that $\iota_!$, being a left adjoint, commutes with homotopy colimits, and 2. that it is computed point-wise on constant diagrams (cf. Lemma 6.11, 2.) allows to conclude.
2-*Cartesian:* Let $\kappa : j \to j'$

$$\begin{tikzcd}
j = (j_1 \arrow[r] \arrow[d, equal] & j_2 \arrow[r] & j_3 \arrow[r] \arrow[d, equal] & j_4) \arrow[d, equal] \\
j' = (j_1 \arrow[r] & j_2' \arrow[r] \arrow[u] & j_3 \arrow[r] & j_4)
\end{tikzcd}$$

be a morphism of type 2 in ${}^{\downarrow\uparrow\uparrow\downarrow}J$. Denote

$$\overline{f} := \widetilde{X}(\pi_{234}(\kappa)) : \widetilde{X}(\pi_{234}(j)) \to \widetilde{X}(\pi_{234}(j'))$$

the corresponding morphism in $\mathcal{S}$. Note that $\overline{f}$ is proper by the properties of induced interior compactifications. We have to show that the induced map

$$j^* \operatorname{pr}_{1,!} \to \overline{f}_*(j')^* \operatorname{pr}_{1,!}$$

is an isomorphism on 2-Cartesian objects. Since pr_1 is an opfibration, this is the same as to show that the natural morphism

$$\operatorname*{hocolim}_{I_{j_1}} e_j^* \to \overline{f}_* \operatorname*{hocolim}_{I_{j_1}} e_{j'}^*$$

is a isomorphism on 2-Cartesian objects. Since $\overline{f}_*$ commutes with homotopy colimits by (F3), this is to say that

$$\operatorname*{hocolim}_{I_{j_1}} e_j^* \to \operatorname*{hocolim}_{I_{j_1}} \overline{f}_* e_{j'}^*$$

is an isomorphism. However the fibers over j and j' in $({}^{\downarrow\uparrow\uparrow\downarrow}J) \times_J I$ are both isomorphic to I_{j_1} and the natural morphism

$$e_j^* \to \overline{f}_* e_{j'}^*$$

is already an isomorphism on 2-Cartesian objects by definition.
4-*coCartesian:* Let $\kappa : j \to j'$

$$\begin{tikzcd} j = (j_1 \arrow[r] \arrow[d, equal] & j_2 \arrow[r] \arrow[d, equal] & j_3 \arrow[r] \arrow[d, equal] & j_4) \arrow[d] \\ j' = (j_1 \arrow[r] & j_2 \arrow[r] & j_3 \arrow[r] & j_4') \end{tikzcd}$$

be a morphism of type 4 in ${}^{\downarrow\uparrow\uparrow\downarrow}J$. Denote

$$g := \widetilde{X}(\pi_{234}(\kappa)) : \widetilde{X}(\pi_{234}(j)) \to \widetilde{X}(\pi_{234}(j'))$$

the corresponding morphism in $\mathcal{S}$.
We have to show that the induced map

$$g^* j^* \operatorname{pr}_{1,!} \to (j')^* \operatorname{pr}_{1,!}$$

is an isomorphism on 4-coCartesian objects. This is the same as to show that the natural morphism

$$g^* \operatorname*{hocolim}_{I_{j_1}} e_j^* \to \operatorname*{hocolim}_{I_{j_1}} e_{j'}^*$$

is an isomorphism on 4-coCartesian objects. Since g^* commutes with homotopy colimits, this is to say that

$$\operatorname*{hocolim}_{I_{j_1}} g^* e_j^* \to \operatorname*{hocolim}_{I_{j_1}} e_{j'}^*$$

is an isomorphism. However, the fibers over j and j' in $({}^{\downarrow\uparrow\uparrow\downarrow}J) \times_J I$ are both isomorphic to I_{j_1} and the natural morphism

$$g^* e_j^* \to e_{j'}^*$$

is already an isomorphism on 4-coCartesian objects by definition. □

Remark 11.4. *For an opfibration α the proof shows that we have actually just*

$$\alpha_!^{(X \hookrightarrow \overline{X})} = ({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!^{(\pi_{234}^*(\widetilde{X}^{\mathrm{op}}))}$$

where the right hand side is the relative left Kan extension in $\mathbb{D}$. Indeed by construction (we omit the bases of the rel. Kan extensions for $\mathbb{D}$)

$$\alpha_!^{(X \hookrightarrow \overline{X})} = \operatorname{pr}_{1,!} q_{2,*} q_{1,!}.$$

However, applying this to an object in the category

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*(({}^{\downarrow\downarrow\uparrow}\alpha)^*\widetilde{X})^{\mathrm{op}}}$$

$q_{2,*}$ receives an object which lies in the essential image of q_2^* and we have proven

$$\mathrm{id} \cong q_{2,*} q_2^* .$$

Hence, taking adjoints, we also have

$$q_{2,!} q_2^* \cong \mathrm{id} .$$

Hence on the essential image of q_2^*, we have an isomorphism $q_{2,!} \cong q_{2,*}$.

Example 11.5. *We need to understand precisely how relative left Kan extensions along an inclusion of an objects $i \hookrightarrow I$ looks like. Note that this is not an obfibration, but a relative left Kan extension exists by the arguments in [13, Theorem 4.2]. It can be computed using the homotopy exact square*

$$\begin{tikzcd} i \times_{/I} I \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swarrow\mu"] & i \arrow[d, hook] \\ I \arrow[r, equal] & I \end{tikzcd}$$

(where π is an opfibration) as

$$\pi_! \mathbb{S}(\mu)_\bullet p^*$$

Here $\pi_!$ and $(-)_\bullet$ are the functors associated with the left fibered multiderivator $\mathbb{E} \to \mathbb{S}^{\mathrm{cor,comp}}$ of Main Theorem 12.1 below.
In this case the functors

$$^{\downarrow\uparrow\uparrow\downarrow}(i \times_{/I} I) \xrightarrow{q_1 = (^{\downarrow\uparrow\uparrow\downarrow}\alpha, \pi_{123})} {}^{\downarrow\uparrow\uparrow\downarrow}I \times_{(^{\downarrow\uparrow\uparrow}I)} {}^{\downarrow\uparrow\uparrow}(i \times_{/I} I) \xrightarrow{q_2 = \mathrm{id} \times \pi_1} {}^{\downarrow\uparrow\uparrow\downarrow}I \times_I (i \times_{/I} I).$$

are both isomorphisms of diagrams. In fact all diagrams are isomorphic to $i \times_{/I,\pi_1} {}^{\downarrow\uparrow\uparrow\downarrow}I$. Let $\widetilde{X}$ be an interior compactification of $X : {}^{\downarrow\uparrow}I \to \mathcal{S}$ on I. Consider the obvious morphisms (where a zero in the index of π signifies that the object i appers at the corresponding position):

$$X_i = \pi_{000}^* \widetilde{X} \xleftarrow{g} \pi_{004} \widetilde{X} \xhookrightarrow{\iota} \pi_{034}^* \widetilde{X} \xrightarrow{f} \pi_{234}^* \widetilde{X}.$$

We have then by construction

$$i_!^{(X \hookrightarrow \overline{X})} \cong (^{\downarrow\uparrow\uparrow\downarrow}\pi)_!^{(\pi_{234}^*(\widetilde{X}^{\mathrm{op}}))} (\pi_{234}^* f)_* (\pi_{234}^* \iota)_! (\pi_{234}^* g)^* (^{\downarrow\uparrow\uparrow\downarrow}p)^*.$$

12 Conclusion

Main Theorem 12.1. *Let $\mathcal{S}$ be a category with compactifications, and let $\mathbb{S}^{\mathrm{op}}$ be the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ with its natural multicategory structure encoding the product. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a left and right fibered multiderivator with domain* Dirlf *satisfying axioms (F1–F6) and (F4m–F5m) of 6.1. Assume that $\mathbb{D}$ is infinite (i.e. satisfies (Der1) also for infinite coproducts).*
The morphism of pre-2-multiderivators

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor,comp}}$$

constructed in Definition 10.13 is a left fibered multiderivator with domain Catlf*.*

Proof. The 2-pre-multiderivator $\mathbb{E}$, as defined in Definition 10.13, satisfies axioms (Der1) and (Der2) because $\mathbb{D}$ satisfies them. The first part of axiom (FDer0 left) was shown in Lemma 10.12 and the second part follows from Lemma 10.7. Instead of Axioms (FDer3–4 left) it is sufficient to show Axioms (FDer3–4 left') (cf. [13, Theorem 4.2]). (FDer3 left') is Lemma 11.3, and axiom (FDer4 left') follows from the proof of Lemma 11.3.
(FDer5 left): Since every push-forward functor along a 1-multimorphism in $\mathbb{S}^{\mathrm{cor,comp}}$ is of the form $\overline{f}_* \iota_! g^*(-,\dots,-)$ this follows from (FDer5 left) for $\mathbb{D} \to \mathbb{S}$, the fact that $\iota_!$ commutes with homotopy colimits (because it is the left adjoint of a morphism of pre-derivators), and that $\overline{f}_*$ commutes with homotopy colimits (F3). □

Remark 12.2. *From the left fibered multiderivator of Main Theorem 12.1, we may construct an* equivalent *left fibered multiderivator with domain* Catlf

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor}}.$$

This uses that $\mathbb{S}^{\mathrm{cor,comp}}$ is equivalent as pre-2-multiderivator to $\mathbb{S}^{\mathrm{cor}}$ by Proposition 5.12. The construction is best seen via the alternative description of a left fibered multiderivator as a pseudo-functor of 2-multicategories (cf. [13, Theorem 4.2])

$$\mathrm{Catlf}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor,comp}}) \to \mathcal{CAT}.$$

The equivalence of pre-2-multiderivators induces an equivalence of 2-multicategories $\mathrm{Catlf}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor,comp}}) \cong \mathrm{Catlf}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ *and by composing with a quasi-inverse functor we get a pseudo-functor*

$$\mathrm{Catlf}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}}) \to \mathcal{CAT}$$

which may be strictified (replacing its values by equivalent categories) to get a strict 2-functor. From that one, a strict morphism (i.e. a morphism of 2-pre-multiderivators)

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor}}.$$

may be reconstructed. We will keep the notation $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ for this equivalent left fibered multiderivator.

Proposition 12.3. *Assume $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is symmetric. Then the left fibered multiderivator of Main Theorem 12.1 (cf. 12.2)*

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$$

has a canonical structure as symmetric *left fibered multiderivator such that the restriction to $\mathbb{S}^{\mathrm{op}}$ is equivalent to $\mathbb{D}$ as a symmetric left multiderivator.*

Proof. A pseudo-inverse $\mathbb{S}^{\mathrm{cor}}(I) \to \mathbb{S}^{\mathrm{cor,comp}}(I)$ can be chosen is such a way that a 1-morphism $S_1, \dots, S_n \to T$ is mapped to a 1-morphism of the form $(\Delta_{1,n}, (\overline{S}_1, \dots, \overline{S}_n, \overline{T}), \overline{\xi})$ compatible with the action of the symmetric group. Then Lemma 10.10 gives the desired action. □

Recall the notions of perfectly generated, well generated, and compactly generated fibers for a fibered derivator [12, Definition 4.8].

Lemma 12.4. *If $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ has stable (perfectly generated, well generated, resp. compactly generated) fibers then the fibers of $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ are right derivators with domain (at least)* Posf*, stable (and perfectly generated, well generated, resp. compactly generated).*

Proof. By Lemma 11.2 there is an equivalence (compatible with pull-backs in J):

$$\mathbb{E}(I \times J)_{\mathrm{pr}_1^* X \hookrightarrow \mathrm{pr}_1^* \overline{X}} \cong \mathbb{D}((^{\downarrow\uparrow\uparrow\downarrow}I) \times J)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\mathrm{pr}_1^* \mathrm{pr}_{234}^* \widetilde{X}}.$$

Hence the statement follows if we can show that for $\alpha : J_1 \to J_2$ in Posf, the right Kan extension functor

$$(\mathrm{id} \times \alpha)_* : \mathbb{D}((^{\downarrow\uparrow\uparrow\downarrow}I) \times J_1) \to \mathbb{D}((^{\downarrow\uparrow\uparrow\downarrow}I) \times J_2)$$

respects the conditions of being 4-coCartesian, strongly 3-coCartesian, and 2-Cartesian. This follows because the commutation with homotopy colimits implies that all functors g^*, f_* and $\iota_!$ involved in the definitions of (strongly) (co)Cartesian are *exact* and hence commute also with *homotopy limits of shape* Posf (actually *homotopy finite* is sufficient, cf. [22, Theorem 7.1]). By [12, Lemma 4.7], the properties of being perfectly, compactly or well-generated can be checked over fibers above actual objects of $\mathbb{S}^{\mathrm{cor}}(\cdot) = \mathcal{S}^{\mathrm{cor}}$ where the fibers are actually equivalent to those of $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$. □

Corollary 12.5. *With the assumptions of Main Theorem 12.1, if $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is infinite and has stable and perfectly generated fibers then there is a (unique up to equivalence) left and right fibered (symmetric) multiderivator $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ with domain* Cat*, in other words, a (symmetric)* derivator six-functor-formalism*, whose restriction to* Catlf *is equivalent to the (symmetric) left fibered multiderivator of Main theorem 12.1 (cf. also 12.2).*

Proof. [12, Theorem 4.10] and the previous Lemma 12.4 show that the already constructed $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ with domain Invlf satisfies (FDer0 right) as well. Therefore by [14, Corollary 1.2], $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ extends to a (symmetric) left fibered multiderivator on all of Cat because $\mathbb{S}^{\mathrm{cor}}$, being representable, clearly extends to Cat. (The reader may check that the techniques of [loc. cit.] go through for the case of pre-2-multiderivators instead of pre-multiderivators.) The same proof as the one of [13, Theorem 7.2] starting from Main Theorem 12.1 thus shows that $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ is also a right fibered (symmetric) multiderivator. □

13 The construction of proper derivator six-functor-formalisms

13.1. The remaining sections are concerned with the construction of a *proper* derivator six-functor-formalism which also encodes a natural morphism $f_! \to f_*$ which is an isomorphism for proper morphisms f. In the classical (non derivator) case it might be included (as explained in [15, Section 8]) by enlarging the 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ to $\mathcal{S}^{\mathrm{cor},0}$. In the latter 2-multicategory also non-invertible 2-morphisms formed by arbitrary proper morphisms are included[15]. On the derivator side this opens the possibility to include lax, resp. oplax, morphisms of diagrams of correspondences which are important, for instance, to encode the classical exact triangles related to a pair of complementary open and closed embeddings, cf. [13, Section 9].
We need some rather technical facts about the existence of Cartesian and coCartesian projectors whose discussion we postpone to Appendix A.

[15]This implies that the correspondences $(S = S \to T)$ and $(T \leftarrow S = S)$ become formally adjoint in the 2-category $\mathcal{S}^{\mathrm{cor},0}$ for a proper morphism $f : S \to T$. Hence $f_!$ becomes right adjoint to f^*, whence a canonical isomorphism $f_! \cong f_*$.

13.2. We illustrate the push-forward and pull-back along (op)lax morphisms in the simplest case based on the underlying diagrams. A 2-commutative square in $\mathcal{S}^{\mathrm{cor},0}$

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow^{\mu}"] & T \arrow[d] \\ S' \arrow[r] & T' \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow_{\mu}"] & T \arrow[d] \\ S' \arrow[r] & T' \end{tikzcd}$$

in which μ is not invertible, that is, an oplax morphism (l.h.s.), or a lax morphism (r.h.s.), from the top diagram of shape Δ_1 to the bottom diagram of shape Δ_1, can be encoded (as in 5.4) by a commutative diagram

$$\begin{tikzcd} S & A \arrow[l] \arrow[r] \arrow[dr, phantom, "1"] & T \\ B \arrow[u] \arrow[d] \arrow[dr, phantom, "2"] & C \arrow[l] \arrow[r] \arrow[u] \arrow[d] & D \arrow[u] \arrow[d] \\ S' \arrow[r] & A' \arrow[r] & T' \end{tikzcd}$$

in which, in the oplax case, the square 1 is Cartesian and the square 2 is weakly Cartesian or, in the lax case, the square 1 is weakly Cartesian and the square 2 is Cartesian.
Consider a weakly Cartesian square (with the corresponding Cartesian square inserted):

$$\begin{tikzcd} S' \arrow[dr, "H"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "G"'] & & \\ & \square \arrow[r, "F'"] \arrow[d, "G'"] & T' \arrow[d, "g"] \\ & S \arrow[r, "f"] & T \end{tikzcd}$$

A proper six-functor-formalism allows for the following two operations:

1. From a morphism

$$G^*\mathcal{E} \to \mathcal{F}$$

applying $F_!$ one gets a morphism

$$(F'H)_!(G'H)^*\mathcal{E} \to F_!\mathcal{F}.$$

Composing it with the morphism $(F'H)_!(G'H)^* \cong F'_! H_! H^*(G')^* \cong F'_! H_* H^*(G')^* \leftarrow F'_!(G')^* \cong g^* f_!$ (using $H'_! \cong H_*$ and the unit for the adjunction H^*, H_*), we get

$$g^*\mathcal{E}' \to \mathcal{F}'$$

for $\mathcal{E}' := f_!\mathcal{E}$ and $\mathcal{F}' := F_!\mathcal{F}$.

2. From a morphism

$$\mathcal{E} \to F^!\mathcal{F}$$

applying G_* one gets a morphism

$$G_*\mathcal{E} \to (G'H)_*(F'H)^!\mathcal{F}.$$

Composing it with the morphism $(G'H)_*(F'H)^! \cong G'_*H_*H^!(F')^! \cong G'_*H_!H^!(F')^! \to G'_*(F')^! \cong f^!g_*$ (using $H_! \cong H_*$ and the counit for the adjunction $H_!$, $H^!$), we get

$$\mathcal{E}' \to f^!\mathcal{F}'$$

for $\mathcal{E}' := G_*\mathcal{E}$ and $\mathcal{F}' := g_*\mathcal{F}$.

In other words, the operation 1. allows for the construction of a push-forward along the oplax morphism, and the operation 2. allows for the construction of a pull-back along the lax morphism, both being computed point-wise. In our approach it is essential to construct the left fibered version (with push-forwards) first. Hence the construction of the lax pull-back has to be a bit indirect. It turns out that the lax pull-back $\xi^\bullet$ *does have* a left adjoint $\xi_\bullet$ which is, however, not computed point-wise anymore (similar to the existence of internal Homs of diagrams which are also not computed point-wise). It is this left adjoint that will be constructed first. The right adjoint (a posteriori constructed via Brown representability) is then indeed computed point-wise as expected which, however, has to be proven by establishing the adjoint formula

$$\alpha_!(\alpha^*\xi)_\bullet \cong \xi_\bullet\alpha_!$$

involving the left adjoint. In the multi-case, the lax pull-back and oplax push-forward exist for n-ary morphisms as well. The oplax push-forward (involving construction 1. above) involves essentially only a 1-ary construction, whereas the lax pull-back involves a multi-version of construction 2. above.
This section is concerned with the derivator analogue of these constructions for arbitrary diagrams in Catlf.

Recall from Proposition 5.12 the (op)lax 2-pre-multiderivators $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I)$, resp. $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$. We proceed as in section 10 and begin with a couple of Lemmas that will be used to construct a 1-opfibration and 2-opfibration $\mathbb{E}'(I) \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I)$, resp. $\mathbb{E}''(I) \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$.

13.3. Recall the notation from 10.5. In the (op)lax case we have the following changes:
Let I be in Catlf, let Δ_T be a tree, and let

$$\xi : \Delta_T \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor},0}) \quad (\text{resp. } \mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0}))$$

be a functor of 2-multicategories. We can associate with them in each case a functor of opmulticategories

$$X : {}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$$

which is only *weakly admissible* (8.5) in general. By the construction in 8.6 it has an interior compactification

$$\widetilde{X} : {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$$

to which we may apply the results of sections 7–8.
Let o be a multimorphism in ${}^{\downarrow\downarrow\uparrow}\Delta_T$. If o is of type 3 (resp. type 2, resp. type 1) again denote by

$$\begin{array}{rccl} \widetilde{g}: & \widetilde{A} & \to & \widetilde{S}_1, \dots, \widetilde{S}_n \\ \widetilde{\iota}: & \widetilde{A} & \to & \widetilde{A}' \\ \widetilde{f}: & \widetilde{A}' & \to & \widetilde{T} \end{array}$$

their images in

$$\mathrm{Fun}({}^{\downarrow\downarrow\uparrow}I, \mathcal{S}).$$

Example 10.6 remains valid, however, with $g = (g_1, \dots, g_n)$ only weakly type 1 admissible as multimorphism, resp. with f only weakly type 2 admissible.

Lemma 13.4. *With the notation as in 13.3.*

1. *If ξ is a functor $\Delta_T \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor},0})$, and o is any morphism of type 3, the multivalued functor*
$$(\pi_{234}^*\widetilde{g})^* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{S}_1^{\mathrm{op}}} \times \cdots \times \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{S}_n^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{A}^{\mathrm{op}}}.$$
is computed point-wise (in ${}^{\downarrow\uparrow\uparrow\downarrow}I$) and on well-supported objects preserves the condition of being 4-coCartesian, well-supported, and 2-Cartesian.

   If ξ is a functor $\Delta_T \to \mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0})$, and any morphism o of type 3, the functor $(\pi_{234}^*\widetilde{g})^*$, on well-supported objects, preserve the condition of being well-supported, and 4-coCartesian, but not necessarily the condition of being 2-Cartesian.

2. *For any morphism o of type 2, in any case, the functor*
$$(\pi_{234}^*\widetilde{\iota})_! : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{A}^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*(\widetilde{A}')^{\mathrm{op}}}$$
i.e. the left adjoint of $(\pi_{234}^*\widetilde{\iota})^*$, which exists by (F1), is computed point-wise (in ${}^{\downarrow\uparrow\uparrow\downarrow}I$) on 4-coCartesian and well-supported objects, and on such it preserves the conditions of being 4-coCartesian, well-supported, and 2-Cartesian.

3. *If ξ is a functor $\Delta_T \to \mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0})$, and o is any morphism of type 1, the functor*
$$(\pi_{234}^*\widetilde{f})_* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*(\widetilde{A}')^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{T}^{\mathrm{op}}}.$$
is computed point-wise (in ${}^{\downarrow\uparrow\uparrow\downarrow}I$) (cf. Definition 6.10) and on well-supported objects it preserves the condition of being 4-coCartesian, well-supported, and 2-Cartesian.

   If ξ is a functor $\Delta_T \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor},0})$, the functor $(\pi_{234}^*\widetilde{f})_*$, on well-supported objects, preserves the condition of being well-supported, and 2-Cartesian, but not necessarily the condition of 4-coCartesian.

Proof. 1. As in Lemma 10.7, 1. In the lax case preservation of coCartesianity is still clear, and the preservation of the condition of being well-supported still follows from Proposition 8.12, 2. In the oplax case, the relevant top squares in the second part of Lemma 8.9 are still Cartesian.
2. As in Lemma 10.7, 2. — note that Proposition 8.12, 2. holds true for weakly admissible diagrams.
3. As in Lemma 10.7, 3. In the lax case, the relevant top squares in the second part of Lemma 7.5 are still Cartesian. In the oplax case preservation of Cartesianity is clear as well and the preservation of strong coCartesianity follows still from Proposition 8.12, 3. □

13.5. Over the morphism $(\pi_{234}^*\widetilde{f})^{\mathrm{op}}$, resp. $(\pi_{234}^*\widetilde{g})^{\mathrm{op}}$, coming from a $\xi : \Delta_T \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor},0})$ (resp. $\mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0})$), there will be in general no (co)Cartesian morphism[16] in

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}.$$

However, we can use the (co)Cartesian projectors from Propositions A.6 and A.12, and say that $\mathcal{E} \to \mathcal{F}$ is

[16] that is, (co)Cartesian w.r.t. $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I) \to \mathbb{S}({}^{\downarrow\uparrow\uparrow\downarrow}I)$.

1. **oplax Cartesian**, if it induces an isomorphism

$$\mathcal{E} \to \Box_* (\pi_{234}^* \widetilde{f})_* \mathcal{F}$$

   in the fiber,

2. **lax coCartesian**, if it induces an isomorphism

$$\Box_! (\pi_{234}^* \widetilde{g})^* \mathcal{E} \to \mathcal{F}$$

   in the fiber.

Lemma 13.6. *Compositions of lax coCartesian morphisms are lax coCartesian and of oplax Cartesian morphisms are oplax Cartesian, if the sources (resp. destinations) are 4-coCartesian, well-supported and 2-Cartesian. (Of course, we only consider morphisms over (multi)morphisms of the form $\pi_{234}^* \widetilde{g}$, resp. $\pi_{234}^* \widetilde{f}$, considered above.)*

Proof. For lax coCartesian, we have to show:

$$\Box_! (\pi_{234}^* \widetilde{g}_1)^* \Box_! (\pi_{234}^* \widetilde{g}_2)^* \cong \Box_! (\pi_{234}^* \widetilde{g}_2 \circ \widetilde{g}_1)^*.$$

For a morphism g denote by g_*^{cart} be the restriction of g_* to the full subcategory of 2-Cartesian objects (note that any g_* preserves the condition of being 2-Cartesian). The left adjoint of g_*^{cart} is the functor $\Box_! g^*$ restricted to the full subcategory of 2-Cartesian objects. The assertions follows therefore as adjoint formula to $(\pi_{234}^* \widetilde{g}_1)_*^{\mathrm{cart}} \circ (\pi_{234}^* \widetilde{g}_2)_*^{\mathrm{cart}} \cong (\pi_{234}^* \widetilde{g}_1 \circ \widetilde{g}_2)_*^{\mathrm{cart}}$.
For lax Cartesian we have to show:

$$\Box_* (\pi_{234}^* \widetilde{f}_1)_* \Box_* (\pi_{234}^* \widetilde{f}_2)_* \cong \Box_* (\pi_{234}^* \widetilde{f}_1 \circ \widetilde{f}_2)_*.$$

On the full subcategories of 2-Cartesian, well-supported and 4-coCartesian objects, this formula can be shown point-wise on objects of the form ${}^{\downarrow\uparrow\uparrow\downarrow}i$. However, by Proposition A.12, $\Box_*$ does nothing over objects of the form ${}^{\downarrow\uparrow\uparrow\downarrow}i$. $\square$

Proposition 13.7 ((Op)lax version of Proposition 10.8)**.** *Let Δ_T be a tree, $\xi : \Delta_T \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor},0})$ (resp. $\mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0})$) be a functor, and let $\widetilde{X} : {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ be any interior compactification of the corresponding $X : {}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$. Consider a diagram of the form*

$$\begin{tikzcd} w \arrow[r, "d"] \arrow[d, "c"'] & z_1, \dots, z_n \arrow[d, "a_1,\dots,a_n"] \\ y \arrow[r, "b"'] & x_1, \dots, x_n \end{tikzcd}$$

in ${}^{\downarrow\downarrow\uparrow}\Delta_T$. Let

$$\begin{tikzcd} \mathcal{H} & \mathcal{E}_1, \dots, \mathcal{E}_n \arrow[l, "\delta"'] \\ \mathcal{G} \arrow[u, "\gamma"] & \mathcal{F}_1, \dots, \mathcal{F}_n \arrow[l, "\beta"] \arrow[u, "\alpha_1,\dots,\alpha_n"'] \end{tikzcd}$$

be a diagram in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)$ above π_{234}^* of $\widetilde{X}^{\mathrm{op}}$ applied to the top square*.*
Then the following holds:

1. *Let the a_i and c be of type 1 and let b and d be multimorphisms of type 3.*

   Assume that the $\mathcal{E}_i$ are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   In the oplax case: If all α_i are oplax Cartesian ($\mathcal{F}_i \cong \Box_* \widetilde{X}(a_i)_* \mathcal{E}_i$) and δ is coCartesian ($\widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is coCartesian if and only if γ is oplax Cartesian or, in other words, the natural exchange

$$\widetilde{X}(b)^*(\Box_* \widetilde{X}(a_1)_* -, \dots, \Box_* \widetilde{X}(a_n)_* -) \to \Box_* \widetilde{X}(c)_* \widetilde{X}(d)^*(-, \dots, -) \tag{17}$$

   is an isomorphism on n-tupels of objects in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   If this is the case then all other objects in the diagram are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   In the lax case: If all α_i are Cartesian ($\mathcal{F}_i \cong \widetilde{X}(a_i)_* \mathcal{E}_i$) and δ is lax coCartesian ($\Box_! \widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is lax coCartesian if and only if γ is Cartesian or, in other words, the natural exchange

$$\Box_! \widetilde{X}(b)^*(\widetilde{X}(a_1)_* -, \dots, \widetilde{X}(a_n)_* -) \to \widetilde{X}(c)_* \Box_! \widetilde{X}(d)^*(-, \dots, -) \tag{18}$$

   is an isomorphism on n-tupels of well-supported objects in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   If this is the case then all other objects in the diagram are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

2. *Let the a_i and c be of type 2 and let b and d be multimorphisms of type 3.*

   Assume that the $\mathcal{E}_i$ are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   In the lax case: If all α_i are strongly coCartesian ($\widetilde{X}(a_i)^* \mathcal{F}_i \cong \mathcal{E}_i$ inducing $\widetilde{X}(a_i)_! \mathcal{E}_i \cong \mathcal{F}_i$) and δ is lax coCartesian ($\Box_! \widetilde{X}(d)^*(\mathcal{E}_1, \dots, \mathcal{E}_n) \cong \mathcal{H}$) then β is lax coCartesian if and only if γ is strongly coCartesian or, in other words, the natural exchange

$$\widetilde{X}(c)_! \Box_! \widetilde{X}(d)^*(-, \dots, -) \to \Box_! \widetilde{X}(b)^*(\widetilde{X}(a_1)_! -, \dots, \widetilde{X}(a_n)_! -) \tag{19}$$

   is an isomorphism on n-tupels of objects in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

   If this is the case then all other objects in the diagram are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

3. *Let a and c be of type 2 and b and d of type 1 (now necessarily 1-ary).*

   In the oplax case: If γ is strongly coCartesian ($\widetilde{X}(c)^* \mathcal{G} \cong \mathcal{H}$ inducing $\widetilde{X}(c)_! \mathcal{H} \cong \mathcal{G}$) and δ is oplax Cartesian ($\mathcal{E} \cong \Box_* \widetilde{X}(d)_* \mathcal{H}$) then β is oplax Cartesian if and only if α is strongly coCartesian, or in other words, the natural exchange

$$\widetilde{X}(a)_! \Box_* \widetilde{X}(d)_* \to \Box_* \widetilde{X}(b)_* \widetilde{X}(c)_! \tag{20}$$

   is an isomorphism.

   If this is the case, and $\mathcal{H}$ is in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$, then all other objects in the diagram are in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}$.

Proof. Oplax cases: It suffices to see that (17) and (20) are isomorphisms on objects of the form ${}^{\downarrow\uparrow\uparrow\downarrow}i$, where i is an object of I, because of the (strong) (co)Cartesianity conditions. However, by Proposition A.12, $\Box_*$ does nothing over objects of the form ${}^{\downarrow\uparrow\uparrow\downarrow}i$.
Lax cases: By Lemmas A.7–A.8, $\Box_!$ commutes with $\widetilde{X}(c)_*$ and $\widetilde{X}(c)_!$. Therefore (18) and (19) are isomorphisms — note that $\widetilde{X}(c)_!$ is also computed point-wise, if the argument is not assumed to be 2-Cartesian, cf. Lemma 10.7, 2. $\Box$

Fundamental Lemma 13.8 ((Op)lax version of Fundamental Lemma 10.9)**.** *With the notation as in 10.3.*

1. *Let Δ_T be a tree, and let $\xi : \Delta_T \to \mathrm{Fun}^{\mathrm{oplax}}(I, \mathcal{S}^{\mathrm{cor},0})$ (resp. $\mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor},0})$) be a functor of multicategories. Let $\widetilde{X} : {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ be an interior compactification (8.6) of the corresponding weakly admissible $X : {}^{\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$.*

   Let $(\mathcal{E}_o)_{o\in\Delta_T}$ be a collection of objects with $\mathcal{E}_o \in \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{X}_o^{\mathrm{op}}}$, where $\widetilde{X}_o$ is the value of $\widetilde{X}$ at ${}^{\downarrow\downarrow\uparrow}o$. Then the category

$$M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \coloneqq \{\mathcal{F} \in \mathrm{Fun}({}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T, \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I))^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{oplax-cart}}_{\pi_{234}^*\widetilde{X}^{\mathrm{op}}}, ({}^{\downarrow\uparrow\uparrow\downarrow}o)^*\mathcal{F} \cong \mathcal{E}_o\}$$

   resp.

$$M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \coloneqq \{\mathcal{F} \in \mathrm{Fun}({}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T, \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I))^{4-\mathrm{lax-cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{X}^{\mathrm{op}}}, ({}^{\downarrow\uparrow\uparrow\downarrow}o)^*\mathcal{F} \cong \mathcal{E}_o\}$$

   is equivalent to a set.

   The superscripts are concerned here with the bifibration $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I) \to \mathbb{S}({}^{\downarrow\uparrow\uparrow\downarrow}I)$ only. The three superscripts here mean that for the functor

$${}^{\downarrow\uparrow\uparrow\downarrow}\Delta_T \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)$$

   multimorphisms of type 4 are mapped to (lax) coCartesian multimorphisms and (neccessarily 1-ary) morphisms of type 3 are mapped to strongly coCartesian morphisms and (neccessarily 1-ary) morphisms of type 2 are mapped to (oplax) Cartesian morphisms.

2. *For $\Delta_T = \Delta_{T_1} \circ_i \Delta_{T_2}$, where i is a source object of Δ_{T_1} (which we identify with the final object of Δ_{T_2}), the square*

$$\begin{tikzcd} M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \arrow[r] \arrow[d] & M_{\widetilde{X}_1}((\mathcal{E}_o)_{o\in\Delta_{T_1}}) \arrow[d] \\ M_{\widetilde{X}_2}((\mathcal{E}_o)_{o\in\Delta_{T_2}}) \arrow[r] & \{ \text{ Iso.class of } \mathcal{E}_i \ \} \end{tikzcd}$$

   is 2-Cartesian. Hence if we consider the M as sets, we have

$$M_{\widetilde{X}}((\mathcal{E}_o)_{o\in\Delta_T}) \cong M_{\widetilde{X}_1}((\mathcal{E}_o)_{o\in\Delta_{T_1}}) \times M_{\widetilde{X}_2}((\mathcal{E}_o)_{o\in\Delta_{T_2}}).$$

3. *(oplax case) For $\Delta_T = \Delta_{1,n}$, we have canonically an isomorphism of sets*

$$M_{\widetilde{X}}(\mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{E}_{n+1}) \cong \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{T}^{\mathrm{op}}}}(\Box_* \widetilde{f}_* \widetilde{\iota}_! \widetilde{g}^*(\mathcal{E}_1, \dots, \mathcal{E}_n); \mathcal{E}_{n+1})$$

   for any choice of pull-back $\widetilde{g}^*$, push-forward $\widetilde{f}_*$, and adjoint $\widetilde{\iota}_!$ to the pull-back $\widetilde{\iota}^*$. Here $\widetilde{T}$ is the restriction of $\widetilde{X}$ to ${}^{\downarrow\downarrow\uparrow}(n+1)$, and $\widetilde{g}$, $\widetilde{\iota}$, and $\widetilde{f}$ are the components of $\widetilde{X}$ as in 10.6. Here $\Box_*$ is the right coCartesian projector of Proposition A.12.

   An object $\mathcal{F}$ on the left hand side is mapped to an isomorphism if and only if it is also Cartesian (or equivalently coCartesian) w.r.t. the projection

$$\pi_{234} \times \mathrm{id} : {}^{\downarrow\uparrow\uparrow\downarrow}(\Delta_{1,n} \times I) \to {}^{\uparrow\uparrow\downarrow}\Delta_{1,n} \times {}^{\downarrow\uparrow\uparrow\downarrow}I.$$

3. *(lax case) For $\Delta_T = \Delta_{1,n}$, we have canonically an isomorphism of sets*

$$M_{\widetilde{X}}(\mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{E}_{n+1}) \cong \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^* \widetilde{T}^{\mathrm{op}}}}(\widetilde{f}_* \widetilde{\iota}_! \,\square_!\, \widetilde{g}^*(\mathcal{E}_1, \dots, \mathcal{E}_n); \mathcal{E}_{n+1})$$

for any choice of pull-back $\widetilde{g}^*$, push-forward $\widetilde{f}_*$, and adjoint $\widetilde{\iota}_!$ to the pull-back $\widetilde{\iota}^*$. Here $\widetilde{T}$ is the restriction of $\widetilde{X}$ to ${}^{\downarrow\downarrow\uparrow}(n+1)$, and $\widetilde{g}$, $\widetilde{\iota}$, and $\widetilde{f}$ are the components of $\widetilde{X}$ as in 10.6. Here $\square_!$ is the left Cartesian projector of Proposition A.6.

An object $\mathcal{F}$ on the left hand side is mapped to an isomorphism if and only if it is also Cartesian (or equivalently coCartesian) w.r.t. the projection

$$\pi_{234} \times \mathrm{id} : {}^{\downarrow\uparrow\uparrow\downarrow}(\Delta_{1,n} \times I) \to {}^{\uparrow\uparrow\downarrow}\Delta_{1,n} \times {}^{\downarrow\uparrow\uparrow\downarrow}I.$$

4. *For each interior compactification $\widetilde{f} : \Delta_1 \times {}^{\downarrow\downarrow\uparrow}(\Delta_T \times I) \to \mathcal{S}$ which comes from a refinement of exterior compactifications $\overline{f} : \overline{X}_1 \to \overline{X}_2$ of S, the functors $(\pi_{234}^* \widetilde{f})^*, (\pi_{234}^* \widetilde{f})_*$ are mutually inverse bijections between*

$$M_{e_0^* \widetilde{f}}((\mathcal{E}_o)_{o \in \Delta_T}) \underset{(\pi_{234}^* \widetilde{f})^*}{\overset{(\pi_{234} \widetilde{f})_*}{\rightleftarrows}} M_{e_1^* \widetilde{f}}((\mathcal{E}_o)_{o \in \Delta_T})$$

Proof. As the proof of Fundamental Lemma 10.9, using Proposition 13.7 instead of Proposition 10.8. Note that by Lemma 13.6 the composition of oplax Cartesian morphisms is oplax Cartesian and the composition of lax coCartesian morphisms is lax coCartesian. □

From now on, we will again use $M_{\widetilde{X}}((\mathcal{E}_o)_o)$ for the corresponding *set*.

Definition 13.9. *With the notation as in 10.3, let I be a diagram in* Catlf*. We define 2-multicategories together with a strict functor between 2-multicategories*

$$\begin{array}{ccc} \mathbb{E}'(I) & \to & \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I) \\ \mathbb{E}''(I) & \to & \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I) \end{array}$$

as follows:

1. *Objects in $\mathbb{E}'(I)$, resp. $\mathbb{E}''(I)$, are pairs $(S \hookrightarrow \overline{S}, \mathcal{E})$ of an exterior compactification $X \hookrightarrow \overline{X}$ in $\mathbb{S}^{\mathrm{cor},\mathrm{comp}}(I)$* [17] *and an object*

$$\mathcal{E} \in \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^* \widetilde{X}^{\mathrm{op}}}$$

where $\widetilde{X}$ is the induced interior compactification.

2. *1-morphisms from $(X_1 \hookrightarrow \overline{X}_1, \mathcal{E}_1), \dots, (X_n \hookrightarrow \overline{X}_n, \mathcal{E}_n)$ to $(Y \hookrightarrow \overline{Y}, \mathcal{F})$ are pairs $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m, x)$ of a 1-morphism $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$ in $\mathbb{S}^{\mathrm{cor},\mathrm{comp},\mathrm{oplax}}(I)$ (resp. $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$), with the given sources and destination, and an element $x \in M_{\widetilde{\Xi}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$. Recall that for the 1-morphism $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m)$, we fixed a compactification $\Xi \hookrightarrow \overline{\Xi}$ of the composition in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}(I)$, resp. $\mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}(I)$. (The compactification $\Xi \hookrightarrow \overline{\Xi}$ is the pullback along $\Delta_{1,n} \to \Delta_T$ of the compactification constructed in Lemma 5.11). The $\widetilde{\Xi}$ appearing is the induced interior compactification.*

[17] or equivalently in $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I)$ or $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$

3. *Any 2-morphism* $(\Delta_T, (\overline{S}_o)_o, (\overline{\xi}_m)_m) \Rightarrow (\Delta_{T'}, (\overline{S}'_o)_o, (\overline{\xi}'_m)_m)$ *in the 2-multicategory* $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I)$*, resp.* $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$*, which can be represented by a morphism of compactifications* $\Xi \to \Xi'$*, gives a morphism* $M_{\Xi}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \to M_{\Xi'}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F})$ *and we declare a 2-morphism from elements in source to the corresponding element in the destination. An arbitrary 2-morphism in* $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I)$*, resp.* $\mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$*, is a morphism in the localization. Since the inverted morphisms are mapped to isomorphisms (cf. Lemma 10.9, 4.) and the association is functorial, the construction uniquely extends to the localization.*

4. *The composition of 1-morphisms is defined precisely as in the plain case.*

Lemma 13.10. *The functors*

$$\begin{array}{rcl} \mathbb{E}'(I) & \to & \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}(I) \\ \mathbb{E}''(I) & \to & \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I) \end{array}$$

constructed in Definition 13.9 are 1-opfibrations and 2-opfibrations with 1-categorical fibers.

Proof. The same as in the plain case. □

Definition 13.11. *With the notation as in 10.3, we now construct strict morphisms of (op)lax pre-2-multiderivators of domain* Catlf

$$\begin{array}{rcl} \mathbb{E}' & \to & \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}} \\ \mathbb{E}'' & \to & \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}} \end{array}$$

precisely as in the plain case.
In the lax case observe that, since $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*$ *preserves 2-Cartesian, we get an exchange morphism*

$$\Box_!({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^* \to ({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*\Box_!$$

and therefore

$$\Box_! \, (\alpha^* g)^*(({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^* -, \dots, ({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^* -) \to ({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^* \, \Box_! \, (\pi_{234}^* \widetilde{g})^*(-, \dots, -) \tag{21}$$

However, this is not an isomorphism, hence (as expected, cf. 13.2) the second part of (FDer0 left) will not hold for $\mathbb{E}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}$ *and arbitrary* $\alpha : I \to J$*!*

The next goal is to establish that the morphism of pre-2-multiderivators

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}$$

is an oplax left fibered multiderivator with domain Catlf, and in the stable case, using Brown representability, that

$$\mathbb{E}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}.$$

is an lax right fibered multiderivator with domain Catlf. Both can be extended to all of Cat.

Proposition 13.12. *Let* $\Delta_T = \Delta_{1,n}$ *(cf. also Example 10.6). The functor* $\Box_!(\pi_{234}^*\widetilde{g})^*$ *from 13.5 is a functor* $\mathbb{E}(J)_{S_1 \hookrightarrow \overline{S}_1} \times \cdots \times \mathbb{E}(J)_{S_n \hookrightarrow \overline{S}_n} \to \mathbb{E}(J)_{A \hookrightarrow \overline{A}}$*. It commutes with relative left Kan extensions for* $\alpha : I \to J$ *an opfibration in the following sense: For all* i*, there are natural isomorphisms*

$$\alpha_!^{(A \hookrightarrow \overline{A})} \, \Box_! \, (\alpha^* \pi_{234}^* \widetilde{g})^*(\alpha^* -, \dots, -, \dots, \alpha^* -) \to \Box_!(\pi_{234}\widetilde{g})^*(-, \dots, \alpha_!^{(S_i \hookrightarrow \overline{S}_i)}, \dots, -).$$

If $\widetilde{g}$ *is 1-ary then the above holds for* $\alpha : i \hookrightarrow J$ *the inclusion of an object.*

If $\widetilde{g}$ is 1-ary and the functors $\Box_!(\pi_{234}^*\widetilde{g})^*$ have right adjoints, then Axiom (Der2) implies that the functor commutes with arbitrary $\alpha_!$ in the sense above. This can probably be shown directly, but we will not need to do this, because we are interested in the right adjoints only, anyway.

Proof. Using the morphism (21) one defines

$$\begin{array}{rl} \alpha_!^{(A\hookrightarrow\overline{A})}\,\Box_!\,(\alpha^*\pi_{234}^*\widetilde{g})^*(\alpha^*-,\dots,-,\dots,\alpha^*-) & \to\ \alpha_!^{(T\hookrightarrow\overline{T})}\,\Box_!\,(\alpha^*\pi_{234}^*\widetilde{g})^*(\alpha^*-,\dots,\alpha^*\alpha_!^{(S_i\hookrightarrow\overline{S}_i)}-,\dots,\alpha^*-) \\ & \to\ \alpha_!^{(A\hookrightarrow\overline{A})}\alpha^*\,\Box_!\,(\pi_{234}^*\widetilde{g})^*(-,\dots,\alpha_!^{(S_i\hookrightarrow\overline{S}_i)}-,\dots,-) \\ & \to\ \Box_!(\pi_{234}^*\widetilde{g})^*(-,\dots,\alpha_!^{(S_i\hookrightarrow\overline{S}_i)}-,\dots,-). \end{array}$$

Hence we have natural transformations even for any α. We first show that it is an isomorphism if α is an opfibration. We have the following commutative diagram

$$\begin{tikzcd} \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{2-\mathrm{cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \arrow[r, hook] \arrow[d, "({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*|_{2-\mathrm{cart}}"] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \arrow[d, "({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*"] \\ \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{2-\mathrm{cart}}_{\pi_{234}^*\alpha^*\widetilde{S}^{\mathrm{op}}} \arrow[r, hook] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \end{tikzcd}$$

Obviously a left adjoint of the restriction $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*|_{2-\mathrm{cart}}$ of $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*$ to the 2-Cartesian subcategory is given by $\Box_!({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!$. We have therefore as adjoint to the above commutative diagram:

$$\Box_!({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!\Box_! \cong \Box_!({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!.$$

If we evaluate this natural transformation on the category

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{4-\mathrm{cocart},3-\mathrm{cocart}^*}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \tag{22}$$

then, since $\Box_!$ preserves the conditions of being strongly 3-coCartesian and 4-coCartesian (cf. Proposition A.6), the $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!$ on the left hand side receives an object in

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{4-\mathrm{cocart},3-\mathrm{cocart}^*,2-\mathrm{cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \tag{23}$$

where it is simply $\alpha_!^{(A\hookrightarrow\overline{A})}$ (cf. Remark 11.4), hence the leftmost $\Box_!$ on the left hand side is superfluous. We arrive at

$$\alpha_!^{(A\hookrightarrow\overline{A})}\Box_! \cong \Box_!({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!$$

(still on the subcategory (22)) and therefore

$$\alpha_!^{(A\hookrightarrow\overline{A})}\,\Box_!\,(\alpha^*\pi_{234}^*\widetilde{g})^*(\alpha^*-,\dots,-,\dots,\alpha^*-) \cong \Box_!({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_!(\pi_{234}^*\widetilde{g})^*(\alpha^*-,\dots,-,\dots,\alpha^*-)$$

on the subcategory (23), where the α^* denotes $\mathbb{E}(\alpha)$, i.e. $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*$, hence by (FDer5 left) for $\mathbb{D}$:

$$\alpha_!^{(A\hookrightarrow\overline{A})}\,\Box_!\,(\alpha^*\pi_{234}^*\widetilde{g})^*(\alpha^*-,\dots,-,\dots,\alpha^*-) \cong \Box_!(\pi_{234}^*\widetilde{g})^*(-,\dots,\alpha_!^{(S_i\hookrightarrow\overline{S}_i)}-,\dots,-).$$

using that $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)_! \cong \alpha_!^{(S_i\hookrightarrow\overline{S}_i)}$ on the subcategory (23).
Now let $\widetilde{g}$ be 1-ary and $i\in J$ be an object. We have to show that

$$i_!^{(A\hookrightarrow\overline{A})}\widetilde{g}_i^* \to \Box_!(\pi_{234}^*\widetilde{g})^*i_!^{(S\hookrightarrow\overline{S})} \tag{24}$$

is an isomorphism. Here $\widetilde{g}_i$ is $\widetilde{g}$ evaluated at ${}^{\downarrow\downarrow\uparrow}i$. This can be checked point-wise at $j \in J$:

$$j^* i_!^{(A\hookrightarrow\overline{A})} \widetilde{g}_i^* \to ({}^{\downarrow\uparrow\uparrow\downarrow}j)^* \pi_{4567,!} f_* \pi_{1267}^* (\pi_{234}^* \widetilde{g})^* i_!^{(S\hookrightarrow\overline{S})}$$

Using that π_{4567} is an opfibration with fiber (over ${}^{\downarrow\uparrow\uparrow\downarrow}j$) equal to ${}^{\downarrow\uparrow\uparrow}J \times_{/J} j \hookrightarrow {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}J$ (subcategory of elements of the form $j_1 \to j_2 \to j_3 \to j = j = j = j$), we get

$$j^* i_!^{(A\hookrightarrow\overline{A})} \widetilde{g}_i^* \to \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}J\times_{/J}j} f_* \pi_{1267}^* (\pi_{234}^* \widetilde{g})^* i_!^{(S\hookrightarrow\overline{S})}$$

Then, inserting the precise calculation from Example 11.5, we get for the right hand side

$$\operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}J\times_{/J}j} f_* \pi_{1267}^* (\pi_{234}^* \widetilde{g})^* ({}^{\downarrow\uparrow\uparrow\downarrow}\pi)_! f_{1,*} \iota_{1,!} g_1^* p^*$$

for

$$\begin{tikzcd}
i \times_{/J,\pi_1} {}^{\downarrow\uparrow\uparrow}(J \times_{/J} j) \arrow[r] \arrow[d] & i \times_{/J,\pi_1} {}^{\downarrow\uparrow\uparrow\downarrow}J \arrow[d, "{}^{\downarrow\uparrow\uparrow\downarrow}\pi"] \\
{}^{\downarrow\uparrow\uparrow}(J \times_{/J} j) \arrow[r, "\pi_{1267}"] & {}^{\downarrow\uparrow\uparrow\downarrow}J
\end{tikzcd}$$

which is homotopy exact.
Using that $(\pi_{234}\widetilde{g})^*$ and f_* commute with homotopy colimits, we can write this as

$$\operatorname*{hocolim}_{i\times_{/J,\pi_1}{}^{\downarrow\uparrow\uparrow}(J\times_{/J}j)} f_{2,*} g_2^* f_{1,*} \iota_{1,!} g_1^* P^*$$

for $P : i\times_{/J,\pi_1}{}^{\downarrow\uparrow\uparrow}(J\times_{/J}j) \to \cdot$, where, denoting an object in the fiber by $i \to j_1 \to j_2 \to j_3 \to j = j = j = j$ the various morphisms are given point-wise by

$$\begin{aligned}
g_1 : \widetilde{S}(i = i \to j) &\to \widetilde{S}(i = i = i) \\
\iota_1 : \widetilde{S}(i = i \to j) &\to \widetilde{S}(i \to j = j) \\
f_1 : \widetilde{S}(i \to j = j) &\to \widetilde{S}(j_2 \to j = j) \\
g_2 : \widetilde{A}(j_2 \to j = j) &\to \widetilde{S}(j_2 \to j = j) \\
f_2 : \widetilde{A}(j_2 \to j = j) &\to \widetilde{A}(j = j = j)
\end{aligned}$$

The argument of $\operatorname{hocolim}_{i\times_{/J,\pi_1}{}^{\downarrow\uparrow\uparrow}(J\times_{/J}j)}$ does not depend on j_1 and j_3. We therefore factor

$$i \times_{/J,\pi_1} {}^{\downarrow\uparrow\uparrow}(J \times_{/J} j) \xrightarrow{\pi_2} (i \times_{/J} J \times_{/J} j)^{\mathrm{op}} \xrightarrow{P_1} \cdot$$

Like in 11.4 one shows that for π_2 we have $\pi_{2,!}\pi_2^* \cong \mathrm{id}$ (it is the composition of an opfibration and a fibration each with contractible fibers). Hence we are left with a homotopy colimit over $(i \times_{/J} J \times_{/J} j)^{\mathrm{op}}$, which splits up into a union over $\mathrm{Hom}_J(i,j)$ and on each component is evaluation at the final object $i = i = i \to j = j = j = j = j$. Hence we can set $j_2 := i$ in the formulas above and replace P with $P_2 : i \times_{/J} j \to \cdot$ (Note: $i \times_{/J} j$ is the discrete category with objects $\mathrm{Hom}_J(i,j)$). Now we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{A}(i = i = i) \arrow[d, "\widetilde{g}_i"] & \widetilde{A}(i = i \to j) \arrow[l, "G"'] \arrow[r, "\iota_3"] \arrow[d, "g_3"] & \widetilde{A}(i \to j = j) \arrow[r, equal] \arrow[d, "g_2"] & \widetilde{A}(i \to j = j) \arrow[r, "f_2"] \arrow[d, "g_2"] & \widetilde{A}(j = j = j) \arrow[d, "\widetilde{g}_j"] \\
\widetilde{S}(i = i = i) & \widetilde{S}(i = i \to j) \arrow[l, "g_1"] \arrow[r, "\iota_1"'] & \widetilde{S}(i \to j = j) \arrow[r, equal, "f_1"'] & \widetilde{S}(i \to j = j) \arrow[r, "f_2"'] & \widetilde{S}(j = j = j)
\end{tikzcd}$$

The left hand square is Cartesian by Lemma 7.4, hence we have

$$g_2^* \iota_{1,!} \cong \iota_{3,!} g_3^*$$

Hence we arrive at

$$\operatorname*{hocolim}_{i \times_{/J} j} f_{2,*} \iota_{3,!} G^* P_2^* \widetilde{g}_i^*.$$

This is the same as the left hand side of (24) (use Kan's formula (FDer4 left)). A tedious check shows that the natural transformations match. □

Main Theorem 13.13. *Let $\mathcal{S}$ be a category with compactifications, and let $\mathbb{S}^{\mathrm{op}}$ be the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$ with its natural multicategory structure encoding the product. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a left and right fibered multiderivator with domain* Dirlf *satisfying axioms (F1–F6) and (F4m–F5m) of 6.1. Assume that $\mathbb{D}$ is infinite (i.e. satisfies (Der1) also for infinite coproducts). The morphism of pre-2-multiderivators*

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}$$

constructed in Definition 13.11 is an oplax left fibered multiderivator with domain Catlf*.*

Proof. As in the plain case, with the following modifications:
The second statement of (FDer0 left) follows from Proposition A.12 because on objects of the form ${}^{\downarrow\uparrow\uparrow\downarrow}i$ the functor $\Box_*$ does not do anything.
(FDer5 left) Because by the strong form of (FDer0 left) the push-forward along oplax morphisms is computed point-wise it suffices to see this for projections $p : I \to \cdot$, i.e. for homotopy colimits. Over a point though the condition "oplax" is vacuous. □

Corollary 13.14. *With the assumptions of Main Theorem 13.13, if $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is infinite and has stable and perfectly generated fibers then $\mathbb{E}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}$ is a lax right fibered multiderivator with domain* Catlf*. Furthermore, the oplax left (resp. lax right) fibered multiderivator $\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{oplax}}$ (resp. $\mathbb{E}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}$) extends (uniquely up to equivalence) to* Cat*.*

Proof. We have to show that the resulting morphism of 2-pre-multiderivators

$$\mathbb{E}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}$$

is a lax right fibered multiderivator: The 1-fiberedness of $\mathbb{E}''(I) \to \mathbb{S}^{\mathrm{cor},0,\mathrm{comp},\mathrm{lax}}(I)$ follows from Brown representability as in the plain case. The second statement of (FDer0 right) — which involves only opfibrations — follows from Proposition 13.12. For 1-ary morphisms the pull-back is even computed point-wise which follows from the additional statement of Proposition 13.12. (FDer5 right) — which (in the lax case) involves only relative right Kan extensions along fibrations — follows from Lemma A.11. The extension to Cat is constructed as in the plain case. □

As in the plain case, one can construct equivalent oplax left (resp. lax right) fibered multiderivators

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}} \quad (\text{resp. } \mathbb{E}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}).$$

Those are canonically symmetric if $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ is symmetric. In other words, we actually get a (symmetric) proper derivator six-functor-formalism as in Definition 10.2. The restrictions of both to Catlf and $\mathbb{S}^{\mathrm{cor}}$ are equivalent to the morphism of (symmetric) 2-pre-multiderivators of Main theorem 12.1 (cf. also 12.2).

A (co)Cartesian projectors

In this appendix the (co)Cartesian projectors needed in the construction of the *proper* derivator six-functor formalism in Section 13 are constructed. It would be in principle possible to construct them using Brown representability techniques (cf. [12, §4.3]). However, in this case, an explicit construction is available with the aid of which many properties become more clearly visible. Recall the notation from 10.5 and 13.3. Let $\widetilde{S}$ be any object in $\mathbb{S}({}^{\downarrow\downarrow\uparrow}I)$ that is the specialization of an interior compactification of a morphism $\Delta_T \to \mathrm{Fun}^{(\mathrm{op})\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor}})$ to an object of ${}^{\downarrow\downarrow\uparrow}\Delta_T$.

A.1. We will show that the fully-faithful inclusion

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \hookrightarrow \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})}$$

has a left adjoint $\Box_!$, which we will call a **left Cartesian projector** (cf. also [12, Section 2.4]). It induces a left adjoint also of the restriction:

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},\mathrm{ws},2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \hookrightarrow \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cart},\mathrm{ws}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})}$$

that is to say: $\Box_!$ preserves the conditions of being simultaneously 4-coCartesian and well-supported. Such a left Cartesian projector (or rather its composition with the fully-faithful inclusion) can be specified by an endofunctor of $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}}$ together with a natural transformation

$$\nu : \mathrm{id} \Rightarrow \Box_!$$

such that

1. $\Box_!\mathcal{E}$ is 2-Cartesian for all objects $\mathcal{E}$,
2. $\nu_{\mathcal{E}}$ is an isomorphism on 2-Cartesian objects $\mathcal{E}$,
3. $\nu_{\Box_!\mathcal{E}} = \Box_!\nu_{\mathcal{E}}$ holds true.

A.2. Furthermore, we will show that the fully-faithful inclusion

$$\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart},\mathrm{ws},2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})} \hookrightarrow \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{\mathrm{ws},2-\mathrm{cart}}_{\pi_{234}^*(\widetilde{S}^{\mathrm{op}})}$$

has a right adjoint $\Box_*$ which we will call a **right coCartesian projector** (cf. also [12, Section 2.4]). A right coCartesian projector (or rather its composition with the fully-faithful inclusion) can be specified by an endofunctor $\Box_*$ of $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{\mathrm{ws},2-\mathrm{cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}}$ together with a natural transformation

$$\nu : \Box_* \Rightarrow \mathrm{id}$$

such that

1. $\Box_*\mathcal{E}$ is 4-coCartesian for all objects $\mathcal{E}$ (in the source category),
2. $\nu_{\mathcal{E}}$ is an isomorphism on 4-coCartesian objects $\mathcal{E}$,
3. $\nu_{\Box_*\mathcal{E}} = \Box_*\nu_{\mathcal{E}}$ holds true.

This, in particular, gives a push-forward functor

$$\Box_*(\pi_{234}^*\widetilde{f})_*: \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*(\widetilde{A}')^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*\widetilde{T}^{\mathrm{op}}}$$

for a point-wise proper morphism

$$\widetilde{f}: \widetilde{A}' \to \widetilde{T}$$

arising from an oplax morphism as in 13.3.
Note that $(\pi_{234}^*\widetilde{f})_*$ preserves automatically the condition of being 2-Cartesian and well-supported by Lemma 13.4, 3. Proposition A.12 below shows that this is still computed point-wise (in the sense of $\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$), i.e. that we have for any $\alpha: I \to J$

$$\alpha^* \Box_* (\pi_{234}^*\widetilde{f})_* \cong \Box_*(\pi_{234}^*({}^{\downarrow\downarrow\uparrow}\alpha)^*\widetilde{f})_*\alpha^*.$$

A.3. We need some technical preparation. Consider the projections:

$$\pi_{1234}, \pi_{1237}, \pi_{1267}, \pi_{1567}, \pi_{4567}: {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow\uparrow\downarrow}I.$$

We have obvious natural transformations

$$\pi_{1234} \Rightarrow \pi_{1237} \Leftarrow \pi_{1267} \Leftarrow \pi_{1567} \Rightarrow \pi_{4567}$$

and therefore

$$\pi_{1234}^* \Rightarrow \pi_{1237}^* \Leftarrow \pi_{1267}^* \Leftarrow \pi_{1567}^* \Rightarrow \pi_{4567}^*$$

If we plug in $\pi_{234}^*\widetilde{S}$ for $\widetilde{S}$ an object in $\mathbb{S}({}^{\downarrow\downarrow\uparrow}I)$ that is the specialization of an interior compactification of a morphism $\Delta_T \to \mathrm{Fun}^{(\mathrm{op})\mathrm{lax}}(I, \mathcal{S}^{\mathrm{cor}})$ to an object of ${}^{\downarrow\downarrow\uparrow}\Delta_T$ (cf. 10.5 and 13.3), we get morphisms of diagrams in $\mathcal{S}$:

$$\begin{tikzcd} \pi_{234}^*\widetilde{S} & \pi_{237}^*\widetilde{S} \arrow[l, "g"'] \arrow[r, hook, "\iota"] & \pi_{267}^*\widetilde{S} \arrow[r, two heads, "\overline{f}"] & \pi_{567}^*\widetilde{S} \arrow[r, equal] & \pi_{567}^*\widetilde{S} \end{tikzcd}$$

and therefore natural transformations

$$\begin{array}{rcl} g^*\pi_{1234}^* & \Rightarrow & \pi_{1237}^* \\ \pi_{1237}^* & \Leftarrow & \iota^*\pi_{1267}^* \\ \overline{f}_*\pi_{1567}^* & \Leftarrow & \pi_{1267}^* \end{array}$$

of functors between fibers.

Lemma A.4. *The natural transformation*

$$\pi_{4567,!}^{(\pi_{234}^*\widetilde{S}^{\mathrm{op}})}\pi_{1567}^* \Rightarrow \mathrm{id}$$

induced by the natural transformation

$$\pi_{1567}^* \Rightarrow \pi_{4567}^*$$

of functors

$$\pi_{1567}^*, \pi_{4567}^*: \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{567}^*S^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)_{\pi_{234}^*S^{\mathrm{op}}}$$

is an isomorphism.

Proof. π_{4567} is an opfibration (cf. 4.2). Denote by $e_i : {}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1) \hookrightarrow {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I$ the inclusion of the fiber over an object

$$i = \{i_1 \to i_2 \to i_3 \to i_4\}.$$

We have

$$i^*\pi_{4567,!}\pi_{1567}^* = \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I}i_1)} e_i^*\pi_{1567}^*$$

We can factor $\pi_{1567} \circ e_i$ in the following way

$${}^{\downarrow\uparrow\uparrow}(I \times_{/I} i) \xrightarrow{\pi_1} I \times_{/I} i_1 \xrightarrow{\rho} {}^{\downarrow\uparrow\uparrow\downarrow}I$$

where ρ maps $i' \to i_1$ to $i' \to i_2 \to i_3 \to i_4$. The functor π_1 is a fibration with fibers of the form $\beta \times_{/{}^{\uparrow\uparrow}(I\times_{/I}i)} {}^{\uparrow\uparrow}(I \times_{/I} i)$. Since these fibers have an initial object, the unit $\mathrm{id} \to \pi_{1,*}\pi_1^*$ is actually an isomorphism. Therefore also the counit $\pi_{1,!}\pi_1^* \to \mathrm{id}$ (which is its adjoint) is an isomorphism. Hence we have

$$i^*\pi_{4567,!}\pi_{1567}^* \cong \operatorname*{hocolim}_{I\times_{/I}i_1} \rho^*$$

Since $I \times_{/I} i_1$ has a final object, the homotopy colimit is actually evaluation at the latter, therefore we get

$$i^*\pi_{4567,!}\pi_{1567}^* \cong i^*.$$

$\square$

If $\mathcal{E}$ is an object in $\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{\mathrm{ws,2-cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}}$ we have that the morphisms

$$\begin{aligned} \pi_{1237}^*\mathcal{E} &\leftarrow \iota^*\pi_{1267}^*\mathcal{E} & (25)\\ \overline{f}_*\pi_{1567}^*\mathcal{E} &\leftarrow \pi_{1267}^*\mathcal{E} & (26) \end{aligned}$$

are isomorphisms.

Lemma A.5. *Assume $\mathbb{D}$ infinite or that I has finite Hom sets.*
If $\mathcal{E}$ is well-supported, then the inverse of (25) induces an isomorphism

$$\iota_!\pi_{1237}^*\mathcal{E} \cong \pi_{1267}^*\mathcal{E}.$$

Proof. The assertion follows if we can show that $\iota_!$ is computed point-wise on $\pi_{1237}^*\mathcal{E}$. Consider the projection $\pi_{12} : {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow}I$. It is an opfibration. Every coCartesian morphism in ${}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I$ w.r.t. this opfibration is mapped by $\pi_{237}^*\widetilde{S}^{\mathrm{op}}$ and $\pi_{267}^*\widetilde{S}^{\mathrm{op}}$ to a proper morphism. Therefore by Lemma 2.5 and Lemma 6.11, 1. $\iota_!$ commutes with the inclusion of the fiber. On the fiber we will check the condition of Lemma 6.11, 3. Let $\widetilde{\iota} : \widetilde{S}' \to \widetilde{S}$ be the morphism (point-wise dense embedding) as in the definition of "well-supported" (10.5). Let $\alpha : i'' \to i'$ and $\mu : i' \to i$ be morphisms in the fiber. Applying π_{237} and π_{267} we get the following situation:

$$\begin{tikzcd}
 & \widetilde{S}(i_2 \to i_3 \to i_7) \arrow[r, hook, "\iota_2"] \arrow[d, hook, "\iota_1"] \arrow[dd, bend right, "\pi_{237}(\mu^{\mathrm{op}})"'] & \widetilde{S}(i_2 \to i_6 \to i_7) \arrow[d, hook, "\iota_3"] \arrow[dd, bend left, "\pi_{267}(\mu^{\mathrm{op}})"] \\
\widetilde{S}'(i_2 = i_2 \to i_7) \arrow[r, hook] \arrow[ur, hook] \arrow[d] & \widetilde{S}(i_2 \to i_3' \to i_7) \arrow[r, hook, "\iota_4"] \arrow[d, "g_1"] & \widetilde{S}(i_2 \to i_6' \to i_7) \arrow[d, "g_2"] \\
\widetilde{S}'(i_2 = i_2 \to i_7') \arrow[r, hook] \arrow[d] & \widetilde{S}(i_2 \to i_3' \to i_7') \arrow[r, hook, "\iota_5"] \arrow[d, "g_3"] \arrow[dd, bend right, "\pi_{237}(\alpha^{\mathrm{op}})"'] & \widetilde{S}(i_2 \to i_6' \to i_7') \arrow[d, "g_4"] \arrow[dd, bend left, "\pi_{267}(\alpha^{\mathrm{op}})"] \\
\widetilde{S}'(i_2 = i_2 \to i_7'') \arrow[r, hook] \arrow[dr, hook] & \widetilde{S}(i_2 \to i_3' \to i_7'') \arrow[r, hook, "\iota_7"] \arrow[d, hook, "\iota_6"] & \widetilde{S}(i_2 \to i_6' \to i_7'') \arrow[d, hook, "\iota_8"] \\
 & \widetilde{S}(i_2 \to i_3'' \to i_7'') \arrow[r, hook, "\iota_9"] & \widetilde{S}(i_2 \to i_6'' \to i_7'')
\end{tikzcd}$$

We have to show

$$\iota_{2,!}\ \iota_1^* g_1^*\ g_3^* \iota_6^*\ \mathcal{E}(i_1 \to i_2 \to i_3'' \to i_7'') \cong \iota_3^* g_2^*\ \iota_{5,!}\ g_3^* \iota_6^*\ \mathcal{E}(i_1 \to i_2 \to i_3'' \to i_7'').$$

Because $\mathcal{E}$ is 3-coCartesian this is the same as

$$\iota_{2,!}\ \iota_1^* g_1^*\ g_3^* \mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'') \cong \iota_3^* g_2^*\ \iota_{5,!}\ g_3^* \mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'').$$

By assumption $\mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'')$ has support in $\widetilde{S}(i_2 = i_2 \to i_7'')$, therefore $g_3^*\mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'')$ has support in $\widetilde{S}(i_2 = i_2 \to i_7')$, and $g_1^* g_3^* \mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'')$ has support in $\widetilde{S}(i_2 = i_2 \to i_7)$, by (the adjoint of) axiom (F5). Note that the relevant squares are Cartesian (Lemma 8.8). Therefore on such an object, we have $\iota_{2,!}\iota_1^* \cong \iota_3^* \iota_{3,!} \iota_{2,!} \iota_1^* \cong \iota_3^* \iota_{4,!} \iota_{1,!} \iota_1^* \cong \iota_3^* \iota_{4,!}$. Inserting this, we get the morphism:

$$\iota_3^* \iota_{4,!} g_1^* g_3^* \mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'') \cong \iota_3^* g_2^* \iota_{5,!} g_3^* \mathcal{E}(i_1 \to i_2 \to i_3' \to i_7'').$$

That this is an isomorphism follows from Proposition 8.12, 2. because of the support condition. Hence $\iota_!$ is also computed point-wise on the fiber. $\square$

Warning: Even if $\mathcal{E}$ is 3-coCartesian (but not well-supported) $\iota_!$ will not be computed point-wise on $\pi_{1237}^* \mathcal{E}$ in general.

Proposition A.6. *Using the notation of A.3, denote* $\Box_! := \pi_{4567;!} \overline{f}_* \pi_{1267}^*$. *This functor, together with*

$$\mathcal{E} \xleftarrow{\sim} \pi_{4567;!} \pi_{1567}^* \mathcal{E} \longrightarrow \pi_{4567;!} \overline{f}_* \pi_{1267}^* \mathcal{E}\ ,$$

defines a left Cartesian projector:

$$\Box_! : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow} I)_{\pi_{234}^* \widetilde{S}^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow} I)^{2-\mathrm{cart}}_{\pi_{234}^* \widetilde{S}^{\mathrm{op}}}$$

$\Box_!$ *preserves the conditions of being (simultaneously) 4-coCartesian and well-supported.*

Proof. We have to check the properties 1.–3. of A.1.
1. Consider a morphism μ of type 2 in ${}^{\downarrow\uparrow\uparrow\downarrow} I$:

$$\begin{tikzcd}
i = (i_1 \arrow[r] \arrow[d, equal] & i_2 \arrow[r] & i_3 \arrow[r] \arrow[d, equal] & i_4) \arrow[d, equal] \\
i' = (i_1 \arrow[r] & i_2' \arrow[r] \arrow[u] & i_3 \arrow[r] & i_4)
\end{tikzcd}$$

We have to see that
$$\widetilde{S}(\pi_{234}\mu)_* i^* \pi_{4567;!} \overline{f}_* \pi_{1267}^* \to (i')^* \pi_{4567;!} \overline{f}_* \pi_{1267}^*$$
is an isomorphism. Using that $\pi_{4567;!}$ is an obfibration, we get
$$\widetilde{S}(\pi_{234}\mu)_* \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I} i_1)} e_i^* \overline{f}_* \pi_{1267}^* \to \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I} i_1)} e_{i'}^* \overline{f}_* \pi_{1267}^*$$
However, we already have an isomorphism
$$(\widetilde{S}(\pi_{234}\mu))_* e_i^* \overline{f}_* \to e_{i'}^* \overline{f}_*.$$
2. follows because for a 2-Cartesian object $\mathcal{E}$ the morphism
$$\overline{f}_* \pi_{1567}^* \mathcal{E} \leftarrow \pi_{1267}^* \mathcal{E}$$
is already an isomorphism.
3. Is proven as for [13, Proposition 8.5].
We now show that $\Box_!$ preserves the conditions of being (simultaneously) 4-coCartesian and well-supported. Let $\mathcal{E}$ be an objects with these properties.
Consider a morphism μ of type 4 in ${}^{\downarrow\uparrow\uparrow\downarrow}I$:
$$\begin{tikzcd} i = (i_1 \arrow[r] \arrow[d, equal] & i_2 \arrow[r] \arrow[d, equal] & i_3 \arrow[r] \arrow[d, equal] & i_4) \arrow[d] \\ i' = (i_1 \arrow[r] & i_2 \arrow[r] & i_3 \arrow[r] & i_4') \end{tikzcd}$$
and let $G := \widetilde{S}(\pi_{234}(\mu))$. We have to see that
$$G^* \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I} i_1)} e_i^* \overline{f}_* \pi_{1267}^* \mathcal{E} \to \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I} i_1)} e_{i'}^* \overline{f}_* \pi_{1267}^* \mathcal{E}$$
is an isomorphism. Since G^* commutes with homotopy colimits (being a left adjoint) it suffices to show that point-wise
$$G^* e_i^* \overline{f}_* \pi_{1267}^* \mathcal{E} \to e_{i'}^* \overline{f}_* \pi_{1267}^* \mathcal{E}$$
is an isomorphism. Pick an object $j_1 \to j_2 \to j_3 \to i_1$ in $({}^{\uparrow\uparrow\downarrow}I \times_{/I} i_1)$ and consider the diagram
$$\begin{tikzcd} \widetilde{S}(j_2 \to i_3 \to i_4') \arrow[r, "G"] \arrow[d, "F"'] & \widetilde{S}(j_2 \to i_3 \to i_4) \arrow[d, "f"] \\ \widetilde{S}(i_2 \to i_3 \to i_4') \arrow[r, "g"] & \widetilde{S}(i_2 \to i_3 \to i_4) \end{tikzcd}$$
We are left to show that
$$g^* f_* \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4} \to F_* \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4'}$$
is an isomorphism. However, $\mathcal{E}$ is 4-coCartesian, hence this is the same as
$$g^* f_* \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4} \to F_* G^* \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4}$$
which is an isomorphism by Proposition 8.12, 1. because $\mathcal{E}$ is also well-supported and thus has support in $\widetilde{S}'(j_2 = j_2 \to i_4)$ (with $\widetilde{S}'$ as in the definition of well-supported, cf. 10.5).

Consider now a morphism μ of type 3

$$\begin{tikzcd} i = (i_1 \arrow[r] & i_2 \arrow[r] & i_3 \arrow[r] & i_4) \\ i' = (i_1 \arrow[r] \arrow[u, equal] & i_2 \arrow[r] \arrow[u, equal] & i_3' \arrow[r] \arrow[u] & i_4) \arrow[u, equal] \end{tikzcd}$$

and let $\iota := \widetilde{S}(\pi_{234}(\mu))$. To show that $\Box_! \mathcal{E}$ is well supported, we have to see that

$$\iota_! \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1)} e_i^* \overline{f}_* \pi_{1267}^* \mathcal{E} \to \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1)} e_{i'}^* \overline{f}_* \pi_{1267}^* \mathcal{E}$$

is an isomorphism and that the right hand side has support in $\widetilde{S}'(i_2 = i_2 \to i_4)$ (with $\widetilde{S}'$ as in the definition of well-supported, cf. 10.5). Since $\iota_!$ commutes with homotopy colimits (being a left adjoint) and is computed point-wise on constant diagrams it suffices to show that point-wise

$$\iota_! e_i^* \overline{f}_* \pi_{1267}^* \mathcal{E} \to e_{i'}^* \overline{f}_* \pi_{1267}^* \mathcal{E}$$

is an isomorphism. Pick an object $j_1 \to j_2 \to j_3 \to i_1$ in $({}^{\uparrow\uparrow\downarrow} I \times_{/I} i_1)$ and consider the diagram

$$\begin{tikzcd} \widetilde{S}(j_2 \to i_3 \to i_4) \arrow[r, "I"] \arrow[d, "F"'] & \widetilde{S}(j_2 \to i_3' \to i_4) \arrow[d, "f"] \\ \widetilde{S}(i_2 \to i_3 \to i_4) \arrow[r, "\iota"] & \widetilde{S}(i_2 \to i_3' \to i_4) \end{tikzcd}$$

We are left to show that

$$\iota_! F_* \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4} \to f_* \mathcal{E}_{j_1 \to j_2 \to i_3' \to i_4}$$

is an isomorphism. However, $\mathcal{E}$ is stongly 3-coCartesian, hence this is the same as

$$\iota_! F_* \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4} \to f_* I_! \mathcal{E}_{j_1 \to j_2 \to i_3 \to i_4}$$

which is an isomorphism by Proposition 8.12, 3. For the condition of being well-supported, it suffices to see that $e_{i'}^* \overline{f}_* \pi_{1267}^* \mathcal{E}$ point-wise has support in $\widetilde{S}'(i_2 = i_2 \to i_4)$. This is shown in the same way. $\square$

Lemma A.7. *For a morphism $\widetilde{\iota}: \widetilde{A} \hookrightarrow \widetilde{A}'$ as in 13.3 the following diagram 2-commutes*

$$\begin{tikzcd} \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow} I)^{4-\mathrm{cocart,ws}}_{\pi_{234}^* \widetilde{A}^{\mathrm{op}}} \arrow[r, "\Box_!"] \arrow[d, "(\pi_{234}^* \widetilde{\iota})_!"'] \arrow[dr, phantom, "\swarrow\sim"] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow} I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^* \widetilde{A}^{\mathrm{op}}} \arrow[d, "(\pi_{234}^* \widetilde{\iota})_!"] \\ \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow} I)^{4-\mathrm{cocart,ws}}_{\pi_{234}^* (\widetilde{A}')^{\mathrm{op}}} \arrow[r, "\Box_!"'] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow} I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^* (\widetilde{A}')^{\mathrm{op}}} \end{tikzcd}$$

(The natural transformation is induced by the fact that $(\pi_{234}^* \widetilde{\iota})^*$ preserves the condition of being 2-Cartesian, all relevant squares being Cartesian.)

Proof. We will show that it $(\pi_{234}^* \widetilde{\iota})_!$ “commutes” with the three functors in $\pi_{4567;!} \overline{f}_{\widetilde{A},*} \pi_{1267}^*$, resp. $\pi_{4567;!} \overline{f}_{\widetilde{A}',*} \pi_{1267}^*$ in the obvious sense. We have

$$(\pi_{234}^* \widetilde{\iota})_! \pi_{4567;!} \cong \pi_{4567;!} (\pi_{567}^* \widetilde{\iota})_!$$

because the adjoint relation follows from (FDer0 left) for $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$.
As in the proof of Lemma 13.4, one shows that $(\pi_{267}^*\widetilde{\iota})_!$ and $(\pi_{567}^*\widetilde{\iota})_!$ are both computed point-wise on objects in the image of the subcategory in question under the functors π_{1267}^*, resp. $\overline{f}_{\widetilde{A},*}\pi_{1267}^*$. We have therefore

$$(\pi_{567}^*\widetilde{\iota})_!\overline{f}_{\widetilde{A},*} \cong \overline{f}_{\widetilde{A}',*}(\pi_{267}^*\widetilde{\iota})_!$$

on the subcategory in question because of axiom (F6).
Finally

$$(\pi_{267}^*\widetilde{\iota})_!\pi_{1267}^* \cong \pi_{1267}^*(\pi_{234}^*\widetilde{\iota})_!$$

on the subcategory in question, again because both $(-)_!$ functors are computed point-wise when restricted to the subcategories in question (cf. Lemma 13.4). □

Lemma A.8. *For a morphism $\widetilde{f}: \widetilde{A}' \hookrightarrow \widetilde{T}$ as in 13.3 the following diagram 2-commutes*

$$\begin{tikzcd}
\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws}}_{\pi_{234}^*(\widetilde{A}')^{\mathrm{op}}} \arrow[r, "\Box_!"] \arrow[d, "(\pi_{234}^*\widetilde{f})_*"'] \arrow[dr, phantom, "\swarrow\sim"] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*(\widetilde{A}')^{\mathrm{op}}} \arrow[d, "(\pi_{234}^*\widetilde{f})_*"] \\
\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws}}_{\pi_{234}^*\widetilde{T}^{\mathrm{op}}} \arrow[r, "\Box_!"'] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*\widetilde{T}^{\mathrm{op}}}
\end{tikzcd}$$

(The natural transformation is the exchange induced by the fact that $(\pi_{234}^*\widetilde{f})_*$ preserves the condition of being 2-Cartesian.)

Proof. This is proven as the previous Lemma. Note that $(\pi_{234}^*\widetilde{f})_*$ is computed point-wise (FDer0 right) and commutes with homotopy colimits by Axiom (F3). □

Lemma A.9. *Let $\alpha: I \to J$ be a fibration in* Catlf*, and consider the sequence of functors:*

$${}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I \xrightarrow{q_1=({}^{\downarrow\uparrow\uparrow\downarrow}\alpha,\pi_{234567})} {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}J \times_{({}^{\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}J)} {}^{\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I \xrightarrow{q_2=\mathrm{id}\times\pi_{3456}} {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}J \times_{{}^{\downarrow\uparrow\uparrow\downarrow}J} {}^{\downarrow\uparrow\uparrow\downarrow}I.$$

1. *The functor q_1 is a fibration. The fiber of q_1 over a pair $j_1 \to j_2 \to j_3 \to j_4 \to j_5 \to j_6 \to j_7$ and $i_2 \to i_3 \to i_4 \to i_5 \to i_6 \to i_7$ is*

$$I_{j_1} \times_{/I_{j_1}} i_1$$

   where i_1 is the source of a Cartesian arrow over $j_1 \to j_2$ with destination i_2.

2. *The functor q_2 is an opfibration. The fiber of q_2 over a pair $j_1 \to j_2 \to j_3 \to j_4 \to j_5 \to j_6 \to j_7$ and $i_4 \to i_5 \to i_6 \to i_7$ (lying over $j_4 \to j_5 \to j_6 \to j_7$) is*

$$(I_{j_2} \times_{/I_{j_2}} I_{j_3} \times_{/I_{j_3}} i_3)^{\mathrm{op}}$$

   where i_3 is the source of a Cartesian arrow over $j_3 \to j_4$ with destination i_4 and the first comma category is constructed via the functor $I_{j_3} \to I_{j_2}$ being the Cartesian pull-back along $j_2 \to j_3$.

Proof. Straightforward. □

Lemma A.10. *For the composition $\kappa = q_2 \circ q_1$ we have that the counit*

$$\kappa_!\kappa^* \to \mathrm{id}$$

is an isomorphism.

Proof. As in the proof of Lemma 11.2, the previous Lemma implies that the unit $\mathrm{id} \to q_{1,*}q_1^*$ and counit $q_{2,!}q_2^* \to \mathrm{id}$ are isomorphisms. Therefore also the counit $q_{1,!}q_1^* \to \mathrm{id}$ is an isomorphism (it is the adjoint of the first unit) and finally also $\kappa_!\kappa^* \to \mathrm{id}$. $\square$

Lemma A.11. *Let $\widetilde{S} \in \mathrm{Fun}({}^{\downarrow\downarrow\uparrow}J, \mathcal{S})$ be as above and let $\alpha : I \to J$ be a fibration. The following diagram 2-commutes:*

$$\begin{tikzcd}
\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{4-\mathrm{cocart,ws}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \arrow[r, "\Box_!"] \arrow[d, "({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*"'] \arrow[dr, phantom, "\swArrow\sim"] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}J)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*(\widetilde{S})^{\mathrm{op}}} \arrow[d, "({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*"] \\
\mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws}}_{\pi_{234}^*({}^{\downarrow\downarrow\uparrow}\alpha)^*\widetilde{S}^{\mathrm{op}}} \arrow[r, "\Box_!"'] & \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*({}^{\downarrow\downarrow\uparrow}\alpha)^*\widetilde{S}^{\mathrm{op}}}
\end{tikzcd}$$

(The natural transformation is the exchange induced by the fact that $({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*$ preserves the condition of being 2-Cartesian.)

Proof. We have $\Box_! = \pi_{4567;!}\overline{f}_*\pi_{1267}^*\mathcal{E}$ by definition. π_{1267}^* and $\overline{f}_*$ clearly commute with arbitrary pullbacks in the obvious sense. Consider now the diagram with Cartesian square

$$\begin{tikzcd}
{}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}I \arrow[dr, "\kappa"] \arrow[drr, "\pi_{4567}"] \arrow[ddr, "{}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}\alpha"'] & & \\
& {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}J \times_{{}^{\downarrow\uparrow\uparrow\downarrow}J} {}^{\downarrow\uparrow\uparrow\downarrow}I \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & {}^{\downarrow\uparrow\uparrow\downarrow}I \arrow[d, "{}^{\downarrow\uparrow\uparrow\downarrow}\alpha"] \\
& {}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}J \arrow[r, "\pi_{4567}"'] & {}^{\downarrow\uparrow\uparrow\downarrow}J
\end{tikzcd}$$

It shows (using Lemma A.10) that

$$\begin{aligned}
\pi_{4567,!}({}^{\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow}\alpha)^* &\cong \mathrm{pr}_{2,!}\,\kappa_!\kappa^*\,\mathrm{pr}_1^* \\
&\cong \mathrm{pr}_{2,!}\,\mathrm{pr}_1^* \\
&\cong ({}^{\downarrow\uparrow\uparrow\downarrow}\alpha)^*\pi_{4567,!}
\end{aligned}$$

$\square$

We now turn to the case of the coCartesian projector.

Proposition A.12. *Using the notation of A.3, denote $\Box_* := \pi_{4567;!}\overline{f}_*\iota_!g^*\pi_{1234}^*$. This functor, together with the composition*

$$\begin{tikzcd}
\mathcal{E} & \pi_{4567;!}\pi_{1567}^*\mathcal{E} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \pi_{4567;!}\overline{f}_*\pi_{1267}^*\mathcal{E} & \pi_{4567;!}\overline{f}_*\iota_!\pi_{1237}^*\mathcal{E} \arrow[l, "\sim"'] & \pi_{4567;!}\overline{f}_*\iota_!g^*\pi_{1234}^*\mathcal{E} = \Box_*\mathcal{E} \arrow[l] \arrow[lll, bend left, "\nu_{\mathcal{E}}"] ,
\end{tikzcd}$$

defines a right coCartesian projector:

$$\Box_* : \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{\mathrm{ws,2-cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}} \to \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}}.$$

This projector has the following property:

- *For each $i \in I$ the natural transformation*

$$({}^{\downarrow\uparrow\uparrow\downarrow}i)^* \square_* \to ({}^{\downarrow\uparrow\uparrow\downarrow}i)^* \tag{27}$$

is an isomorphism.

Proof. First note that the statement of the Proposition makes sense, because

$$\pi_{4567;!}\pi_{1567}^*\mathcal{E} \to \pi_{4567;!}\overline{f}_*\pi_{1267}^*\mathcal{E}$$

is an isomorphism on 2-Cartesian objects by Proposition A.6. We have to show the assertions 1–3. of A.2.
1. We have to show that

$$\square_*\mathcal{E} \in \mathbb{D}({}^{\downarrow\uparrow\uparrow\downarrow}I)^{4-\mathrm{cocart,ws,2-cart}}_{\pi_{234}^*\widetilde{S}^{\mathrm{op}}}$$

for a 2-Cartesian and well-supported object $\mathcal{E}$.
Consider a morphism μ of type 4 in ${}^{\downarrow\uparrow\uparrow\downarrow}I$

$$\begin{tikzcd}
i = (i_1 \arrow[r] \arrow[d, equal] & i_2 \arrow[r] \arrow[d, equal] & i_3 \arrow[r] \arrow[d, equal] & i_4) \arrow[d] \\
i' = (i_1 \arrow[r] & i_2 \arrow[r] & i_3 \arrow[r] & i_4')
\end{tikzcd}$$

and let $G_0 := \widetilde{S}(\pi_{234}(\mu))$. We have to see that

$$G_0^* \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I}i_1)} e_i^*\overline{f}_*\iota_!g^*\pi_{1234}^*\mathcal{E} \to \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I\times_{/I}i_1)} e_{i'}^*\overline{f}_*\iota_!g^*\pi_{1234}^*\mathcal{E}$$

is an isomorphism. Since G_0^* commutes with homotopy colimits (being a left adjoint) it suffices to show that point-wise

$$G_0^*e_i^*\overline{f}_*\iota_!g^*\pi_{1234}^*\mathcal{E} \to g^*e_{i'}^*\overline{f}_*\iota_!g^*\pi_{1234}^*\mathcal{E}$$

is an isomorphism. Note that by the proof of Lemma A.5 $\iota_!$ is computed point-wise at the given input. Pick an object $j_1 \to j_2 \to j_3 \to i_1$ in $({}^{\uparrow\uparrow\downarrow}I \times_{/I} i_1)$ and consider the diagram

$$\begin{tikzcd}
\widetilde{S}(j_2 \to j_3 \to i_1) \arrow[r, equal] & \widetilde{S}(j_2 \to j_3 \to i_1) \\
\widetilde{S}(j_2 \to j_3 \to i_4') \arrow[r, "G_2"] \arrow[u, "G"] \arrow[d, hook, "I"'] & \widetilde{S}(j_2 \to j_3 \to i_4) \arrow[u, "g"'] \arrow[d, hook, "\iota"] \\
\widetilde{S}(j_2 \to i_3 \to i_4') \arrow[r, "G_1"] \arrow[d, "F"'] & \widetilde{S}(j_2 \to i_3 \to i_4) \arrow[d, "f"] \\
\widetilde{S}(i_2 \to i_3 \to i_4') \arrow[r, "G_0"] & \widetilde{S}(i_2 \to i_3 \to i_4)
\end{tikzcd}$$

We are left to show that

$$G_0^*f_*\iota_!g^*\mathcal{E}_{j_1\to j_2\to j_3\to i_1} \to F_*I_!G^*\mathcal{E}_{j_1\to j_2\to j_3\to i_1}$$

is an isomorphism. This is the composition

$$\begin{aligned} G_0^* f_* \iota_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} &\to F_* G_1^* \iota_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} \\ &\to F_* I_! G_2^* g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} \\ &\to F_* I_! G^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} \end{aligned}$$

That these morphisms are isomorphisms follows from Proposition 8.12, because $\mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1}$ has support in $\widetilde{S}'(j_2 = j_2 \to i_1)$ and hence $g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1}$ has support in $\widetilde{S}'(j_2 = j_2 \to i_4)$.
Consider a morphism μ of type 3 in ${}^{\downarrow\uparrow\uparrow\downarrow} I$:

$$\begin{tikzcd} i = (i_1 \arrow[r] & i_2 \arrow[r] & i_3 \arrow[r] & i_4) \\ i' = (i_1 \arrow[r] \arrow[u, equal] & i_2 \arrow[r] \arrow[u, equal] & i_3' \arrow[r] \arrow[u] & i_4) \arrow[u, equal] \end{tikzcd}$$

and let $I := \widetilde{S}(\pi_{234}(\mu))$. We have to see that

$$I_! \underset{{}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1)}{\mathrm{hocolim}} e_i^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E} \to \underset{{}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1)}{\mathrm{hocolim}} e_{i'}^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E}$$

is an isomorphism, and the the right hand side has support in $\widetilde{S}'(i_2 = i_2 \to i_4)$ (with $\widetilde{S}'$ as in the definition of well-supported, cf. 10.5).
Since $I_!$ commutes with homotopy colimits (being a left adjoint) and is computed point-wise on constant diagrams it suffices to show that point-wise

$$I_! e_i^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E} \to e_{i'}^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E}$$

is an isomorphism. Note that by the proof of Lemma A.5 $\iota_!$ is computed point-wise at the given input. Pick an object $j_1 \to j_2 \to j_3 \to i_1$ in $({}^{\uparrow\uparrow\downarrow} I \times_{/I} i_1)$ and consider the diagram

$$\begin{tikzcd} \widetilde{S}(j_2 \to j_3 \to i_1) \arrow[r, equal] & \widetilde{S}(j_2 \to j_3 \to i_1) \\ \widetilde{S}(j_2 \to j_3 \to i_4) \arrow[r, equal] \arrow[u, "g"] \arrow[d, hook, "I"'] & \widetilde{S}(j_2 \to j_3 \to i_4) \arrow[u, "g"'] \arrow[d, hook, "\iota"] \\ \widetilde{S}(j_2 \to i_3' \to i_4) \arrow[r, hook, "I_1"] \arrow[d, "F"'] & \widetilde{S}(j_2 \to i_3 \to i_4) \arrow[d, "f"] \\ \widetilde{S}(i_2 \to i_3' \to i_4) \arrow[r, hook, "I_0"] & \widetilde{S}(i_2 \to i_3 \to i_4) \end{tikzcd}$$

We are left to show that

$$I_{0,!} F_* I_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} \to f_* \iota_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} \cong f_* I_{1,!} I_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1}$$

is an isomorphism. This is true by axiom (F6). Similarly one sees that if $\mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1}$ has support in $\widetilde{S}'(j_2 = j_2 \to i_1)$ (with $\widetilde{S}'$ as in the definition of well-supported, cf. 10.5) then $f_* \iota_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1}$ has support in $\widetilde{S}'(i_2 = i_2 \to i_4)$.

Consider now a morphism μ of type 2 in ${}^{\downarrow\uparrow\uparrow\downarrow}I$:

$$\begin{tikzcd}
i = (i_1 \arrow[r] \arrow[d, equal] & i_2 \arrow[r] & i_3 \arrow[r] \arrow[d, equal] & i_4) \arrow[d, equal] \\
i' = (i_1 \arrow[r] & i_2' \arrow[r] \arrow[u] & i_3 \arrow[r] & i_4)
\end{tikzcd}$$

and let $F_0 := \widetilde{S}(\pi_{234}(\mu))$. We have to see that

$$F_{0,*} \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1)} e_i^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E} \to \operatorname*{hocolim}_{{}^{\downarrow\uparrow\uparrow}(I \times_{/I} i_1)} e_{i'}^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E}$$

is an isomorphism.
By axiom (F3) $F_{0,*}$ commutes with homotopy colimits and is computed point-wise. Therefore it suffices to show that point-wise

$$F_{0,*} e_i^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E} \to e_{i'}^* \overline{f}_* \iota_! g^* \pi_{1234}^* \mathcal{E}$$

is an isomorphism. Note that by the proof of Lemma A.5 $\iota_!$ is computed point-wise at the given input.
Pick an object $j_1 \to j_2 \to j_3 \to i_1$ in $({}^{\uparrow\uparrow\downarrow}I \times_{/I} i_1)$ and consider the diagram

$$\begin{tikzcd}
\widetilde{S}(j_2 \to j_3 \to i_1) \arrow[r, equal] & \widetilde{S}(j_2 \to j_3 \to i_1) \\
\widetilde{S}(j_2 \to j_3 \to i_4) \arrow[r, equal] \arrow[u, "g"] \arrow[d, hook, "\iota"'] & \widetilde{S}(j_2 \to j_3 \to i_4) \arrow[u, "g"'] \arrow[d, hook, "\iota"] \\
\widetilde{S}(j_2 \to i_3 \to i_4) \arrow[r, equal] \arrow[d, "F"'] & \widetilde{S}(j_2 \to i_3 \to i_4) \arrow[d, "f"] \\
\widetilde{S}(i_2' \to i_3 \to i_4) \arrow[r, hook, "F_0"] & \widetilde{S}(i_2 \to i_3 \to i_4)
\end{tikzcd}$$

We have to show that

$$F_{0,*} F_* \iota_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1} \to f_* \iota_! g^* \mathcal{E}_{j_1 \to j_2 \to j_3 \to i_1}$$

is an isomorphism which is clear.
2. follows beause for a 4-coCartesian object $\mathcal{E}$ the morphism

$$g^* \pi_{1234}^* \mathcal{E} \to \pi_{1237}^* \mathcal{E}$$

is an isomorphism.
3. is proven as for [13, Proposition 8.5].
To see that (27) is an isomorphism, it suffices to see that *on the fiber (of π_{4567}) over an object of the form ${}^{\downarrow\uparrow\uparrow\downarrow}i$*, the morphism

$$g^* \pi_{1234}^* \mathcal{E} \to \pi_{1237}^* \mathcal{E}$$

is always an isomorphism. However, the natural transformation $\pi_{1234}^* \Rightarrow \pi_{1237}^*$ restricts to an identity on this fiber. $\square$

References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] *Théorie des topos et cohomologie étale des schémas. Tome 2.* Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[3] *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[4] J. Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp., 2007.

[5] J. Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, (315):vi+364 pp., 2007.

[6] D. C. Cisinski. Images directes cohomologiques dans les catégories de modèles. *Ann. Math. Blaise Pascal*, 10(2):195–244, 2003.

[7] D. C. Cisinski and F. Déglise. Triangulated categories of mixed motives. arXiv: 0912.2110, 2009.

[8] B. Drew. Motivic Hodge modules. arXiv: 1801.10129, 2018.

[9] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:No. 4, 107–131, 2003.

[10] D. Gaitsgory and N. Rozenblyum. A study in derived algebraic geometry. book project in progress: http://www.math.harvard.edu/~gaitsgde/GL/, 2016.

[11] J. Gillespie. The flat model structure on complexes of sheaves. *Trans. Amer. Math. Soc.*, 358 (7):2855–2874, 2006.

[12] F. Hörmann. Fibered multiderivators and (co)homological descent. *Theory Appl. Categ.*, 32 (38):1258–1362, 2017.

[13] F. Hörmann. Derivator Six-Functor-Formalisms — Definition and Construction I. arXiv: 1701.02152, 2017.

[14] F. Hörmann. Enlargement of (fibered) derivators. arXiv: 1706.09692, 2017.

[15] F. Hörmann. Six-Functor-Formalisms and Fibered Multiderivators. *Sel. Math.*, 24(4):2841–2925, 2018.

[16] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[17] M. Hovey. Cotorsion pairs and model categories. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 277–296. Amer. Math. Soc., Providence, RI, 2007.

[18] Y. Laszlo and M. Olsson. The six operations for sheaves on Artin stacks. I. Finite coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (107):109–168, 2008.

[19] Y. Laszlo and M. Olsson. The six operations for sheaves on Artin stacks. II. Adic coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (107):169–210, 2008.

[20] J. Lipman and M. Hashimoto. *Foundations of Grothendieck duality for diagrams of schemes*, volume 1960 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[21] Y. Liu and W. Zheng. Enhanced six operations and base change theorem for Artin stacks. arXiv: 1211.5948, 2012.

[22] K. Ponto and M. Shulman. The linearity of traces in monoidal categories and bicategories. *Theory Appl. Categ.*, 31:Paper No. 23, 594–689, 2016.

[23] R. Recktenwald. The flat cotorsion pair for ringed sites. Work in progress, 2019.

[24] O. M. Schnürer. Six operations on dg enhancements of derived categories of sheaves. *Selecta Math. (N.S.)*, 24(3):1805–1911, 2018.

[25] J.-L. Verdier. Catégories dérivées. In *Cohomologie Etale (SGA $4\frac{1}{2}$*, volume 569 of *Lecture Notes in Mathematics*, pages 262–311. Springer, 1977.

[26] W. Zheng. Six operations and Lefschetz-Verdier formula for Deligne-Mumford stacks. *Sci. China Math.*, 58(3):565–632, 2015.