F. Hörmann

Homotopy Limits and Colimits in Nature
A Motivation for Derivators

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An introduction to the notions of homotopy limit and colimit is given. It is explained how they can be used to neatly describe the "old" distinguished triangles and shift functors of derived categories resp. cofiber and fiber sequences in algebraic topology. One of the goals is to motivate the language of *derivators* from the perspective of classical homological algebra. Another one is to give *elementary* proofs (one brute-force in the exercises, and one a bit more abstract) that in the category of unbounded chain complexes of an (AB4, resp. AB4*) abelian category all homotopy limits (resp. colimits) exist and that this situation leads to a (stable) derivator. The heart of these proofs is an explicit formula for homotopy limits and colimits, the Bousfield-Kan formula. Later it is explained how these results fit in the framework of model categories. We sketch proofs that any model category gives rise to a derivator. We also rediscuss Bousfield-Kan's formula and outline the proof that it is valid in any *simplicial* model category (even a slightly weaker structure). In the end the homotopy theory of (homotopy) limits and colimits is discussed. In particular we explain that any derivator is a module over $\mathbb{H}$ (the derivator associated with the homotopy theory of spaces).
The reader is assumed to have seen some algebraic topology and/or homological algebra (here in particular the construction of abstract derived functors) although we will briefly recall everything. Some knowledge of category theory (limits, colimits, adjoints) is helpful but most facts are listed in an appendix. For the second part it is helpful to be familiar with model categories, but they will be briefly motivated and results presented as a black box.
The notes of the fourth talk at the summer school, on *fibered derivators, (co)homological descent and Grothendieck's six functors* is to be found in a subsequent document [16].
I would like to thank all participants of the summer school for the nice week, and for their very useful questions, comments and remarks.

Contents

1 Homological algebra

References: [2, 11, 28, 31, 32]

We first focus on homological algebra. Some general facts about localizations of categories are moved to section 4 to not interrupt the discussion. That section should be read alongside with the present one.

1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, for example the category of R-modules for a ring R. One starting point for doing homological algebra emerges from the fact that functors $F : \mathcal{A} \to \mathcal{B}$ might not be exact but only left- or right-exact. Let us recall what this means: In $\mathcal{A}$ there is a distinguished class of sequences isomorphic to

$$\left\{ X \underset{\text{(mono)}}{\overset{\alpha}{\hookrightarrow}} Y \longrightarrow \operatorname{coker}(\alpha) \right\}$$

— or equivalently — the class of sequences isomorphic to

$$\left\{ \ker(\alpha) \longrightarrow Y \underset{\text{(epi)}}{\overset{\alpha}{\twoheadrightarrow}} Z \right\}$$

which are called **exact sequences**. Actually the fact that these two classes coincide is the *characterising axiom* which makes a category (with zero object and existence of certain (co)limits) abelian. Recall that in any category with zero-object (**??**):

$$\ker(X \to Y) := \lim \begin{pmatrix} & 0 \\ & \downarrow \\ X \longrightarrow & Y \end{pmatrix} \qquad \operatorname{coker}(X \to Y) := \operatorname{colim} \begin{pmatrix} X & \longrightarrow Y \\ \downarrow & \\ 0 & \end{pmatrix}$$

If $F : \mathcal{A} \to \mathcal{B}$ is now a functor between abelian categories which is only left exact, i.e. it preserves kernels (and hence monomorphisms) but not cokernels resp. epimorphisms, the situation is "repaired" by constructing **right derived functors** $R^iF : \mathcal{A} \to \mathcal{B}$ which have the property that an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \tag{1}$$

leads to a long exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow R^1F(X) \longrightarrow R^1F(Y) \longrightarrow R^1F(Z) \longrightarrow R^2F(X) \longrightarrow \cdots$$

Recall how R^iFX is constructed. First one chooses an injective resolution (supposed to exist in $\mathcal{A}$), i.e. an exact sequence

$$0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

where the I_i are injective. This might be seen as a quasi-isomorphism in the category of chain complexes $C(\mathcal{A})$ of $\mathcal{A}$:

$$X \overset{q.i.}{\hookrightarrow} I_\bullet.$$

R^iFX is then defined as the homology of the complex $F(I_\bullet)$. The homology groups do not depend on the choice of $I_\bullet$. The long exact sequence arises because one can in fact find an exact sequence

$$0 \longrightarrow I_{X,\bullet} \longrightarrow I_{Y,\bullet} \longrightarrow I_{Z,\bullet} \longrightarrow 0$$

which is quasi-isomorphic to sequence (1). The sequence

$$0 \longrightarrow F(I_{X,\bullet}) \longrightarrow F(I_{Y,\bullet}) \longrightarrow F(I_{Z,\bullet}) \longrightarrow 0$$

now remains exact (because exact sequences of injectives split).

Let W be the class of quasi-isomorphisms in $C(\mathcal{A})$. Actually one can form a functor to a category $\iota : C(\mathcal{A}) \to C(\mathcal{A})[W^{-1}]$ which is universal (see 4.1) w.r.t. to functors with the property that all quasi-isomorphisms become isomorphisms. Considering F as a functor $C(\mathcal{A}) \to C(\mathcal{B})[W^{-1}]$, its property of being not exact can be expressed by saying that quasi-isomorphisms are not mapped to isomorphisms.

One can even for an *arbitrary* (bounded below; otherwise see 6.3) complex $X_\bullet$ find a quasi-isomorphism

$$X_\bullet \overset{q.i.}{\hookrightarrow} I_{X,\bullet},$$

where $I_{X,\bullet}$ is a complex of injectives. In Remark 4.1 it is explained that RF defined as $X_\bullet \mapsto F(I_{X,\bullet})$ can be seen as a functor $C(\mathcal{A})[W^{-1}] \to C(\mathcal{B})[W^{-1}]$ together with a universal (see 4.3) natural transformation $F \to \iota RF$. It is uniquely determined (up to unique isomorphism) by this property and is called the **total right derived functor** of F.

1.2. *What properties does RF have, reminiscent of the fact that F was left exact, and which imply, for example, the existence of the long exact sequence of right derived functors?*

The problem is that an intrinsic notion of exact sequence does not exist in $C(\mathcal{A})[W^{-1}]$ anymore. If

$$0 \longrightarrow X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

is an exact sequence of chain complexes and

$$0 \longrightarrow X'_\bullet \longrightarrow Y'_\bullet \longrightarrow Z'_\bullet \longrightarrow 0$$

is another sequence, isomorphic to the previous *in* $C(\mathcal{A})[W^{-1}]$, then the latter does not need to be exact, and the long exact sequence of the former cannot be reconstructed. Since in $C(\mathcal{A})[W^{-1}]$ both sequences are isomorphic, it is not clear how to proceed. The classical approach of **triangulated categories** starts from the observation that the connecting morphism

$$H_i(Z_\bullet) \xrightarrow{\delta} H_{i+1}(X_\bullet)$$

actually lifts to a morphism in $C(\mathcal{A})[W^{-1}]$ (we will review this in 1.7)

$$Z_\bullet \xrightarrow{\widetilde{\delta}} X_\bullet[1],$$

where $(X_\bullet[1])_i$ is X_{i+1}. The "old" strategy of Grothendieck and Verdier [31] is to not consider the bare category $C(\mathcal{A})[W^{-1}]$ alone but to equip it with the following *additional structures*:

1. **Shift functors** $[+1], [-1] : C(\mathcal{A})[W^{-1}] \to C(\mathcal{A})[W^{-1}]$.
2. A distinguished set of sequences (so called **distinguished triangles**) isomorphic to

   $$X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \xrightarrow{\widetilde{\delta}} X_\bullet[1]$$

   coming from an exact sequence of complexes.

The answer to the question raised is that RF is *exact* in the sense that it maps distinguished triangles to distinguished triangles. Observe that this implies the existence of the long exact sequence of right derived functors.
Grothendieck and Verdier found (cumbersome but workable) axioms that these two extra structures should satisfy. One of them states that *for any* morphism $\alpha : X_\bullet \to Y_\bullet$ there should be a triangle

$$X_\bullet \longrightarrow Y_\bullet \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow X_\bullet[1]$$

completing it. (That means in our situation in particular that any morphism has to be quasi-isomorphic to a monomorphism, which we will review shortly). Another axiom states that $\mathrm{Cone}(\alpha)$ is up to (non-unique!) isomorphism determined by α. A drawback of this approach is that $\mathrm{Cone}(\alpha)$ is *not* functorial in α, i.e. there is no functor

$$\mathrm{Cone} : \mathrm{Fun}(\to, C(\mathcal{A})[W^{-1}]) \to C(\mathcal{A})[W^{-1}].$$

A further problem with using triangulated categories is that, for example, fundamental constructions like the formation of the *total complex* of a complex of complexes can *not* be performed in $C(\mathcal{A})[W^{-1}]$ equipped with its triangulated structure alone. Considering a complex of complexes as a sequence of objects in $C(\mathcal{A})[W^{-1}]$ looses most information.

1.3. The idea of **derivators** starts from the observation that actually the category $C(\mathcal{A})[W^{-1}]$, its triangulated structure, and also the derived functors, abstractly depend only on the pair $(C(\mathcal{A}), W)$ in a sense that we will now describe. For $C(\mathcal{A})[W^{-1}]$ this is explained in 4.1 (cf. also 4.9), and RF for a functor F does only depend on $(C(\mathcal{A}), W)$ by the abstract total-derived-functor-property 4.3 (cf. also 4.1). The fundamental observation is, that also the functors Cone and $[\pm 1]$ *can* be characterized using only $(C(\mathcal{A}), W)$ and accordingly the distinguished triangles can be defined. The functor Cone is actually just the derived functor of the cokernel[1], as we will now explain.
Consider the classical *snake lemma*: A commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \end{array}$$

yields an exact sequence:

$$0 \longrightarrow \ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow \operatorname{coker}(\beta) \longrightarrow \operatorname{coker}(\gamma) \longrightarrow 0$$

If we consider coker as a functor

$$\operatorname{coker} : \operatorname{Fun}(\to, C(\mathcal{A})) \to C(\mathcal{A})$$

(note that the left hand side is again the category of chain complexes of an abelian category, namely of $\operatorname{Fun}(\to, \mathcal{A})$) then the snake lemma suggests the assertion

$$L^1 \operatorname{coker} = \ker \qquad L^i \operatorname{coker} = 0 \text{ for all } i > 1.$$

But how do we compute $L \operatorname{coker}$, i.e. the total derived functor of coker in the sense of Defintion 4.3? Lemma 4.7 tells us that we have to replace $\alpha : X_\bullet \to Y_\bullet$ (functorially) by a quasi-isomorphic morphism $Q\alpha : X'_\bullet \to Y'_\bullet$ such that $\operatorname{coker} \circ Q$ maps quasi-isomorphisms to quasi-isomorphisms. This would be the case if $Q\alpha : X'_\bullet \to Y'_\bullet$ consists of monomorphisms:

Lemma 1.4. *Consider a diagram of chain complexes*

$$\begin{array}{ccc} X_\bullet & \overset{\alpha}{\hookrightarrow} & Y_\bullet \\ \downarrow{\scriptstyle q.i.} & & \downarrow{\scriptstyle q.i.} \\ X'_\bullet & \overset{\alpha'}{\hookrightarrow} & Y'_\bullet \end{array}$$

where the horizontal maps consists of monomorphisms and the vertical maps are quasi-isomorphisms. Then the induced morphism $\operatorname{coker}(\alpha) \to \operatorname{coker}(\alpha')$ *is a quasi-isomorphism.*

[1] or, up to shift, the derived functor of the kernel

Proof. The morphism of exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\bullet & \xrightarrow{\alpha} & Y_\bullet & \longrightarrow & \operatorname{coker}(\alpha)_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X'_\bullet & \xrightarrow{\alpha'} & Y'_\bullet & \longrightarrow & \operatorname{coker}(\alpha')_\bullet & \longrightarrow & 0 \end{array}$$

gives rise to a morphism of long exact sequences

$$\begin{array}{ccccccccc} H_i(X_\bullet) & \longrightarrow & H_i(Y_\bullet) & \longrightarrow & H_i(\operatorname{coker}(\alpha)_\bullet) & \longrightarrow & H_{i+1}(X_\bullet) & \longrightarrow & H_{i+1}(Y_\bullet) \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ H_i(X'_\bullet) & \longrightarrow & H_i(Y'_\bullet) & \longrightarrow & H_i(\operatorname{coker}(\alpha)_\bullet) & \longrightarrow & H_{i+1}(X'_\bullet) & \longrightarrow & H_{i+1}(Y'_\bullet) \end{array}$$

By the 5-Lemma the statement follows. □

1.5. Now we proceed to construct a functorial replacement which consists of monomorphisms: Recall that the cylinder of a morphism

$$\alpha : X_\bullet \to Y_\bullet$$

is the complex

$$\operatorname{Cyl}(\alpha)_\bullet := X_\bullet \oplus Y_\bullet \oplus X[1]_\bullet \qquad d = \begin{pmatrix} d_{X_\bullet} & & -\operatorname{id} \\ & d_{Y_\bullet} & \alpha \\ & & -d_{X_\bullet[1]} \end{pmatrix}$$

It is the colimit of

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\quad\alpha\quad} & Y_\bullet \\ \downarrow{\scriptstyle \delta_0} & & \vdots \\ \Delta_1^\circ \otimes X_\bullet & \cdots\cdots\to & \operatorname{Cyl}(\alpha) \end{array}$$

(see 4.9 for the definition of Δ_1°). This is analogous to the construction in topology (see 3.2) where the name comes from.
There is an injection $X_\bullet \hookrightarrow \operatorname{Cyl}(\alpha)$ induced by δ_1. Furthermore there are morphisms

$$p : \operatorname{Cyl}(\alpha) \to Y_\bullet \qquad \iota : Y_\bullet \to \operatorname{Cyl}(\alpha)$$

where p is induced by the pair of maps $Y_\bullet = Y_\bullet$ and $\Delta_1^\circ \otimes X_\bullet \to X_\bullet \to Y_\bullet$ and ι is the projection $Y_\bullet \to \operatorname{Cyl}(\alpha)$.

Lemma 1.6. *Let $\mathcal{A}$ be an abelian category and consider the pair $(C(\mathcal{A}), \mathcal{W})$. Then the association*

$$(\alpha : X_\bullet \to Y_\bullet) \mapsto (\widetilde{\alpha} : X_\bullet \mapsto \operatorname{Cyl}(\alpha)_\bullet)$$

is a replacement adapted to coker *(see Definition 4.6) which has values in monomorphisms.*

Proof. We have a functorial diagram

$$\begin{array}{ccc} X_\bullet & \hookrightarrow & \mathrm{Cyl}(X_\bullet) \\ \| & & \downarrow p \\ X_\bullet & \longrightarrow & Y_\bullet \end{array}$$

and it is easy to see that p and ι constitute a homotopy equivalence hence p is a quasi-isomorphism. The replacement is adapted to coker by Lemma 1.4. $\square$

1.7. Now consider the exact sequence

$$0 \longrightarrow X_\bullet \longrightarrow \mathrm{Cyl}(\alpha)_\bullet \longrightarrow \mathrm{Cone}(\alpha)_\bullet \longrightarrow 0$$

where $\mathrm{Cone}(\alpha)_\bullet$ is the cokernel, called **cone** of α.
Hence Lemma 1.6 implies finally:

$$\boxed{L\,\mathrm{coker} = \mathrm{Cone}}$$

Explicitly $\mathrm{Cone}(\alpha)$ is the complex:

$$\mathrm{Cone}(\alpha)_\bullet := Y_\bullet \oplus X_\bullet[1] \qquad d = \begin{pmatrix} d_{Y_\bullet} & \alpha \\ & -d_{X_\bullet[1]} \end{pmatrix}$$

Furthermore, if α is a monomorphism then one has a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\bullet & \longrightarrow & \mathrm{Cyl}(\alpha)_\bullet & \longrightarrow & \mathrm{Cone}(\alpha)_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow p & & \\ 0 & \longrightarrow & X_\bullet & \xrightarrow{\alpha} & Y_\bullet & \longrightarrow & Z_\bullet & \longrightarrow & 0 \end{array}$$

whose vertical morphisms are quasi-isomorphisms (for the third use Lemma 1.4). Furthermore the composition (in $C(\mathcal{A})[W^{-1}]$)

$$Z_\bullet \xrightarrow[\sim]{p^{-1}} \mathrm{Cone}(\alpha)_\bullet \xrightarrow{\mathrm{pr}_2} X_\bullet[1]$$

is the sought-for lift of the connecting homomorphisms of the long exact sequence. Finally, we have $X_\bullet[1] = \mathrm{Cone}(X_\bullet \to 0)$.

In particular:

Corollary 1.8. *The distinguished triangles in $C(\mathcal{A})[W^{-1}]$ are precisely the sequences isomorphic to*

$$\left\{ X_\bullet \xrightarrow{\alpha} Y_\bullet \longrightarrow L\,\mathrm{coker}(\alpha) \longrightarrow L\,\mathrm{coker}(X_\bullet \to 0) \right\} \tag{2}$$

where α runs over all *morphisms in $C(\mathcal{A})$.*

Observe that $L\,\mathrm{coker}$ is in fact a functor

$$L\,\mathrm{coker} : \mathrm{Fun}(\to, C(\mathcal{A}))[W_{\to}^{-1}] \to C(\mathcal{A})[W^{-1}]$$

where $W_{\to}$ is the class of morphisms in $\mathrm{Fun}(\to, C(\mathcal{A}))$ which are point-wise a quasi-isomorphism. The first category is in general different from $\mathrm{Fun}(\to, C(\mathcal{A})[W^{-1}])$!

1.9. Dually the functor

$$\ker : \mathrm{Fun}(\to, C(\mathcal{A})) \to C(\mathcal{A})$$

is not exact either. A dual version of Lemma 1.4 shows that a replacement will be adapted to ker if it has values in epimorphisms. The dual construction of the cylinder yields the cocylinder as the kernel of

$$\begin{array}{ccc} \mathrm{coCyl}(\alpha) & \dashrightarrow & X_\bullet \\ \vdots & & \downarrow \\ \underline{\mathrm{Hom}}(\Delta_1^\circ, Y_\bullet) & \xrightarrow{\delta_0} & Y_\bullet \end{array}$$

where $\underline{\mathrm{Hom}}(\Delta_1^\circ, -)$ is the transposed version of $\Delta_1^\circ \otimes -$.
The Fiber or Cocone $F(\alpha)$ of $\alpha : X_\bullet \to Y_\bullet$ is defined as the limit (kernel) of

$$\begin{array}{ccc} F(\alpha) & \dashrightarrow & \mathrm{coCyl}(\alpha) \\ \vdots & & \downarrow \widetilde{\alpha} \\ 0 & \longrightarrow & Y_\bullet \end{array}$$

Hence:

$$\boxed{R\ker = F}$$

If we carry out the computation of the explicit complex we get that

$$F(\alpha)[1] \cong \mathrm{Cone}(\alpha)$$

and Exercise 1.2 shows that the class of sequences (isomorphic to)

$$\left\{ R\ker(0 \to Y) \longrightarrow R\ker(\alpha) \longrightarrow X_\bullet \xrightarrow{\alpha} Y_\bullet \right\} \tag{3}$$

is also equal to the class of distinguished triangles — or equivalently to the class of sequences (2). Later, in the language of derivators, this fact will be encoded in the notion of **stability**. Note the analogy with the definition of an abelian category.

Exercises

Exercise 1.1. *Prove explicitly that the class of sequences isomorphic to (2) is the same as the class of sequences isomorphic to (3).*

Exercise 1.2. *Prove that the composition*

$$Z_\bullet \xrightarrow[\sim]{p^{-1}} \mathrm{Cone}(\alpha)_\bullet \xrightarrow{\mathrm{pr}_2} X_\bullet[1]$$

is a lift of the connecting homomorphism of the long exact sequence.

2 Derivators

References: [7, 12, 13, 18, 19, 23, 25]

2.1. All additional structures of the triangulated structure on the derived category are determined via the notions of homotopy kernel and cokernel by the pair $(C(\mathcal{A}), W)$. Therefore we could take such pairs as objects of our theory. This is what we do, in principle, when we work with *model categories*. There are however two problems:

- Different pairs $(\mathcal{C}, W)$ might lead to the same triangulated category and even to the same "homotopy theory" altogether. Considering in addition a model category structure makes this issue even worse.

- There might be interesting contexts where there is a triangulated category and a "homotopy theory" but they do not arise from any apparent pair $(\mathcal{C}, W)$.

Grothendieck's idea of a **derivator** starts from the observation that the only thing that we used of the pair $(\mathcal{C}, W)$ was the notion of derived kernel and cokernel. It turns out that to prove all the axioms of a triangulated category and to do many more constructions which are not possible in the world of triangulated categories, all we need are more general homotopy limits and colimits (and also homotopy left and right Kan extensions).

2.2. Let $(\mathcal{C}, W)$ be a localizing pair (see section 4). Let I be a diagram (i.e. a small category). Consider the category $\mathrm{Fun}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$ and natural transformations between them. To each object X in $\mathcal{C}$ we may associate the constant functor $c(X) : i \mapsto X$. By definition the limit is the right adjoint functor to c and the colimit is the left adjoint functor to c:

$$\mathrm{Fun}(I,\mathcal{C}) \underset{\lim}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathcal{C}. \quad (c : \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C}))$$

In other words lim and colim are equipped with functorial isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, D, X) &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(D, c(X)), \\ \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(c(X), D) &\cong \mathrm{Hom}_{\mathcal{C}}(X, \lim D). \end{aligned}$$

The class W distinguishes a class of morphisms W_I in $\mathrm{Fun}(I, \mathcal{C})$, too, which consists of those natural transformations $w : D_1 \to D_2$ such that $w(i) \in W$ for all $i \in I$. We also say w is point-wise in W. Again c induces a functor $\mathcal{C}[W^{-1}] \to \mathrm{Fun}(I, \mathcal{C})[W_I^{-1}]$. The **homotopy limit** is just its right adjoint and the **homotopy colimit** is its left adjoint:

$$\mathrm{Fun}(I,\mathcal{C})[W_I^{-1}] \underset{\mathrm{holim}}{\overset{\mathrm{hocolim}}{\rightleftarrows}} \mathcal{C}[W^{-1}]. \quad (c : \mathcal{C}[W^{-1}] \to \mathrm{Fun}(I,\mathcal{C})[W_I^{-1}])$$

Proposition 2.3. *If the limit* $\lim : \mathrm{Fun}(\mathcal{C}, I) \to \mathcal{C}$ *exists and has an absolute right derived functor* $R\lim$ *(see Definition 4.3) then*

$$\mathrm{holim} = R\lim .$$

If the colimit $\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, I) \to \mathcal{C}$ *exists and has an absolute left derived functor* $L\,\mathrm{colim}$ *(see Definition 4.3) then*

$$\mathrm{hocolim} = L\,\mathrm{colim} .$$

Proof. This is but a consequence of the derived adjunction proposition 2.6 below. □

This also encompasses $L\,\mathrm{coker}$, resp. $R\ker$ of before:

Lemma 2.4. *For* $(C(\mathcal{A}), W)$ *we have:*

$$L\,\mathrm{colim}\begin{pmatrix} X_\bullet \longrightarrow Y_\bullet \\ \downarrow \qquad\qquad \\ 0 \qquad\qquad \end{pmatrix} = L\,\mathrm{coker}(X_\bullet \to Y_\bullet)$$

(and similarly for $R\ker$*).*

Proof. Since

$$\mathrm{colim}\begin{pmatrix} X_\bullet \longrightarrow Y_\bullet \\ \downarrow \qquad\qquad \\ Z_\bullet \qquad\qquad \end{pmatrix} = \mathrm{coker}(X_\bullet, Y_\bullet \oplus Z_\bullet)$$

the replacement

$$\begin{pmatrix} X_\bullet \xrightarrow{\alpha} Y_\bullet \\ \downarrow \qquad\qquad \\ Z_\bullet \qquad\qquad \end{pmatrix} \mapsto \begin{pmatrix} X_\bullet \xhookrightarrow{\widetilde{\alpha}} \mathrm{Cyl}(\alpha) \\ \downarrow \qquad\qquad\qquad \\ Z_\bullet \qquad\qquad\qquad \end{pmatrix}$$

is adapted to colim. For $X_\bullet \hookrightarrow \mathrm{Cyl}(\alpha) \oplus Z_\bullet$ is still a monomorphism. □

2.5. For a functor $\alpha : I \to J$ of diagrams, we get a functor $\alpha^* : \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ by composition with α. It induces again a functor

$$\alpha^* : \mathrm{Fun}(J, \mathcal{C})[W_J^{-1}] \to \mathrm{Fun}(I, \mathcal{C})[W_I^{-1}].$$

The left (resp. right) adjoint to this functor is called a **homotopy left (resp. right) Kan extension**. Note that the homotopy limit (colimit) is the special case of the projection $p : I \to \{\cdot\}$. It turns out that the knowledge of all homotopy left and right Kan extensions suffices to prove that $\mathcal{C}[W^{-1}]$ is a triangulated category. All axioms of a triangulated category follow as theorems from rather simple axioms concerning these homotopy Kan extensions. Conversely, however, homotopy left and right Kan extensions cannot be reconstructed in the triangulated category alone. Grothendieck proposes therefore to consider the whole strict 2-functor

$$\mathbb{D} : I \mapsto \mathrm{Fun}(I, \mathcal{C})[W_I^{-1}]$$

as the fundamental datum, where I runs over all diagrams (small categories), together with functors α^* for all $\alpha : I \to J$ (and together with a 2-functoriality for natural transformations $\alpha \Rightarrow \beta$). Such a datum will be called a **derivator** provided certain axioms hold (in particular left and right adjoints for the α^* should exist!). We will see them in detail in the lectures of Moritz Groth. For convenience we will list them below. Note that *no extra data* is needed to reconstruct the triangulated structure.

Above we used the following:

Proposition 2.6. *Let $(\mathcal{C}, W_{\mathcal{C}})$ and $(\mathcal{D}, W_{\mathcal{D}})$ be localizing pairs and*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be an adjunction. If F has an absolute left derived functor LF and G has an absolute right derived functor RG (see Definition 4.3) then

$$\mathcal{C}[W^{-1}] \underset{RG}{\overset{LF}{\rightleftarrows}} \mathcal{D}[W^{-1}]$$

is an adjunction.

Proof. By Lemma 4.8 the total right derived functor of $\mathrm{Hom}_{\mathcal{D}}(FX, -)$ is $\mathrm{Hom}_{\mathcal{D}[W^{-1}]}(FX, -)$. Therefore by Exercise 4.2 the total right derived functor of $\mathrm{Hom}_{\mathcal{D}}(F-, -)$ is $\mathrm{Hom}_{\mathcal{D}[W^{-1}]}(LF-, -)$ because LF is absolute. By the same reasoning it is equal to $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(-, RG-)$. The statement follows because of the uniqueness of total derived functors (up to unique isomorphism). □

2.7. We'll list here the axioms of a derivator for convenience: A **pre-derivator** is just an association (technically: a strict 2-functor) as mentioned above

$$\mathbb{D} : \mathrm{Dia} \to \mathcal{CAT}$$

from a category of diagrams (could be all small categories for example) to the "category" of categories.
It is called a **left derivator** if the following axioms hold true:

(Der1) For I, J in Dia, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence. Moreover $\mathbb{D}(\varnothing)$ is not empty.

(Der2) For $I \in \mathrm{Dia}$ the 'underlying diagram' functor

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Hom}(I, \mathbb{D}(\cdot))$$

is conservative.

(Der3 left) For each morphism $\alpha : I \to J$ in Dia the functor

$$\alpha^* : \mathbb{D}(J) \to \mathbb{D}(I)_{\alpha^* S}$$

has a left-adjoint $\alpha_!$ (*Existence* of homotopy left Kan extensions, in particular, homotopy colimits!).

(Der4 left) For each morphism $\alpha : I \to J$ in Dia, an object $j \in J$, and the 2-cell (see appendix A.3 and A.9)

$$\begin{array}{ccc} I \times_{/J} j & \xrightarrow{\iota} & I \\ {\scriptstyle \alpha_j}\downarrow & \swArrow^{\mu} & \downarrow{\scriptstyle \alpha} \\ \{j\} & \xhookrightarrow{j} & J \end{array}$$

we get that the induced natural transformation of functors $\alpha_{j!}\iota^* \to j^*\alpha_!$ is an isomorphism.

There are obvious dual variants of (Der3–4 left) defining a right derivator. If it is both left and right, we just call it a **derivator**.
Axioms (Der1) and (Der2) are rather technical and avoid pathological situations. There are immediately clear in the examples. Axiom (Der3) states the existence of homotopy (co)limits and homotopy right (left) Kan extensions. Axiom (Der4) is a derivator version of Kan's formula (A.9) computing Kan extensions in usual (co)complete categories. It is therefore very natural to impose. It is striking that all axioms of a triangulated category follow as easy propositions from these axioms, and the following two[2]:
We call $\mathbb{D}$ **pointed** if the following axiom holds:

(Der6) The category $\mathbb{D}(\cdot)$ has a zero object.

We call $\mathbb{D}$ **stable** if the following axiom holds:

(Der7) In the category $\mathbb{D}(\Box)$ an object is homotopy cartesian if and only if it is homotopy cocartesian.

Roughly, if we imagine an object in $\mathbb{D}(\Box)$ as a square of objects in $\mathbb{D}(\cdot)$ (by which, of course, we would loose its coherence!)

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

[2]plus strongness, a technical condition which we won't state here.

homotopy cartesian means that D is the homotopy colimit of the upper left diagram $\ulcorner$ and **homotopy cocartesian** means that A is the homotopy limit of the lower right diagram $\lrcorner$. Details will be given in the lectures of Moritz Groth, cf. also [12].

3 Algebraic topology

References: [14, 26, 27]

3.1. There is a strong similarity between the derived category of an abelian category and the homotopy category of topological spaces (resp. spectra) as we will explain in this section. As in section 1, we are given a localizing pair $(\mathcal{TOP}, W)$, where $\mathcal{TOP}$ is the category of topological spaces and W is the class of **weak equivalences**, i.e. the class of those continuous maps $\alpha : X \to Y$ which induce isomorphisms of $\pi_i(X, x) \to \pi_i(Y, \alpha(x))$ for all i and $x \in X$. This leads again to notions of derived functors and in particular of homotopy limit and colimit. It turns out that homotopy limit and colimit exist in this context, too, and are related to fundamental constructions in algebraic topology. Also with the pair $(\mathcal{TOP}, W)$ there is an associated derivator, which however is *not stable*, in particular, the classes of sequences (4) and (5) are not equal. However, passing to the theory of spectra, we get a stable derivator. Thus also their homotopy category is a triangulated category. We will, however, not discuss spectra in this lecture. In this section, we consider the case of *pointed* topological spaces $(\mathcal{TOP}_*, W)$[3] and examine first again the notion of homotopy kernel and cokernel.
We will use the fact that $(\mathcal{TOP}_*, W)$ is a localizer. Later, when we discuss the abstraction of both of the situations to model categories, we will see a reason for this.
As in the case of chain complexes, the cokernel as map from

$$\mathrm{Fun}(\to, \mathcal{TOP}_*) \to \mathcal{TOP}_*$$

is not exact, i.e. it maps point-wise weak equivalences not necessarily to weak equivalences. Hence we may try to construct a *total left derived functor* in the sense of 4.3. We will see that this is completely analogous to the construction of $L\,\mathrm{coker}$ for chain complexes.

3.2. Again there is the notion of cylinder and cone associated with a map $\alpha : X \to Y$. The cylinder is defined as the colimit of

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow & & \vdots \\ I \times X & \cdots\cdots\to & \mathrm{Cyl}(\alpha) \end{array}$$

where I is the intervall $[0, 1] \subset \mathbb{R}$ and the vertical map is the injection $x \mapsto (0, x)$. There is a map $\widetilde{\alpha} : X \to \mathrm{Cyl}(\alpha)$ given by $x \mapsto (1, x)$. The cone $\mathrm{Cone}(\alpha)$ is defined by the pushout of

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{\alpha}} & \mathrm{Cyl}(\alpha) \\ \downarrow & & \vdots \\ 0 & \cdots\cdots\to & \mathrm{Cone}(\alpha) \end{array}$$

[3] where the maps in W do, of course, have to respect the base point

where 0 is the one point space (it is a zero object of $\mathcal{TOP}_*$). Observe that this is literally the same construction as in 1.5 except that $\Delta_1^\circ \otimes -$ has been replaced by $I \times -$.
Furthermore there are morphisms

$$p : \mathrm{Cyl}(\alpha) \to Y \qquad \iota : Y \to \mathrm{Cyl}(\alpha)$$

where p is induced by the pair of maps $Y = Y$ and $X \times I \to X \to Y$ and ι is the projection $Y \to \mathrm{Cyl}(\alpha)$.

Again, we claim that

Proposition 3.3. *The replacement*

$$(\alpha : X \to Y) \mapsto (\widetilde{\alpha} : X \to \mathrm{Cyl}(\alpha))$$

is a left replacement functor for coker*.*

It follows again that

$$\boxed{L\,\mathrm{coker} = \mathrm{Cone}}$$

Proof. Again there is the morphism

$$\begin{array}{ccc} X & \hookrightarrow & \mathrm{Cyl}(X) \\ \| & & \downarrow p \\ X & \longrightarrow & Y \end{array}$$

and again homotopic maps become equal in $\mathcal{TOP}_*[W^{-1}]$ (see **??**), hence in particular homotopy equivalences are in W. For the fact that the replacement is adapted to coker, I do not know of a completely elementary proof. The proof from homological algebra 1.4 does not translate to this setting, because the sequence $X \to \mathrm{Cyl}(X) \to \mathrm{Cone}(X)$ does *not* induce a long exact sequence of homotopy groups. For the dual construction of homotopy kernels, however, the same proof works, as we will shortly see. A partial proof of this Proposition will be given using the machinery of model categories, see Proposition 6.10. □

Remark 3.4. *Again we have:*

$$\mathrm{hocolim}\left(\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \\ 0 & & \end{array}\right) = L\,\mathrm{coker}(X \to Y)$$

which will, for example, follow from Proposition 6.10, or the explicit description of hocolim *over this diagram in 7.8.*

3.5. Dually, as in the case of complexes, the kernel as map from

$$[\to, \mathcal{TOP}_*] \to \mathcal{TOP}_*$$

is not exact, i.e. it maps point-wise weak equivalences not necessarily to weak equivalences. Hence we may consider its *right derived functor* in the sense of 4.3.
The **cocylinder** (or **mapping path space**) of $X \to Y$ is defined to be the limit

$$\begin{array}{ccc} \mathrm{coCyl}(\alpha) & \dashrightarrow & X \\ \vdots & & \downarrow \\ \underline{\mathrm{Hom}}(I,Y) & \xrightarrow{\delta_0} & Y \end{array}$$

where $\underline{\mathrm{Hom}}(I,Y)$ is the path space of Y.
The Fiber or Cocone $F(\alpha)$ of $\alpha : X \to Y$ is defined to be the limit (kernel) of

$$\begin{array}{ccc} F(\alpha) & \dashrightarrow & \mathrm{coCyl}(\alpha) \\ \vdots & & \downarrow \widetilde{\alpha} \\ 0 & \longrightarrow & Y \end{array}$$

where $\widetilde{\alpha}$ is induced by the map $\delta_1 : \underline{\mathrm{Hom}}(I,Y) \to Y$. Note that these constructions are completely dual to those of the cylinder and cone.

Again, we claim that

Proposition 3.6. *The replacement*

$$(\alpha : X \to Y) \mapsto (\widetilde{\alpha} : \mathrm{coCyl}(\alpha) \mapsto Y)$$

is a right replacement functor adapted to ker *(see Definition 4.6).*

It follows that

$$\boxed{R\ker = F}$$

Proof. Here we sketch an elementary proof, which is completely analogous to the one for chain complexes.
1. We have the functorial commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow & & \| \\ \mathrm{coCyl}(\alpha) & \xrightarrow{\widetilde{\alpha}} & Y \end{array}$$

where the vertical maps are homotopy equivalences, therefore are in W. The vertical map $X \to \mathrm{coCyl}(\alpha)$ is given by the pair of maps $X \to Y \to Y^I$ (constant path) and $X = X$ respectively. The homotopy inverse is given by the natural map $\mathrm{coCyl}(\alpha) \to X$.
2. Now there is an exact sequence of pointed sets ($\sim$ = morphisms up to homotopy)

$$\mathrm{Hom}(A, F(\alpha))/\sim \longrightarrow \mathrm{Hom}(A, X)/\sim \longrightarrow \mathrm{Hom}(A, Y)/\sim$$

which, for the case $A = S_n$, for $n > 0$, can be shown to give an exact sequence *of groups*

$$\pi_n(F(\alpha)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y).$$

3. Denote $\beta : F(\alpha) \to X$ the canonical map. There is a homotopy equivalence $F(\beta) \to \Omega Y$, where ΩY is the loop space of Y. This gives a map

$$\mathrm{Hom}(A, \Omega Y)/\sim \longrightarrow \mathrm{Hom}(A, F(\alpha))/\sim$$

and an exact sequence

$$\pi_{n+1}(Y) \longrightarrow \pi_n(F(\alpha)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y)$$

Furthermore the fiber of the map $\Omega Y \to F(\alpha)$ is homotopy equivalent to ΩX. Therefore we actually get a long exact sequence of homology groups.

$$\pi_{n+1}(X) \longrightarrow \pi_{n+1}(Y) \longrightarrow \pi_n(F(\alpha)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y)$$

4. An extended version of the 5-lemma (to include the case $n = 0$) shows that if there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\alpha'} & Y' \end{array}$$

where the vertical arrows are in W then also the induced map $F(\alpha) \to F(\alpha')$ is in W. □

3.7. For the pair $(\mathcal{TOP}_*, W)$ we call the class of sequences in $\mathcal{TOP}_*[W^{-1}]$ isomorphic to

$$\left\{ X \xrightarrow{\alpha} Y \longrightarrow L\,\mathrm{coker}(\alpha) \longrightarrow L\,\mathrm{coker}(X \to 0) \right\} \tag{4}$$

cofiber sequences and those isomorphic to

$$\left\{ R\ker(0 \to X) \longrightarrow R\ker(\alpha) \longrightarrow X \xrightarrow{\alpha} Y \right\} \tag{5}$$

fiber sequences. Unlike for the case of chain complexes these classes are not equal! They become equal in the homotopy category of spectra.

Exercises

Exercise 3.1. *Prove that* $F(0 \to X)$ *is the loop space of* X *and that* $\mathrm{Cone}(X \to 0)$ *is the suspension of* X.

4 Localizing categories

References: [24]

Let $\mathcal{C}$ be a category and W a class of morphisms of $\mathcal{C}$. As for rings one can try to construct a functor $\iota : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that all morphisms $\iota(w)$ for $w \in W$ are invertible, and such that ι is universal w.r.t. this property. In other words given any functor F such that $F(w)$ for $w \in W$ is invertible, there is a unique functor F' such that

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle\iota}\downarrow & \nearrow_{\exists! F'} & \\ \mathcal{C}[\mathcal{W}^{-1}] & & \end{array}$$

is commutative (on the nose, not up to isomorphism of functors).

Proposition 4.1. *$\iota : \mathcal{C} \to \mathcal{C}[W^{-1}]$ with the above universal property exists.*

Proof (sketch). $\mathcal{C}[W^{-1}]$ is defined by

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}[W^{-1}]) &= \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}(\mathcal{C}[W^{-1}]) &= \left\{ \begin{array}{c} \text{finite chains of morphisms in } \mathcal{C}\text{:} \\ X \xleftarrow{\in W} X_1 \longrightarrow X_2 \xleftarrow{\in W} \cdots \longrightarrow X_{n-2} \xleftarrow{\in W} X_{n-1} \longrightarrow Y \end{array} \right\} / \sim \end{aligned}$$

where composition of morphisms is given by composition of chains. Here $\sim$ is an appropriate equivalence relation. It is the finest implying that the obvious map $\iota : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a functor and that for every $w \in W$ the compositions $X \xleftarrow{w} Y \xrightarrow{w} X$ and $Y \xrightarrow{w} X \xleftarrow{w} Y$ are the identity. We leave it to the reader to fill in the details. □

From the construction in the proposition follows that $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y)$ might not be a set but a proper class which is, in this generality, rather impossible to determine. Therefore we make the following definition

Definition 4.2. *The pair $(\mathcal{C}, W)$ is called a* **localizing pair** *whenever $\mathcal{C}[W^{-1}]$ is locally small and we have $w \in W$ if and only if $\iota(w)$ is an isomorphism.*

The second condition has been added for convenience. Obviously one can always enlarge W such that it becomes true without changing $\mathcal{C}[W^{-1}]$. In our examples there will always be means of ensuring that a given pair $(\mathcal{C}, W)$ is localizing, for example, because it will be part of a model structure.
The most fundamental definition about localizing categories is the notion of derived functor:

Definition 4.3. *Let $(\mathcal{C}, \mathcal{W})$ be a localising pair and $F : \mathcal{C} \to \mathcal{D}$ be a functor. A* **right derived functor** *RF of F is a functor together with a natural transformation $\eta : F \Rightarrow RF \circ \iota$*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle \iota} & \nearrow_{RF} & \\ \mathcal{C}[\mathcal{W}^{-1}] & & \end{array}$$

RF is called **total**, *if it is universal in the sense that applying ι and composing with η induces an isomorphism:*

$$\mathrm{Hom}(RF, G) \cong \mathrm{Hom}(F, G \circ \iota)$$

for all functors $G : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$.
RF is called **absolute** *if $(\kappa \circ RF, \kappa * \eta)$ is a total right derived functor of $\kappa \circ F$ for any category $\mathcal{E}$ and functor $\kappa : \mathcal{D} \to \mathcal{E}$.*

Note that a total right derived functor RF is characterized up to a unique natural isomorphism by its defining property and moreover RF is actually a *left* Kan extension of F along ι (A.9).

Remark 4.4. *If F maps elements of W already to isomorphisms then by the universal property there is a functor F' with $F = F'\iota$. It follows from the definition that F' is an absolute left and right derived functor and η is in both cases the identity.*

Remark 4.5. *If we have two localizing pairs $(\mathcal{C}, \mathcal{W}_\mathcal{C})$ and $(\mathcal{D}, \mathcal{W}_\mathcal{D})$ and a functor $F : \mathcal{C} \to \mathcal{D}$ then by abuse of notation we denote by RF a total derived functor of $\iota \circ F$.*

In most cases RF is computed by means of a replacement functor:

Definition 4.6. *A* **right replacement functor** *adapted to F is a functor $Q : \mathcal{C} \to \mathcal{C}$ together with a natural transformation: $\nu : \mathrm{id}_\mathcal{C} \to Q$ object-wise in W and such that $FQ(w)$ is an isomorphism for each $w \in W$.*

If (P, ν) is a right replacement functor adapted to F then FQ is equal to $\widetilde{FQ}\iota$ for some functor $\widetilde{FQ}$ by the universal property of $\mathcal{C}[\mathcal{W}^{-1}]$.

Lemma 4.7. *$\widetilde{FQ}$ is the absolute right derived functor for F.*

Proof. First of all the morphism $\eta : F \to \widetilde{FQ}$ is given as the composition

$$F \xrightarrow{F\nu} FQ = \widetilde{FQ}\iota.$$

Let $\alpha : F \to G\iota$ be given. For any X we consider the diagram

$$\begin{array}{ccccc} FX & \xrightarrow{F\nu_X} & FQX & = & \widetilde{FQ}X \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_{QX}} & & \\ GX & \xrightarrow{G\iota\nu_X} & GQX & & \end{array}$$

Since $\iota\nu_X$ is invertible in $\mathcal{C}[W^{-1}]$ we get a morphism

$$\widetilde{FQ}X \xrightarrow{\alpha_{QX}} GQX \xrightarrow{G(\iota\nu_X)^{-1}} GX.$$

Obviously this is inverse to the map in the definition of right derived functor.
Note that $\widetilde{FQ}$ is absolute because GFQ is equal to $G\widetilde{FQ}\iota$ and $G\widetilde{FQ}$ is a total derived functor of GF by the first part of the proof. □

Lemma 4.8. *The right derived functors of* $X \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$*,* $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$ *and* $(X,Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$ *considered as functors from* $\mathcal{C} \to \mathcal{SET}$*,* $\mathcal{C}^{op} \to \mathcal{SET}$*, and* $\mathcal{C} \times \mathcal{C}^{op} \to \mathcal{SET}$ *exist and are all equal to* $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}$*.*

Proof. We have

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(\iota(X),\cdot),G) \cong G(\iota(X))$$

and also

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(X,\cdot),G\iota) \cong G(\iota(X))$$

by Yoneda's Lemma. The statement about the functor of two variables follows from Exercise 4.2 considering that $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}$ already maps W in both arguments to isomorphisms. □

Remark 4.9. *Consider the case of chain complexes as in section 1. Actually* $C(\mathcal{A})[W^{-1}]$*, which is called the* **derived category** *of* $\mathcal{A}$*, is usually constructed in a two-step process. First, one defines on* $\mathrm{Hom}(X,Y)$ *an equivalence relation, where two morphisms* f *and* g *are equivalent (or homotopic) if there is a homotopy* $f \Rightarrow g$*, and obtains a category* $K(\mathcal{A})$*. Let us recall what a homotopy is. This is analogously defined as in topology. First, one can "realize" any simplex in the category of bounded chain complexes over* $\mathbb{Z}$*.*

$$(\Delta_i^\circ)_{-n} = \mathbb{Z}[\{ \text{ strictly increasing maps } \{0,\dots,n\} \hookrightarrow \{0,\dots,i\}\}]$$

with boundary maps given by the alternating sum of the compositions with the $n+2$ *maps* δ_n^i : $\{0,\dots,n\} \to \{0,\dots,n+1\}$*. For example*

$$\Delta_0^\circ = \{\mathbb{Z}\} \qquad \Delta_1^\circ = \left\{ \mathbb{Z} \xrightarrow{\begin{pmatrix}-1\\1\end{pmatrix}} \mathbb{Z}^2 \right\} \qquad \Delta_2^\circ = \left\{ \mathbb{Z} \xrightarrow{\begin{pmatrix}1\\-1\\1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} & 1 & 1\\ 1 & & -1\\ -1 & -1 & \end{pmatrix}} \mathbb{Z}^3 \right\}$$

where always the rightmost entry sits in degree 0. This is a reduced version of the chain complex associated with the standard simplices and actually a cosimplicial object in $C^b(\mathbb{Z}-\mathcal{MOD})$ *(cf. 5.5 for details). All* $X \otimes \Delta_i^\circ$ *are quasi-isomorphic to* X*.*

A homotopy $\mu : f \Rightarrow g$ is a map $\mu : X \otimes \Delta_1^\circ \to Y$ such that the two compositions are respectively f and g:

$$X \underset{\delta_1}{\overset{\delta^0}{\rightrightarrows}} X \otimes \Delta_1^\circ \xrightarrow{\mu} Y$$

Now in the diagram

$$X \underset{\delta_1}{\overset{\delta^0}{\rightrightarrows}} X \otimes \Delta_1^\circ \xrightarrow{p} X$$

the composition is the identity and p is a quasi-isomorphism! Therefore the first two morphisms δ_0 and δ_1 are the same in $\mathrm{Hom}_{C(\mathcal{A})[W^{-1}]}(X, X \otimes \Delta_1^\circ)$, therefore also f and g. We conclude that it makes no difference, whether we consider $C(\mathcal{A})[W^{-1}]$ or $K(\mathcal{A})[W^{-1}]$. The advantage of constructing $K(\mathcal{A})$ first is that the chains of morphisms in the construction of $K(\mathcal{A})[W^{-1}]$ are actually all of the form $X \leftarrow X' \to Y$, where the first arrow is a quasi-isomorphism. See e.g. [11]. This is a common behavior if there is a model category structure (see [17]), in which case even X' is w.l.o.g. a fixed replacement of X.

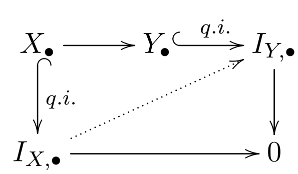
Exercises

Exercise 4.1. *Explain that $RF : C^+(\mathcal{A})[W^{-1}] \to C^+(\mathcal{B})[W^{-1}]$ defined by means of an injective replacement of (bounded below) complexes is actually a functor and even an absolute right derived functor of $\iota \circ F : C(\mathcal{A}) \to C(\mathcal{B})[W^{-1}]$.*

Hint. By Lemma 4.7 we need to check that

1. the injective replacement $Q : X_\bullet \to I_\bullet$ can be made functorial (actually is suffices functoriality up to homotopy because F respects the relation of homotopy of morphisms, hence we may replace $C(\mathcal{A})$ with $K(\mathcal{A})$ from the beginning),
2. $F \circ Q$ maps quasi-isomorphisms to quasi-isomorphisms.

First of all by exercise 4.3 we have a lift in the following diagram:



To show functoriality it suffices to show that two lifts are homotopic. This follows from the existence of a lift in the diagramm

$$\begin{array}{ccc} \frac{I_{X,\bullet}\oplus I_{X,\bullet}}{X_\bullet} & \longrightarrow & I_{Y,\bullet} \\ {\scriptstyle q.i.}\big\downarrow & & \big\downarrow \\ \frac{I_{X,\bullet}\otimes \Delta_1^\circ}{X_\bullet} & \longrightarrow & 0 \end{array}$$

It remains to show that if $I_\bullet \to J_\bullet$ is a quasi-isomorphism between complexes of injectives then $F(I_\bullet) \to F(J_\bullet)$ is a quasi-isomorphism, too. For this use the exact sequence

$$0 \longrightarrow I_\bullet \longrightarrow \mathrm{Cyl}(\alpha)_\bullet \longrightarrow \mathrm{Cone}(\alpha)_\bullet \longrightarrow 0$$

and exercise 4.4. □

Exercise 4.2. *Consider two localizing pairs* $(\mathcal{C}_i, \mathcal{W}_i)$, $i = 1, 2$ *and a functor*

$$F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}.$$

Suppose that the total right derived functor $R(F_Y)$ *of the functor* $F_Y : X \mapsto F(X, Y)$ *exists for all* Y*. Denote* $R_X F : Y \mapsto R(F_Y)$*. Show that a total right derived functor of* $R_X F$ *is also a total right derived functor of* F*, i.e.* $R_Y R_X F = RF$*.*
Suppose that the derived functor $R(F_X)$ *of the functor* $F_X : Y \mapsto F(X, Y)$ *exists for all* X*. Denote* $R_Y F : X \mapsto R(F_X)$*. Show that a total right derived functor of* $R_Y F$ *is also a total right derived functor of* F*, i.e.* $R_X R_Y F = RF$*.*

Exercise 4.3. *Let* $I_\bullet$ *be a bounded below complex of injectives in an abelian category. Prove that* $I_\bullet \to 0$ *has the right lifting property w.r.t. monomorphisms which are quasi-isomorphisms, i.e. for any square*

$$\begin{array}{ccc} X_\bullet & \longrightarrow & I_\bullet \\ {\scriptstyle q.i.}\big\downarrow & & \big\downarrow \\ Y_\bullet & \longrightarrow & 0 \end{array}$$

there is a lift as indicated.

Exercise 4.4. *Let* $I_\bullet$ *be an acyclic bounded below complex of injectives in an abelian category and let* F *be a left exact functor. Prove that also* $F(I_\bullet)$ *is acyclic.*

5 General homotopy limits and colimits — explicit construction

References: [1, 7, 10]

In this section we show how general homotopy (co)limits may be computed for a localizing pair $(\mathcal{C}, W)$ provided that $\mathcal{C}$ is *roughly* a simplicial category which is tensored and cotensored, and such that some axioms hold. The pairs considered in sections 1 and 3 are of this form. Later we will see more abstract ways of constructing homotopy limits and colimits using model categories. In particular, in this section, we prove *without* using model categories that the category $C(\mathcal{A})$ of unbounded chain complexes in an abelian category admits homotopy limits and colimits, as well as left and right Kan extensions.
To show the existence of the left adjoints (colimit, left Kan extension) we have to assume that $\mathcal{A}$ satisfies axiom (AB4) and for the existence of the right adjoints that $\mathcal{A}$ satisfies axiom (AB4*)[4].

5.1. Let I be a diagram and $F : I \to C(\mathcal{A})$ be a functor.
We define a double complex $C_I(F)_{\bullet,\bullet}$ as follows:

$$C_I(F)_{-p,q} := \bigoplus_{\mu:\Delta_p\to I} F(\mu(0))^q$$

for $p \geq 0$. Here Δ_p is the category $0 \to 1 \to \cdots \to p$. The vertical differential is given by d on the complexes $F(i)$, and the horizontal by the boundary operations

$$\begin{array}{rcl} C_I(F)^{-p,q} & \to & C_I(F)^{-p+1,q} \\ (x)_\mu & \mapsto & \sum_{k=0}^{p} (-1)^k (F(\mu(0 \to \delta_k(0)))x)_{\mu\delta_k} \end{array}$$

where $\delta_0, \dots, \delta_p$ are all injective functors $\Delta_{p-1} \to \Delta_p$, numbered such that δ_k omits k. Define

$$\mathrm{Cone}_I(F)_\bullet := \mathrm{Tot}^{\oplus}(C_I(F)_{\bullet,\bullet}).$$

One goal of this section is to explain

Theorem 5.2. *If $\mathcal{A}$ is an (AB4) abelian category, I is a diagram and $F \in \mathrm{Fun}(I, C(\mathcal{A}))$ then*

$$\mathrm{hocolim}_I(F) = \mathrm{Cone}_I(F),$$

i.e. the functor $F \mapsto \mathrm{Cone}_I(F)$ is left-adjoint to $p^* : C(\mathcal{A})[W^{-1}] \to \mathrm{Fun}(I, C(\mathcal{A}))[W_I^{-1}]$.

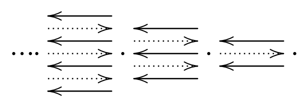
[4] For *coherently* bounded below (lim-case), resp. *coherently* bounded above (colim-case) diagrams of complexes one doesn't need these assumptions.

In the exercises (cf. 5.9) a brute-force proof of this result is given. In the sequel, we will sketch a bit more conceptual, yet elementary proof (which however assumes also AB4*) and covers also the case of topological spaces. In section 7 we will review the construction from the point of view of model categories.
The nice feature is that the construction is actually functorial in diagrams on the nose (i.e. not only in the derived category), see 5.17 and also Exercise 5.6

5.3. To proceed, we review a tiny bit from the general theory of simplicial sets and (co)simplicial objects in categories: Let Δ be the category

$$\begin{aligned} \mathrm{Ob}(\Delta) &= \{\Delta_n := \{0,1,\dots,n\} \mid n \in \mathbb{N}_0\} \\ \mathrm{Mor}(\Delta) &= \{\text{order preserving maps } \Delta_n \to \Delta_m\} \end{aligned}$$

which may be depicted schematically as:



(where only some morphisms are depicted).
A **simplicial object** in a category is a functor $X : \Delta^{op} \to \mathcal{C}$, in particular a **simplicial set** is a functor $X : \Delta^{op} \to \mathcal{SET}$. A **cosimplicial object** in a category is a functor $A : \Delta \to \mathcal{C}$.
If $\mathcal{C}$ is complete, any cosimplicial object A defines an adjunction[5]:

$$\mathcal{SET}^{\Delta^{op}} \underset{N_A}{\overset{R_A}{\rightleftarrows}} \mathcal{C}$$

(R_A is the left adjoint) where the functors are defined by (cf. section A.6 in the appendix)

$$\begin{aligned} R_A : S &\mapsto \int^n (\coprod_{S_n} A_n) \\ N_A : C &\mapsto (\Delta_n \mapsto \mathrm{Hom}(A_n, C)) \end{aligned}$$

R_A is uniquely determined by the property of preserving colimits and mapping Δ_n (representable simplicial set) to A_n. This follows from Exercise A.7 in the appendix.

Example 5.4. *If $\mathcal{C} = \mathcal{TOP}$, there is a canonical cosimplicial object*

$$A : \Delta_n \mapsto \{\textit{standard } n\textit{-simplex in } \mathbb{R}^n\ \}$$

The associated functor

$$R : \mathcal{SET}^{\Delta^{op}} \to \mathcal{TOP}$$

[5] R stands for **realization** and N for **nerve**.

is called the geometric realization. Its right adjoint

$$N : \mathcal{TOP} \to \mathcal{SET}^{\Delta^{op}}$$

is the singular simplicial set associated with a topological space. These actually can be used to define a Quillen equivalence between associated model categories (see 6.8).

Example 5.5. *If $\mathcal{C} = \mathbb{Z} - \mathcal{MOD}$ there is also a canonical functor*

$$R : \mathcal{SET}^{\Delta^{op}} \to \mathbb{Z} - \mathcal{MOD}$$

$$X \mapsto R(X) \quad R(X)_n := \mathbb{Z}[X_{-n}] \qquad \text{for } n \le 0$$

with differential given for $x \in X_n$ by

$$[x] \mapsto \sum_{i=0}^{n} (-1)^i [\delta_i^n x]$$

from which a cosimplicial object $A : \Delta_n \mapsto R(\Delta_n)$ may be reconstructed. The $R(\Delta_n)$ are however a bit different from the Δ_n° of 4.9. In contrast to the latter the $R(\Delta_n)$ are unbounded *(but bounded above) complexes. In this section it will be convenient to work with the unbounded version.*

Example 5.6. *If $\mathcal{C} = \mathcal{SCAT}$ there is also a canonical (even tautological) simplicial object given by*

$$A : \Delta_n \mapsto \Delta_n = (0 \to 1 \to \cdots \to n).$$

The right adjoint functor N_A is called the **nerve***.*

$$\mathbf{N} : \mathcal{CAT} \to \mathcal{SET}^{\Delta^{op}}$$

$$\mathbf{N}(A)_n := \{x_0 \to \cdots \to x_n \mid x_i \in \mathrm{Ob}(A)\}$$

The functor $\mathbf{N}$, and its left adjoint $\mathbf{R}$, can be used to define a Quillen equivalence between associated model categories (see 6.8). (As they stand, they do not yet.)

5.7. Let now $\mathcal{C}$ be either $\mathcal{TOP}$ or $C(\mathcal{A})$. Recall our canonical cosimplicial object $A \in \mathcal{TOP}^\Delta$ (see 5.4), resp. $A \in C(\mathbb{Z}-\mathcal{MOD})^\Delta$ (see 5.5). Write $\otimes$ for the natural tensor product with $C(\mathbb{Z}-\mathcal{MOD})$ for chain complexes, and for $\times$ for topological spaces. We actually get an adjunction of two variables given by

$$\begin{aligned} \otimes : \mathcal{SET}^{\Delta^{op}} \times \mathcal{C} &\to \mathcal{C} \\ S, Y &\mapsto R_A(S) \otimes Y \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_r : \mathcal{C}^{op} \times \mathcal{C} &\to \mathcal{SET}^{\Delta^{op}} \\ X, Y &\mapsto (n \mapsto \mathrm{Hom}(A_n \otimes X, Y)) \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_l : (\mathcal{SET}^{\Delta^{op}})^{op} \times \mathcal{C} &\to \mathcal{C} \\ S, Y &\mapsto \underline{\mathrm{Hom}}(R_A(S), Y) \end{aligned}$$

For $\mathcal{TOP}$, $\underline{\mathrm{Hom}}$ is the homomorphism space, and for $C(\mathcal{A})$ the external hom $C(\mathbb{Z})^{op} \times C(\mathcal{A}) \to C(\mathcal{A})$ (transposed version of $\otimes$). In other words:

$$\mathrm{Hom}(S \otimes Y, Z) = \mathrm{Hom}(S, \mathrm{Hom}_r(Y, Z)) = \mathrm{Hom}(Y, \mathrm{Hom}_l(S, Z))$$

This renders $\mathcal{TOP}$ even into a *tensored and cotensored simplicial category.* For $C(\mathcal{A})$ this would be true, if we replaced $R(\Delta_n)$ with the bounded Δ_n°.

5.8. Now, if I is a diagram, we can extend the adjunction of 5.7 in two variables. This is a fairly general process (see Exercise A.5 in the appendix) and is given by:

$$\begin{array}{rcl} \otimes : \mathrm{Fun}(I^{op}, (\mathcal{SET}^{\Delta^{op}})) \times \mathrm{Fun}(I, \mathcal{C}) & \to & \mathcal{C} \\ S, Y & \mapsto & \int^i S(i) \otimes Y(i) \end{array}$$

$$\begin{array}{rcl} \mathrm{Hom}_r : \mathrm{Fun}(I, \mathcal{C})^{op} \times \mathcal{C} & \to & \mathrm{Fun}(I^{op}, \mathcal{SET}^{\Delta^{op}}) \\ X, Y & \mapsto & \{i \mapsto \mathrm{Hom}_r(X(i), Y)\} \end{array}$$

$$\begin{array}{rcl} \mathrm{Hom}_l : \mathrm{Fun}(I^{op}, (\mathcal{SET}^{\Delta^{op}}))^{op} \times \mathcal{C} & \to & \mathrm{Fun}(I, \mathcal{C}) \\ S, Y & \mapsto & \{i \mapsto \mathrm{Hom}_l(S(i), Y)\} \end{array}$$

Example 5.9. *Consider the case $\mathcal{C} = C(\mathcal{A})$. There is the tautological functor $\delta : \Delta \mapsto \mathcal{SET}^{\Delta^{op}}$. Let $X \in \mathrm{Fun}(\Delta^{op}, C(\mathcal{A}))$. We can associate with it a double complex*

$$X_{n,m} := X(\Delta_{-n})_m$$

where the horizontal differential is given by the alternating sum of the differentials δ_i^n. Then we have

$$\delta \otimes X \cong \mathrm{Tot}^{\oplus} X_{\bullet,\bullet}.$$

5.10. Later (cf. section 7) we will consider these adjunctions from the point of view of model categories. To prove the existence of homotopy (co)limits in the category of chain complexes and to get an elementary formula for them we actually only need the following axioms:

1. $\mathrm{Hom}_l(\Delta_0, X)$ and X are weakly equivalent.
2. The maps $\mathrm{Hom}_l(\Delta_i, X) \leftarrow \mathrm{Hom}_l(\Delta_0, X)$ induced by the unique maps $\Delta_i \to \Delta_0$ are weak equivalences.
3. For a simplicial object $X \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ the functor $X \mapsto \delta \otimes X$ maps weak equivalences to weak equivalences.

4. $\operatorname{Hom}_l(S,-)$ maps weak equivalences to weak equivalences for any simplicial set S.

5. Weak equivalences are stable under coproducts.

If $\mathcal{C} = C(\mathcal{A})$ where $\mathcal{A}$ satisfies (AB4) and (AB4*), these properties can be shown in an elementary way, see exercise 5.3. If $\mathcal{C} = \mathcal{TOP}$ the validity of 1.–5. will be shown in section 7 using model categories under the extra assumption that all occurring spaces are *cofibrant* (cf. the next chapter for this notion). Actually, with the language of 7, we can give a much more conceptual proof of the Bousfield-Kan formula.

Theorem 5.11 (Bousfield-Kan)**.** *Let $(\mathcal{C}, W)$ be a localizing pair such that an adjunction like in 5.7 exists with the properties 1.–5. of 5.10. Consider the functor $N(- \times_{/I} I) \in \operatorname{Fun}(I^{op}, \mathcal{SET}^{\Delta^{op}})$. Recall that it is given by (cf. A.3 for the definition of comma category):*

$$N(i \times_{/I} I)_n = \left\{ \begin{array}{c} i \\ \swarrow \quad \swarrow \quad \searrow \\ i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_n \end{array} \middle| \, i_j \in I \right\}$$

Then for all $X \in \operatorname{Fun}(I, \mathcal{C})[W_I^{-1}]$ we have:

$$\operatorname{hocolim}_I X = \int^i N(i \times_{/I} I) \otimes X(i). \qquad (6)$$

Remark 5.12. *Note that (at least for topological spaces):*

$$\operatorname{colim}_I X = \int^i \Delta_0 \otimes X(i).$$

The mechanism behind the Bousfield-Kan formula is therefore: Instead of replacing X by a "good replacement", replace Δ_0 by a "good replacement". We will come back to this point of view in section 7.

Example 5.13. *If $\mathcal{C} = \mathcal{TOP}$ we get:*

$$\operatorname{hocolim}_I X = \int^i R(N(i \times_{/I} I)) \times X(i)$$

where R is the geometric realization, provieded X is point-wise cofibrant (cf. the next chapter for this notion).

Example 5.14. *In the case $\mathcal{C} = C(\mathcal{A})$, we get analogously* $\operatorname{hocolim}_I X_\bullet = \int^i R(N(i \times_{/I} I)) \otimes X_\bullet(i)$, *where R is the realization in $C(\mathbb{Z} - \mathcal{MOD})$. Recall that*

$$R(N(i \times_{/I} I))_n = \bigoplus_{\Delta_{-n} \to i \times_{/I} I} \mathbb{Z} \quad \textit{for } n \leq 0$$

with differential given by the alternating sum of face maps. From this and from the explicit formula of the coend it follows that

$$\operatorname{hocolim}_I X_\bullet = \int^i R(N(i \times_{/I} I)) \otimes X_\bullet(i) = \operatorname{Cone}_I(X_\bullet).$$

5.15. For the proof of Bousfield-Kan, we will need the construction of a category Δ^{op}/S for any simplicial set S, which is the Grothendieck construction applied to the functor $S : \Delta^{op} \to \mathcal{SET} \subset \mathcal{SCAT}$, where sets are considered as discrete categories.

$$\begin{array}{rcl} \operatorname{Ob}(\Delta^{op}/S) & = & \{\Delta_n \in \Delta, x \in X_n\} \\ \operatorname{Hom}_{\Delta^{op}/S}((\Delta_n, x), (\Delta_m, x')) & = & \{\mu : \Delta_m \to \Delta_n \mid \mu(x) = x'\}. \end{array}$$

Proof of theorem 5.11. Call F the functor in the RHS of (6).
Consider the correspondence of functors

$$\begin{array}{ccc} & (\Delta^{op}/\mathbf{N}(I))^{op} & \\ {}^{\iota^{op}}\swarrow & & \searrow^{p^{op}} \\ I^{op} & & \Delta \end{array}$$

where ι is the functor mapping $(\Delta_n, \mu : \Delta_n \to I)$ to $x(0)$ and $p = \Delta^{op}/(\mathbf{N}(I) \to \Delta_0)$ is the projection. By Kan's formula, we have $N(i \times_{/I} I) = (\iota^{op})_!(p^{op})^*\delta$. Here $\delta \in \operatorname{Fun}(\Delta, \mathcal{SET}^{\Delta^{op}})$ is the canonical object which maps Δ_n to the simplicial set represented by Δ_n. Using Exercise A.6 it follows that

$$F(X) := \int^i N(i \times_{/I} I) \otimes X(i) = \int^n \Delta_n \otimes (p_! \iota^* X)(\Delta_n).$$

Since p is a Grothendieck op-fibration $p_!$ is the same as the colimit over the fibers, which are discrete. Therefore by property 3 we have that this functor maps weak equivalences to weak equivalences. Now the functor

$$F : X \mapsto \int^i N(i \times_{/I} I) \otimes X(i)$$

is adjoint to

$$G : Z \mapsto \big(i \mapsto \operatorname{Hom}_l(N(i \times_{/I} I), Z)\big)$$

which both map weak equivalences to weak equivalences and hence coincide with their derived functors. Therefore they also induce an adjuction (Lemma 2.6)

$$\operatorname{Fun}(I, \mathcal{C})[W_I^{-1}] \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}[W^{-1}].$$

It suffices to show that the functor G (as a functor between homotopy categories) is actually isomorphic to the constant functor. Now $N(i \times_{/I} I)$ is homotopy equivalent to Δ_0 in $\mathcal{SET}^{\Delta^{op}}$ (use exercise A.1 in the appendix). Therefore by axiom 2. $\operatorname{Hom}_l(N(i \times_{/I} I), Z)$ is weakly equivalent to $\operatorname{Hom}_l(\Delta_0, Z)$. Therefore by axiom 1. $\operatorname{Hom}(N(i \times_{/I} I), Z)$ is weakly equivalent to Z. □

Example 5.16. *Resuming Example 5.9, we see from the proof of the Bousfield-Kan formula that for $X \in \mathrm{Fun}(\Delta^{op}, C(\mathcal{A}))$ there is the easier formula:*

$$\mathrm{hocolim}\, X = \mathrm{Tot}^{\oplus} X_{\bullet,\bullet}$$

(Compare this to the formula: $\mathrm{colim}\, X = H_0^{horz}(X_{\bullet,\bullet})$*).*

5.17. The nice feature about these explicit formulas is that we can write down left and right Kan extensions *using Kan's formula* as for ordinary categories. Indeed, for a morphism $\alpha : I \to J$ of diagrams, the association

$$F(X) : j \mapsto \int^{\mu} N(\mu \times_{/(I \times_{/J} j)} (I \times_{/J} j)) \otimes X(\iota(\mu))$$

is *functorial in* j and therefore defines an object in $\mathrm{Fun}(J, \mathcal{C})$. Since the right hand side represents the homotopy colimit over $I \times_{/J} j$ of $\iota^* X$, the Kan formula holds automatically, provided we are able to show that this object *is* a homotopy Kan extension.

Theorem 5.18. *Under the hypothesis of Theorem 5.11, F defines a homotopy left Kan extension.*

Proof. We have that

$$\int^{\mu} N(\mu \times_{/(I \times_{/J} j)} (I \times_{/J} j)) \otimes \iota^* X(\mu) = \int^{i} N(i \times_{/I} I \times_{/J} j) \otimes X(\mu)$$

because of Exercise A.6 and the easy calculation $(\iota^{op})_! N(\mu \times_{/(I \times_{/J} j)} (I \times_{/J} j)) = N(i \times_{/I} I \times_{/J} j)$. By elementary properties of ends and coends (Exercise A.5) the functor

$$F : X \mapsto (j \mapsto \int^{i} N(i \times_{/I} I \times_{/J} j) \otimes X(i))$$

is adjoint to

$$G : Z \mapsto (i \mapsto \int_j \mathrm{Hom}_l(N(i \times_{/I} I \times_{/J} j), Z(j)).$$

Now there is a homotopy equivalence of *diagrams of* simplicial sets (cf. also Exercise A.2)

$$(j \mapsto N(i \times_{/I} I \times_{/J} j)) \to (j \mapsto N(i \times_{/J} j))$$

This implies (using the properties 1.–5. of 5.10) that $\int_j \mathrm{Hom}_l(N(i \times_{/I} I \times_{/J} j), Z(j))$ is weakly equivalent to

$$\int_j \mathrm{Hom}_l(N(i \times_{/J} j), Z(j)) = \int_j \prod_{\mathrm{Hom}(\alpha(i), j)} \mathrm{Hom}_l(\Delta_0, Z(j)) = \mathrm{Hom}_l(\Delta_0, Z(\alpha(i))).$$

By property 1. this functor is weakly equivalent to $\alpha^* Z : i \mapsto Z(\alpha(i))$. Since this functor is exact, also G is exact and as derived functor isomorphic to $\alpha^* Z$. □

Corollary 5.19. *Let $(\mathcal{C}, W)$ be a localizing pair such that an adjunction like in 5.7 exists with the properties as in 5.10. Then the association*

$$\mathbb{D} : I \mapsto \mathrm{Fun}(I, \mathcal{C})[W_I^{-1}]$$

defines a left-derivator on all diagrams I.

Remark 5.20. *The assumption (of a localizing pair) that $\mathcal{C}[W^{-1}]$ be locally small was not used in the proof. If one is content with working with derivators which have values in categories which are not necessarily locally small, one can relax this condition.*

We leave it to the reader to state the dual version of the results in this section using a dual version of the properties in 5.10.

Exercises

For the exercises you'll need the following folklore theorem:

Theorem 5.21. *Let $\mathcal{A}$ be an abelian category (cf. A.4).*

1. *Let $X^{\bullet,\bullet}$ be a double-complex in $\mathcal{A}$ which is concentrated in the left semiplane. If A satisfies (AB4) then the (horizontal) spectral sequence*

$$H^q(X^{p,\bullet}) \Rightarrow \mathrm{Tot}^{\oplus, p+q}(X^{\bullet,\bullet})$$

   converges.

2. *Dually (and rotated by $180°$), let $X^{\bullet,\bullet}$ be a double-complex in $\mathcal{A}$ which is concentrated in the right semiplane. If A satisfies (AB4*) then the (horizontal) spectral sequence*

$$H^q(X^{p,\bullet}) \Rightarrow \mathrm{Tot}^{\Pi, p+q}(X^{\bullet,\bullet})$$

   converges.

Exercise 5.1. *Verify the following formula for simplicial sets:*

$$N(i \times_{/I} I) = (\iota^{op})_!(p^{op})^* \delta.$$

Exercise 5.2. *For $X \in \mathrm{Fun}(\Delta^{op}, C(\mathcal{A}))$ check the formula*

$$\delta \otimes X = \mathrm{Tot}^{\oplus} X_{\bullet,\bullet}$$

from Example 5.9. For $X_\bullet \in \mathrm{Fun}(I, C(\mathcal{A}))$ check the formula

$$\int^i R(N(i \times_{/I} I)) \otimes X_\bullet(i) = \mathrm{Cone}_I(X_\bullet)$$

from Example 5.14.

Hint. In both cases an easy explicit computation using the explicit formula A.11 for computing ends as cokernels. □

Exercise 5.3. *Let $\mathcal{A}$ be an (AB4) and (AB4*) abelian category. Prove using Theorem 5.21 that properties 1.–5. of 5.10 hold true. You will need both spectral sequences.*

Exercise 5.4. *Give a meaning to the following statement and prove it: The category of cosimplicial objects* $\mathrm{Fun}(\Delta, \mathcal{C})$ *is equivalent to the category of adjunctions*

$$\mathcal{SET}^{\Delta^{op}} \underset{N}{\overset{R}{\rightleftarrows}} \mathcal{C} \; .$$

The exercises in the sequel give an alternative proof of the existence of homotopy Kan extensions using only (AB4):

Exercise 5.5. *Let $\mathcal{A}$ be an (AB4) abelian category. Prove that for a natural transformation* $\gamma : F \to G$ *we get a natural morphism* $\mathrm{Cone}_I(\gamma) : \mathrm{Cone}_I(F) \to \mathrm{Cone}_I(G)$*. Moreover show that if γ is a point-wise quasi-isomorphism then γ' is a quasi-isomorphism.*

Hint. Use the spectral sequence 1. from Theorem 5.21. □

Exercise 5.6. *Let $\alpha : I \to J$ be a functor between diagrams, and let $F : J \to C(\mathcal{A})$ be a functor. Define a morphism* $\alpha' : \mathrm{Cone}_I(\alpha^* F) \to \mathrm{Cone}_J(F)$ *which is functorial in F.*

Exercise 5.7. *Let $\alpha, \beta : I \to J$ be two functors between diagrams, let $F : J \to C(\mathcal{A})$ be a functor, and let $\nu : \alpha \Rightarrow \beta$ be a natural transformation. We get an associated natural transformation* $\nu' : \alpha^* F \Rightarrow \beta^* F$.
Prove that

$$\alpha' : \mathrm{Cone}_I(\alpha^* F) \to \mathrm{Cone}_J(F)$$

and

$$\beta' \circ \mathrm{Cone}_I(\nu') : \mathrm{Cone}_I(\alpha^* F) \to \mathrm{Cone}_I(\beta^* F) \to \mathrm{Cone}_J(F)$$

are homotopic maps of complexes.

Exercise 5.8. *Let $\mathcal{A}$ be an (AB4) abelian category. Prove the following two assertions:*

1. *There is natural map* $\mathrm{can}. : \mathrm{Cone}_{\{\cdot\}}(F) \to F(\cdot)$ *which is a quasi-isomorphism.*
2. $s' : \mathrm{Cone}_{\{\cdot\}} s^* F \to \mathrm{Cone}_I F$ *is a quasi-isomorphism if I contains a final object* $s : \{\cdot\} \to I$.

Hint. For 1. use the spectral sequence from Theorem 5.21 again. For 2., use 5.7. □

Exercise* 5.9. *Let $\mathcal{A}$ be an (AB4) abelian category, let I be a diagram, and let $F \in \mathrm{Fun}(I, C(\mathcal{A}))$. Give a direct proof that*

$$\mathrm{hocolim}_I(F) = \mathrm{Cone}_I(F),$$

i.e. the functor $F \mapsto \mathrm{Cone}_I(F)$ is left-adjoint to $p^* : C(\mathcal{A})[W^{-1}] \to \mathrm{Fun}(I, C(\mathcal{A}))[W_I^{-1}]$.

Hint. Let $p : I \to \{\cdot\}$ be the projection. Define counit and unit as the following compositions:

$$c\colon\ \mathrm{Cone}_I\, p^* F \xrightarrow{p'} \mathrm{Cone}_{\{\cdot\}}\, F \xrightarrow{\mathrm{can.}} F\ ,$$

$$u\colon\ F \xleftarrow{t_F} X_F \xrightarrow{o_F} p^*\, \mathrm{Cone}_I\, F\ .$$

Here X_F is defined as follows: For $i \in I$ define:

$$X_F(i) := \mathrm{Cone}_{I \times_{/I} \{i\}}\, \iota_i^* F$$

and for $\mu : i \to j$ define

$$X_F(\mu) := \mu' : \mathrm{Cone}_{I \times_{/I} i}\, \iota_i^* F \to \mathrm{Cone}_{I \times_{/I} j}\, \iota_j^* F,$$

where μ denotes, by abuse of language, the functor $I \times_{/I} i \to I \times_{/I} j$.
Consider the 2-cartesian diagram

$$\begin{array}{ccc} I \times_{/I} i & \xrightarrow{\iota_i} & I \\ \downarrow p_i & \swarrow \nu & \| \\ \{i\} & \longrightarrow & I \end{array}$$

Define $t_F(i)$ to be the composition

$$\mathrm{Cone}_{I \times_{/I} i}\, \iota_i^* F \xrightarrow{\mathrm{Cone}_I(\nu')} \mathrm{Cone}_{I \times_{/I} i}\, p_i^* F(i) \xrightarrow{p'} \mathrm{Cone}_{\{\cdot\}}\, F(i) \xrightarrow{\mathrm{can.}} F(i).$$

and

$$o(i) := \iota_i' : \mathrm{Cone}_{I \times_{/I} i}\, \iota_i^* F \to \mathrm{Cone}_I(F),$$

which is compatible with the constant morphisms and the $X_F(\mu)$.
Prove that t_F constitutes a morphism of diagrams and is point-wise a homotopy equivalence hence a quasi-isomorphism. (Note that however the homotopy-inverse can not be arranged as a morphism of diagrams.)
Show the unit/counit equations. The first equation will be true even in $\mathcal{C}(A)$. The second equation is true up to homotopy. Construct this homotopy explicitly. □

6 Model categories and homotopy (co)limits

References: [3, 8, 9, 15, 17, 20, 21, 29]

6.1. The replacements needed to compute derived functors and in particular homotopy (co)limits are mostly special cases of a much richer structure, the structure of a *model category*. This structure also implies that $(\mathcal{C}, W)$ is indeed a localizing pair, in particular that $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(-,-)$ are actually sets. As a motivation from the point of view of homological algebra recall that injective, resp. projective objects were a suitable replacement to compute the derived functors of a left- resp. right-exact functor. Let us focus on the case of injectives for a moment. By definition I is injective if and only if in a diagram

$$\begin{array}{ccc} X & \xrightarrow{\beta} & I \\ {\scriptstyle\alpha}\big\downarrow & \overset{\widetilde{\beta}}{\nearrow} & \big\downarrow \\ Y & \longrightarrow & 0 \end{array}$$

where α is a monomorphism, there is a lift $\widetilde{\beta}$ making the diagram commutative. We say that $X \to Y$ has the left-lifting property w.r.t. $I \to 0$ or equivalently that $I \to 0$ has the right-lifting property w.r.t. $X \to Y$. One might ask, which complexes $I_\bullet$ do have the property that in a diagram

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\beta} & I_\bullet \\ {\scriptstyle\alpha}\big\downarrow & \nearrow & \big\downarrow \\ Y_\bullet & \longrightarrow & 0 \end{array}$$

were α is a monomorphism (= degree-wise monomorphism) *and* a weak-equivalence there exists a lift as indicated. This depends a little on whether we consider bounded or unbounded complexes. If we consider bounded below complexes then the property translates precisely to the condition that I_n is injective for all n (see Exercises 4.3 and 6.3).
If we consider unbounded complexes (at least in the case of $\mathcal{A} = R - \mathcal{MOD}$) then we get so called DG-injective complexes (cf. [3, 17]) and it turns out that this is the better notion of injective replacements to work with. At least DG-injective complexes have the property that I_n is injective for all n but not conversely! More generally we call $\gamma : I_\bullet \to J_\bullet$ an **injective fibration** (denoted $\alpha \in \mathrm{Fib}^{inj}$), if in a diagram

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\beta} & I_\bullet \\ {\scriptstyle\alpha}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle\gamma} \\ Y_\bullet & \longrightarrow & J_\bullet \end{array}$$

there exists a lift. We might formulate this in the following way: Let S be a class of morphisms of $\mathcal{C}$. Then call $r(S)$ (resp. $l(S)$) the class of morphisms which have the right (resp. left) lifting property w.r.t. all morphisms in S. If we call a morphism α of complexes an injective cofibration if it is a monomorphism (denoted $\alpha \in \mathrm{Cof}^{inj}$) then we have

$$r(\mathrm{Cof}^{inj} \cap W) = \mathrm{Fib}^{inj}\,. \tag{7}$$

Additionally, we also have the on the first sight rather surprising symmetric property:

$$l(\mathrm{Fib}^{inj} \cap W) = \mathrm{Cof}^{inj} \tag{8}$$

as well as

$$W = r(\mathrm{Cof}^{inj}) \cdot l(\mathrm{Fib}^{inj}), \tag{9}$$

where $\cdot$ means composition.

Definition 6.2. *A structure*

$$(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$$

where $\mathcal{C}$ is a complete and cocomplete[6] *category and* Cof*,* Fib*, and W are subclasses of morphisms (called cofibrations, fibrations and weak equivalences, respectively) of $\mathcal{C}$, is called a* **model category***, if the following axioms hold:*

1. *Properties (7) and (8) are satisfied.*
2. *Any morphism $\alpha : X \to Y$ can be factored as $\alpha = \beta \circ \gamma$, where γ is a weak equivalence and a cofibration (we say a trivial cofibration) and β is a fibration, as well as $\alpha = \beta' \circ \gamma'$ where γ' is a cofibration and β' is a fibration and a weak equivalence (we say a trivial fibration). This factorization can be made functorial.*
3. Cof, Fib *and W are closed under retracts.*
4. *W satisfies the 2-out-of-3 property.*

For the precise meaning of axioms 3. and 4. see [17].

A consequence of these axioms is that $(\mathcal{C}, W)$ form a localizing pair, in particular the associated **homotopy category** $\mathcal{C}[W^{-1}]$ is locally small. Also (9) follows.

6.3. The injective structure $(C(\mathcal{A}), \mathrm{Cof}^{inj}, \mathrm{Fib}^{inj}, W)$ forms a model category for $\mathcal{A} = R - \mathcal{MOD}$ [17] or, if we consider bounded below complexes, for arbitrary $\mathcal{A}$ [29]. The injective replacement of $X_\bullet$ (in the case of 6.1) can be reconstructed as the factorization required by axiom 2.:

$$X_\bullet \xrightarrow{\text{trivial cofibration}} I_\bullet \xrightarrow{\text{fibration}} 0$$

[6] we assume this as in [17] for simplicity

of the morphism $X_\bullet \to 0$. In general in a model category, an object X such that $X \to \cdot$ is a fibration is called **fibrant**. An object X such that $\varnothing \to X$ is a cofibration is call **cofibrant**. Here $\varnothing$ is the initial and $\cdot$ is the final object.
Analogously, there is a projective model structure $(C(\mathcal{A}), \mathrm{Cof}^{proj}, \mathrm{Fib}^{prop}, W)$ with

$$\begin{aligned} \mathrm{Cof}^{proj} &= \text{dimension-wise split mono with DG-projective cokernel} \\ \mathrm{Fib}^{proj} &= \text{point-wise epimorphisms} \end{aligned}$$

at least if $\mathcal{A} = R-\mathcal{MOD}$ or if we consider bounded above complexes. There are also model category structures for $(C(\mathcal{A}), W)$ for *unbounded* complexes in more general abelian categories [3].

6.4. The fact that injective resp. projective resolutions were useful to compute derived functors generalizes as follows:
Assume that F and G are adjoint functors between abelian categories

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

(observe that this implies that F is right-exact and G is left-exact).
It follows

1. $G(\mathrm{Fib}^{inj}) \subset \mathrm{Fib}^{proj}$ (this expresses that G preserves surjections with injective kernel, too.)

2. $G(\mathrm{Fib}^{inj} \cap W) \subset \mathrm{Fib}^{proj} \cap W$ (this expresses roughly that on injectives G preserves quasi-isomorphisms, too.)

or equivalently, dually (cf. Exercise 6.1)

3. $F(\mathrm{Cof}^{proj}) \subset \mathrm{Cof}^{inj}$ (this expresses that F preserves injections with projective cokernel, too.)

4. $F(\mathrm{Cof}^{proj} \cap W) \subset \mathrm{Cof}^{inj} \cap W$ (this expresses roughly that on projectives F preserves quasi-isomorphisms, too.)

This obviously generalizes to situations where we have two arbitrary model structures:

Definition 6.5. *An adjunction*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that

$$(\mathcal{C}, \mathrm{Cof}_\mathcal{C}, \mathrm{Fib}_\mathcal{C}, W_\mathcal{C}) \qquad (\mathcal{D}, \mathrm{Cof}_\mathcal{D}, \mathrm{Fib}_\mathcal{D}, W_\mathcal{D})$$

are model categories is called a **Quillen adjunction** *if the properties (analogous to) 1.–4. above are satisfied. F is called a* **left Quillen** *functor and G is called a* **right Quillen** *functor.*

Now, we get derived functors for free:

Lemma 6.6 (Ken Brown)**.** *If F, G form a Quillen adjunction then the fibrant, resp. cofibrant resolution is a replacement functor adapted to G resp. to F.*

Hence by 2.6 and 4.7 LF and RG exist and form an adjunction:

$$\mathcal{C}[W_{\mathcal{C}}^{-1}] \underset{RG}{\overset{LF}{\rightleftarrows}} \mathcal{D}[W_{\mathcal{D}}^{-1}] .$$

Example 6.7. *Also $(\mathcal{TOP}, W)$ is part of (in fact several) model category structures. The most common is called the* **Quillen model structure** *and consists of*

$$\begin{aligned} \mathrm{Fib}_q &= \textit{Serre fibrations}, \\ \mathrm{Cof}_q &= \textit{Retracts of relative cell complexes.} \end{aligned}$$

See [17] for the meaning of relative cell complex. In particular CW-complexes are cell complexes relative to $*$ hence cofibrant.

Example 6.8. *There is also the* **Quillen model structures** *on $(\mathcal{SET}^{\Delta^{op}}, W)$, where W is the class of w such that $R(w)$ is a weak equivalence of topological spaces. Here all objects are cofibrant and fibrations are the Kan fibrations of simplicial sets.*
And there is the **Thomason model structure** *on $(\mathcal{SCAT}, W_\infty)$, where W_∞ is the class of w such that $R(\mathbf{N}(w))$ is a weak equivalence of topological spaces. Between those we have the adjunctions (cf. 5.3)*

$$\mathcal{SCAT} \underset{\mathbf{R}\mathrm{Sd}^2}{\overset{\mathrm{Ex}^2\mathbf{N}}{\rightleftarrows}} \mathcal{SET}^{\Delta^{op}} \underset{R}{\overset{N}{\leftrightarrows}} \mathcal{TOP}$$

which all induce equivalences *of the corresponding homotopy categories (see [30] for the definition of* Ex *and* Sd*).*

Using model category structures, the constructions of sections 1 and 3 appear in a much more clear fashion:

Definition 6.9. *A model category $(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$ is called* **left proper** *(resp.* **right proper***) if weak equivalences are preserved by pushout along cofibrations (resp. by pullback along fibrations).*

The model category structures $(C(\mathcal{A}), \mathrm{Cof}^{proj/inj}, \mathrm{Fib}^{proj/inj}, W)$, if they exist, are left and right proper. Also $(\mathcal{TOP}_*, \mathrm{Cof}_q, \mathrm{Fib}_q, W)$ is left and right proper.

Proposition 6.10. *Let $(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$ be a left proper model category with zero object. Then* $\mathrm{coker} : \mathrm{Fun}(\to, \mathcal{C}) \to \mathcal{C}$ *maps weak equivalences to weak equivalences on the subcategory consisting*

of cofibrations. Also colim : $\mathrm{Fun}(\ulcorner, \mathcal{C}) \to \mathcal{C}$ *maps weak equivalences to weak equivalences on the subcategory consisting of diagrams where the top morphism is a cofibration.*
Let $(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$ *be a right proper model category with zero object. Then* ker : $\mathrm{Fun}(\to, \mathcal{C}) \to \mathcal{C}$ *maps weak equivalences to weak equivalence on the subcategory consisting of fibrations. Also* lim : $\mathrm{Fun}(\lrcorner, \mathcal{C}) \to \mathcal{C}$ *maps weak equivalences to weak equivalences on the subcategory consisting of diagrams where the bottom morphism is a fibration.*

From this proposition the following observations can be made:

- This proves Proposition 3.3 in case that X and Y are cofibrant, for example CW-complexes. Then the map $X \hookrightarrow \mathrm{Cyl}(\alpha)$ is in Cof_q.

- The factorization from axiom 3. of a model category

$$\begin{array}{ccc} X & \xrightarrow{\text{cofib.}} & X' \\ \| & & \downarrow{\scriptstyle \text{trivial fib.}} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

  is a replacement functor adapted to coker.

- The (first three terms of the) cofiber sequences are just given by[7]

$$\left\{ X \xrightarrow{\alpha \text{ cofib.}} Y \longrightarrow \mathrm{coker}(\alpha) \right\}$$

- We have

$$\mathrm{hocolim} \begin{pmatrix} X & \longrightarrow & Y \\ \downarrow & & \\ 0 & & \end{pmatrix} = L\,\mathrm{coker}(X \to Y)$$

and dually:

- The factorization from axiom 3. of a model category

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle \text{trivial cofib.}}\downarrow & & \| \\ X' & \xrightarrow{\text{fib.}} & Y \end{array}$$

  is a replacement functor adapted to ker.

[7]If $\mathcal{C}$ is not left proper then X and Y have to be cofibrant, too.

- The (last three terms of the) fiber sequences are just given by[8]

$$\left\{ \ker(\alpha) \longrightarrow X \xrightarrow{\alpha \text{ fib.}} Y \right\}$$

- We have

$$\operatorname{holim}\left(\begin{array}{ccc} & & 0 \\ & & \downarrow \\ X & \longrightarrow & Y \end{array} \right) = R\ker(X \to Y)$$

In the case of chain complexes we have model categories in which cofibrations resp. fibrations are monomorphisms resp. epimorphisms. We thus also reobtain the observations of section 1.

6.11. Our next goal is to show that for every model category there exist arbitrary homotopy limits and colimits and left and right Kan extensions satisfying the axioms of a derivator. The main idea is to realize the adjunction characterizing the (co)limit

$$\operatorname{Fun}(I,\mathcal{C}) \underset{c}{\overset{\operatorname{colim}}{\rightleftarrows}} \mathcal{C}$$

as a Quillen adjunction!
By the observation above this will be satisfied if we can find a model structure on $\operatorname{Fun}(I,\mathcal{C})$ such that W_I (point-wise weak equivalences) are the weak equivalences and $\operatorname{Fib}_I^{proj}$ are the *point-wise* fibrations. ($\operatorname{Cof}_I^{proj}$ would then be determined by the property $\operatorname{Cof}_I^{proj} = l(\operatorname{Fib}_I^{proj} \cap W_I)$.) Reason: Obviously *const.* will preserve weak-equivalences and fibrations and hence the above will be a Quillen adjunction (see 6.1). The hocolim exists!
In the same way one defines $(\operatorname{Fun}(I,\mathcal{C}), \operatorname{Cof}_I^{inj}, \operatorname{Fib}_I^{inj}, W_I)$ which makes lim a (potential) right Quillen functor.
The remaining question is: Are $(\operatorname{Fun}(I,\mathcal{C}), \operatorname{Cof}_I^{proj}, \operatorname{Fib}_I^{proj}, W_I)$ and $(\operatorname{Fun}(I,\mathcal{C}), \operatorname{Cof}_I^{inj}, \operatorname{Fib}_I^{inj}, W_I)$ model categories?
We will sketch an unconditional proof for this, if I is **directed**, i.e. there is a functor $I \to \mathbb{N}$ such that preimages of identities are identities (in other words: objects can be indexed by natural numbers such that every non-trivial morphism decreases the index), or if I is **inverse** i.e. I^{op} is directed. Most model categories of interest are in fact **cofibrantly generated** and *presentable*. Such model categories are called **combinatorial**. Under this condition, there is a formal proof that the above structure defines a model category which does not use any assumption on I. Cofibrantly generated means that there are *sets* $C \subset \operatorname{Cof}$ and $T \subset \operatorname{Cof} \cap W$ such that already $\operatorname{Fib} = r(T)$ and $\operatorname{Fib} \cap W = r(C)$. Obviously there is the dual notion of "fibrantly generated" which however is seldom used because most naturally occurring model category structures do not satisfy this property.

[8] If $\mathcal{C}$ is not right proper then X and Y have to be fibrant, too.

Theorem 6.12 ([20, Appendix]). *If I is a diagram (small category) and*

$$(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$$

is a combinatorial *model category, then*

$$(\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{proj}, \mathrm{Fib}_I^{proj}, W_I) \qquad (\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{inj}, \mathrm{Fib}_I^{inj}, W_I)$$

are combinatorial model categories, too.

These model structures are called the **projective** and **injective** structure, respectively. This is a bit unfortunate because they should not be confused with the injective and projective model structures on chain complexes, although there is a certain analogy.

Theorem 6.13. *If*

$$(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$$

is any model category then

$$(\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{proj}, \mathrm{Fib}_I^{proj}, W_I)$$

is a model category if I is directed and

$$(\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{inj}, \mathrm{Fib}_I^{inj}, W_I)$$

is a model category if I is inverse.

These are actually special cases of the so called **Reedy model structures** [17].

Proof (idea): We focus on the case of directed categories. The approach is completely elementary. In this case it is possible to describe the class Cof_I^{proj} explicitly, as follows. Consider an object $i \in I$ and the category I_i of morphisms $j \to i$ in I which are different from the identity. It is a full subcategory of the slice category $I \times_{/I} i$ and comes equipped with a functor $\iota_i : I_i \to I$. For an functor $F \in \mathrm{Fun}(I,\mathcal{C})$, we define the **latching object**

$$L_i F := \mathrm{colim}_{I_i}\, \iota_i^* F.$$

There is a natural morphism $L_i F \to F(i)$. The sought-for description of projective cofibrations is then

$$\mathrm{Cof}_I^{proj} = \left\{ \alpha : F \to G \;\middle|\; \forall i \quad \begin{array}{ccccc} L_i F & \xrightarrow{L_i \alpha} & L_i G & & \\ \downarrow & & \downarrow & & \\ F(i) & \longrightarrow & \text{push-out} & \overset{\text{cofib.}}{\dashrightarrow} & G(i) \end{array} \right\}$$

In other words, cofibrations are those morphisms $\alpha : F \to G$ such that for all $i \in I$ the induced dotted arrow is a cofibration. It is easy to see that properties (7) and (8) are satisfied. The point is that morphisms $\alpha : F \to G$ in $\mathrm{Fun}(I,\mathcal{C})$ can be constructed inductively using the indexing of the objects of I. If for all indexes $m < n$ we have constructed $\alpha(j)$ for objects $j \in I$ of index m, and if i is an object of I of index n then we may define $\alpha(i)$ as an arbitrary morphism which makes

$$\begin{array}{ccc} L_i F & \xrightarrow{L_i \alpha} & L_i G \\ \downarrow & & \downarrow \\ F(i) & \xrightarrow{\alpha(i)} & \mathrm{G(i)} \end{array}$$

commute. Note that in the definition of L_i only objects j of index $< n$ occur. □

Corollary 6.14. *If* Dia *is a class of diagrams (small categories) such that*

$$(\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{proj}, \mathrm{Fib}_I^{proj}, W_I)$$

is a model structure for all $I \in \mathrm{Dia}$*, then the associated pre-derivator*

$$\mathbb{D}_{\mathcal{C}} : I \mapsto \mathrm{Fun}(I,\mathcal{C})[W_I^{-1}]$$

is a left derivator with domain Dia*.*

Of course, there is the analogous dual statement. In particular, if $(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$ is a combinatorial model category then $\mathbb{D}_{\mathcal{C}}$ is a (left and right) derivator.

Proof. The validity of (Der1) and (Der2) is relatively obvious.
(Der3 left) We have seen the existence of colimits. The existence of left Kan extensions associated with a functor $\alpha : I \to J$ between diagrams in Dia follows the same way, just because the functor α^* is a left Quillen functor, too:

$$\alpha^* : (\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{proj}, \mathrm{Fib}_I^{proj}, W_I) \to (\mathrm{Fun}(I,\mathcal{C}), \mathrm{Cof}_I^{proj}, \mathrm{Fib}_I^{proj}, W_I)$$

(Der4 left) It remains to see that for $\alpha : I \to J$ in Dia and all $j \in J$ Kan's formula

$$\mathrm{hocolim}_I\, \iota^* \cong j^* L\alpha_!$$

holds. We know that this is true for the underived functors. Hence it suffices to show that in the following diagram of model categories (all equipped with the projective model structure)

$$\begin{array}{ccc} \mathrm{Fun}(I,\mathcal{C}) & \xrightarrow{\iota^*} & \mathrm{Fun}(I \times_{/J} j, \mathcal{C}) \\ \downarrow{\scriptstyle \alpha_!} & & \downarrow{\scriptstyle \mathrm{colim}} \\ \mathrm{Fun}(J,\mathcal{C}) & \xrightarrow{j^*} & \mathcal{C} \end{array}$$

the two compositions of the derived functors are also isomorphic (via the map induced by the 2-commutative square). Since $\alpha_!$ and colim are left Quillen functors, it suffices to see that i^* and ι^* are left Quillen functors, too (see Exercise 6.2). Or, equivalently, that j_* and ι_* are right Quillen functors.

By definition of the projective model structure, it suffices to show that j_* and ι_* respect point-wise fibrations and trivial fibrations.

By Kan's formula (underived version) the exchange morphism associated with the 2-commutative diagram

$$\begin{array}{ccc} k \times_{/J} j & \xrightarrow{\iota'} & \{j\} \\ \downarrow & \swArrow & \downarrow{\scriptstyle j} \\ \{k\} & \xhookrightarrow{k} & J \end{array}$$

$$k^* j_* \to \lim (\iota')^*$$

is an isomorphism. Now $k \times_{/J} j = \mathrm{Hom}_J(k, j)$ is a discrete category, hence the limit respects fibrations and trivial fibrations, because the latter are stable under taking arbitrary products in any model category [15, Proposition 7.2.5].

By definition of the projective model structure, we need to show that ι_* respects point-wise fibrations and trivial fibrations. By Kan's formula the exchange morphism associated with the 2-commutative diagram

$$\begin{array}{ccc} i \times_{/I} I \times_{/J} j & \xrightarrow{\iota''} & I \times_{/J} j \\ \downarrow & \swArrow & \downarrow{\scriptstyle \iota} \\ i & \xhookrightarrow{i} & I \end{array}$$

$$i^* \iota_* \to \lim (\iota'')^*$$

is an isomorphism.

Now there is an adjunction (see exercise A.2)

$$\alpha(i) \times_{/J} j \underset{p}{\overset{s}{\rightleftarrows}} i \times_{/I} I \times_{/J} j$$

Therefore also p^* and s^* are adjoint, and hence $p_* = s^*$, or, applying lim to this equation:

$$\lim_{i \times_{/I} I \times_{/J} j} = \lim_{\alpha(i) \times_{/J} j} s^*.$$

Now $\mathrm{Hom}_J(\alpha(i), j)$ is again a discrete category hence, as above, the limit respects fibrations and trivial fibrations. $\square$

Remark 6.15. *Cisinski proved [4] that for* any *model category*

$$(\mathcal{C}, \mathrm{Cof}, \mathrm{Fib}, W)$$

the associated pre-derivator

$$\mathbb{D}_{\mathcal{C}} : I \mapsto \mathrm{Fun}(I, \mathcal{C})[W_I^{-1}]$$

is a derivator with domain $\mathcal{SCAT}$.
The idea is as follows: We have seen that $\mathbb{D}_{\mathcal{C}}$ is, say, a left *derivator with domain the* directed *diagrams. The case of a general diagram I is traced back to this case. For this argument a similar correspondence*

$$\begin{array}{ccc} & (\Delta^\circ)^{op}/\mathbf{N}^\circ(I) & \\ {}^{\iota}\swarrow & & \searrow^{p} \\ I & & (\Delta^\circ)^{op} \end{array}$$

to that used in the proof of 5.11 plays the key role. The only difference is that everything is constructed w.r.t. the inverse *diagram Δ° which has* strictly *increasing maps $\{0, \dots, n\} \hookrightarrow \{0, \dots, m\}$ as morphisms. $(\Delta^\circ)^{op}/\mathbf{N}^\circ(I)$ is then an directed diagram!*
The first step is to prove that

$$\mathrm{hocolim}_I X \coloneqq \mathrm{hocolim}_{(\Delta^\circ)^{op}/\mathbf{N}^\circ(I)} \iota^* X$$

defines *a homotopy limit of X over I. This is reasonable to expect because ι is $\mathbb{D}$-coacyclic for any derivator (we will learn about this notion in 8.3).*

Exercises

Exercise 6.1. *Prove that properties 3.–4. of a Quillen adjunction (see 6.4) are formally equivalent to properties 1.–2. using the lifting properties and the adjunction.*

Exercise 6.2. *Prove that a composition $F_2 \circ F_1$ of left-Quillen functors is a left-Quillen functor and that we have*

$$LF_2 \circ LF_1 = L(F_2 \circ F_1).$$

Similarly for right-Quillen functors.

Exercise 6.3. *Show the converse of Exercise 4.3: Suppose that $\alpha : X_\bullet \to Y_\bullet$ is a morphism of bounded below complexes in an abelian category (with enough injectives) which has the left lifting property w.r.t. morphisms $I_\bullet \to J_\bullet$ of bounded below complexes with point-wise injective kernel*

$$\begin{array}{ccc} X_\bullet & \longrightarrow & I_\bullet \\ {\scriptstyle\alpha}\downarrow & \nearrow & \downarrow \\ Y_\bullet & \longrightarrow & J_\bullet \end{array}$$

Show that α is a point-wise monomorphism and a quasi-isomorphism.

Exercise 6.4. *Prove Ken Brown's Lemma 6.6.*

Exercise* 6.5. *Prove Proposition 6.10.*

7 Bousfield Kan revisited

References: [10]

Let $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ be model categories (we will not explicitly indroduce notation for the classes of fibrations, cofibrations and weak equivalences). In this section, we will give a more conceptual explanation of Bousfield-Kan's formula. The reader is assumed to familiar with model categories.

Definition 7.1. *An adjunction in two variables*

$$\begin{aligned} \mathcal{C} \times \mathcal{D} &\to \mathcal{E} \\ A, B &\mapsto A \otimes B \end{aligned}$$

$$\begin{aligned} \mathcal{D}^{op} \times \mathcal{E} &\to \mathcal{C} \\ A, B &\mapsto \mathrm{Hom}_r(A, B) \end{aligned}$$

$$\begin{aligned} \mathcal{C}^{op} \times \mathcal{E} &\to \mathcal{D} \\ A, B &\mapsto \mathrm{Hom}_l(A, B) \end{aligned}$$

is called a **Quillen adjunction in 2 variables** *(or $\otimes$ is called a left Quillen bifunctor), if the following equivalent conditions hold:*

1. *If $\alpha : X \to X'$, and $\beta : Y \to Y'$, are cofibrations in $\mathcal{C}$, and $\mathcal{D}$, respectively, then the induced dotted arrow*

$$\begin{array}{ccccc} X \otimes Y & \longrightarrow & X' \otimes Y & & \\ \downarrow & & \downarrow & & \\ X \otimes Y' & \longrightarrow & \textit{push-out} & \xrightarrow{\textit{cofib.}} & X' \otimes Y' \end{array}$$

   is a cofibration. It is a trivial cofibration if, in addition, either α or β is a weak equivalence.

2. *If $\alpha : X \to X'$ is a cofibration, and $\beta : Y \to Y'$ is a fibration in $\mathcal{D}$, and $\mathcal{E}$, respectively, then the induced dotted arrow*

$$\begin{array}{ccccc} \mathrm{Hom}_r(X', Y) & \xrightarrow{\textit{fib.}} & \textit{pull-back} & \longrightarrow & \mathrm{Hom}_r(X', Y') \\ & & \downarrow & & \downarrow \\ & & \mathrm{Hom}_r(X, Y) & \longrightarrow & \mathrm{Hom}_r(X, Y') \end{array}$$

   is a fibration. It is a trivial fibration if, in addition, either α or β is a weak equivalence.

3. *If $\alpha : X \to X'$ is a cofibration, and $\beta : Y \to Y'$ is a fibration in $\mathcal{C}$, and $\mathcal{E}$, respectively, then the induced dotted arrow*

$$\begin{array}{ccccc} \mathrm{Hom}_l(X',Y) & \overset{fib.}{\dashrightarrow} & \textit{pull-back} & \longrightarrow & \mathrm{Hom}_l(X',Y') \\ & & \downarrow & & \downarrow \\ & & \mathrm{Hom}_l(X,Y) & \longrightarrow & \mathrm{Hom}_l(X,Y') \end{array}$$

is a fibration. It is a trivial fibration if, in addition, either α or β is a weak equivalence.

We have

Theorem 7.2. *Let $\mathcal{C}$ be either $C(\mathcal{A})$ or $\mathcal{TOP}_*$, equipped with one of the model structures of* **??**. *Consider $\mathcal{SET}^{\Delta^{op}}$ equipped with the Quillen model category structure (cf. 6.8).*
Then the functors of 5.7

$$\begin{aligned} \otimes : \mathcal{SET}^{\Delta^{op}} \times \mathcal{C} &\to \mathcal{C} \\ S, Y &\mapsto R_A(S) \otimes Y \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_r : \mathcal{C}^{op} \times \mathcal{C} &\to \mathcal{SET}^{\Delta^{op}} \\ X, Y &\mapsto (n \mapsto \mathrm{Hom}(A_n \otimes X, Y)) \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_l : (\mathcal{SET}^{\Delta^{op}})^{op} \times \mathcal{C} &\to \mathcal{C} \\ S, Y &\mapsto \underline{\mathrm{Hom}}(R_A(S), Y) \end{aligned}$$

form a Quillen adjunction in 2 variables.

Remark 7.3. *This is almost saying that $\mathcal{C}$ is a simplicial model category, except that we do not care about the precise relation of $\mathrm{Hom}_r(X,Y)$ to $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.*

Corollary 7.4. *Also the extension-to-diagrams adjunction of 5.8*

$$\begin{aligned} \otimes : \mathrm{Fun}(I^{op}, (\mathcal{SET}^{\Delta^{op}})) \times \mathrm{Fun}(I, \mathcal{C}) &\to \mathcal{C} \\ S, Y &\mapsto \int^i S(i) \otimes Y(i) \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_r : \mathrm{Fun}(I, \mathcal{C})^{op} \times \mathcal{C} &\to \mathrm{Fun}(I^{op}, \mathcal{SET}^{\Delta^{op}}) \\ X, Y &\mapsto \{i \mapsto \mathrm{Hom}_r(X(i), Y)\} \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_l : \mathrm{Fun}(I^{op}, (\mathcal{SET}^{\Delta^{op}}))^{op} \times \mathcal{C} &\to \mathrm{Fun}(I, \mathcal{C}) \\ S, Y &\mapsto \{i \mapsto \mathrm{Hom}_l(S(i), Y)\} \end{aligned}$$

form a Quillen adjunction in 2 variables where $\mathrm{Fun}(I^{op}, (\mathcal{SET}^{\Delta^{op}}))$ *is equipped with the projective model structure and* $\mathrm{Fun}(I, \mathcal{C})$ *is equipped with the injective model structure.*

Proof. We leave it as an exercise to derive this from Theorem 7.2. □

Remark 7.5. *Actually the corollary does also hold with projective and injective exchanged, see [10].*

Note that if we have a left Quillen functor $\otimes$ in two variables then, in particular, for a fixed X the functor $X \otimes -$ is a left Quillen functor. Using this, we can easily prove the following: (for $(C(\mathcal{A}), W)$ this had been shown in Exercise 5.3 in an elementary way)

Corollary 7.6. $(\mathcal{C}, W)$ *satisfies the properties 2.–5. of 5.10, provided we restrict to cofibrant objects.*

Proof. 2. follows from the fact that the maps $\Delta_i \to \Delta_0$ are trivial Kan fibrations. 3. follows because $\delta \otimes -$ maps trivial cofibrations to trivial cofibrations, hence by Ken Brown's Lemma it maps also all weak equivalences between cofibrant objects to weak equivalences. 4. follows because $\mathrm{Hom}_l(S, -)$ maps trivial cofibrations to trivial cofibration, hence by Ken Brown's Lemma it maps also all weak equivalences between cofibrant objects to weak equivalences. 5. holds in any model category. □

Remark 7.7. *The Bousfield-Kan formula, Theorem 5.11, now also follows directly, provided we know that the object* $i \mapsto N(i \times_{/I} I)$ *is a cofibrant replacement of the constant diagram* $i \mapsto \Delta_0$ *in the projective model structure on* $\mathrm{Fun}(I^{op}, \mathcal{SET}^{\Delta^{op}})$*. See e.g. [15, Proposition 14.8.8] for a direct proof of this fact.*

7.8. Consider the pair $(\mathcal{TOP}_*, W)$ and the diagram

$$\ulcorner := \left(\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & & \\ \cdot & & \end{array} \right)$$

Let us (re)calculate the homotopy colimit over this diagram (from a very high-brow perspective, indeed...):
Consider a diagram (point-wise cofibrant):

$$X := \left(\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle \beta} & & \\ C & & \end{array} \right)$$

Since we have

$$\mathrm{colim}_{\ulcorner} X = \left(\begin{array}{ccc} \Delta_0 & \longleftarrow & \Delta_0 \\ \uparrow & & \\ \Delta_0 & & \end{array} \right) \otimes \left(\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array} \right) \qquad (10)$$

we may compute the homotopy colimit by replacing the left hand side diagram by a cofibrant diagram.
Claim: The replacement

$$\left(\begin{array}{ccc} \Delta_1 & \xleftarrow{\delta_0} & \Delta_0 \\ \uparrow{\scriptstyle \delta_1} & & \\ \Delta_0 & & \end{array}\right)$$

is a cofibrant replacement in $\mathrm{Fun}(\ulcorner^{\mathrm{op}}, \mathcal{SET}^{\Delta^{\mathrm{op}}})$ of the constant diagram with value Δ_0. *Proof:* Considering the explicit description of cofibrant objects in the projective model structure for directed diagrams (see proof of Proposition 6.13) we get the conditions:

- everthing has to be cofibrant (this is automatic for $\mathcal{SET}^{\Delta^{\mathrm{op}}}$),
- the induced morphism $\Delta_0 \coprod \Delta_0 \to \Delta_1$ has to be a cofibration ($\Delta_0 \coprod \Delta_0$ is the latching object for the upper left corner object),
- and, of course, everything has to be contractible (because we want a replacement of the constant diagram with value Δ_0).

Δ_0 and Δ_1 are contractible and, by definition, $\Delta_0 \coprod \Delta_0 \to \Delta_1$ is a cofibration of simplical sets!
We proceed to compute the tensor product (10). Inserting geometric realization and using the explicit formula for coends (cf. A.11) we get

$$(B \sqcup A \times I \sqcup C)/\sim$$

with $(a, 0)$ identified with $\alpha(a)$ in B and $(a, 1)$ identified with $\beta(c)$ in C. This shows nicely how the (topological) homotopy push-out looks like in general. We have seen the special case in which $C = 0 = \Delta_0$ in section 3.

8 The homotopy theory of (homotopy) limits and colimits

References: [4–6, 12]

8.1. Consider a morphism $\alpha : I \to J$. One of the questions, we want to adress in this section, is:

"When do we have isomorphisms $p_{I,!}\alpha^* \to p_{J,!}$, resp. $p_{I,*}\alpha^* \to p_{J,*}$"?

To determine this, we assume only the abstract axioms of a derivator. In particular our reasoning will be valid for (co)limits as well as for homotopy (co)limits.
$p_{I,!}\alpha^* \to p_{J,!}$ is, by definition, an isomorphism precisely if the diagram

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & J \\ \downarrow & & \downarrow \\ \{\cdot\} & = & \{\cdot\} \end{array} \tag{11}$$

is homotopy exact.

Lemma 8.2. *If α is right adjoint then (11) is homotopy exact.*

Proof. If β, α are an adjunction then also α^*, β^* are an adjunction (the characterization 2. of adjoints of Lemma A.1 is preserved under a 2-functor), hence $\alpha^* \cong \beta_!$, hence $p_{I,!}\alpha^* \cong p_{I,!}\beta_! \cong p_{J,!}$. □

Actually we can give a precise characterization (provided that all $\alpha_!$ and α_* exist). For this we need a definition:

Definition 8.3. *A morphism $\alpha : I \to J$ is called a $\mathbb{D}$-**equivalence** if the induced morphism $p_{I,*}p_I^* \leftarrow p_{J,*}p_J^*$ (or equivalently the morphism $p_{I,!}p_I^* \to p_{J,!}p_J^*$) is an isomorphism.*
*A morphism $\alpha : I \to J$ is $\mathbb{D}$-**acyclic** if for all $j \in J$ the projection $I \times_{/J} j \to \{\cdot\}$ is a $\mathbb{D}$-equivalence.*
*A morphism $\alpha : I \to J$ is $\mathbb{D}$-**coacyclic** if for all $j \in J$ the projection $j \times_{/J} I \to \{\cdot\}$ is a $\mathbb{D}$-equivalence.*

Proposition 8.4.

$$\begin{array}{ccc} I & \longrightarrow & J \\ \downarrow & & \downarrow \\ \{\cdot\} & = & \{\cdot\} \end{array}$$

is homotopy exact if and only if α is $\mathbb{D}$-coacyclic.

Proof. The homotopy exactness is equivalent to the condition of

$$p_J^* \to \alpha_* p_I^*$$

being an isomorphism. This is turn is equivalent to

$$\mathrm{id} \to j^* \alpha_* p_I^*$$

being an isomorphism for all j which in turn is equivalent to

$$\mathrm{id} \to p_{j \times_{/J} I, *} p_{j \times_{/J} I}^*$$

being an isomorphism by (Der4). □

8.5. The first observation is rather trivial. If there is a natural transformation between $\alpha \to \beta$ then the maps

$$p_{I,!} p_I^* \to p_{J,!} p_J^*$$

(induced by the counit $\beta_! \beta^* \to \mathrm{id}$ resp. $\alpha_! \alpha^* \to \mathrm{id}$) are actually equal or equivalently the corresponding maps:

$$p_{J,*} p_J^* \to p_{I,*} p_I^*.$$

Therefore: If α is a $\mathbb{D}$-equivalence then so is β. We call a **homotopy equivalence** between I and J functors $\alpha : I \to J$ and $\beta : J \to I$ such that there are chains of natural transformations

$$\alpha\beta \Rightarrow \mu_1 \Leftarrow \mu_2 \cdots \Rightarrow \mathrm{id}_J$$

and

$$\beta\alpha \Rightarrow \nu_1 \Leftarrow \nu_2 \cdots \Rightarrow \mathrm{id}_J .$$

Therefore: homotopy equivalences are $\mathbb{D}$-equivalences for all $\mathbb{D}$.

Example 8.6.
1. *Left and Right adjoints are homotopy equivalences and hence $\mathbb{D}$-equivalences for any $\mathbb{D}$.*
2. *If a category I has a final resp. initial object then $p_I : I \to \{\cdot\}$ is in fact a left resp. right adjoint and hence p_I is a $\mathbb{D}$-equivalence.*

Remark 8.7. *The proposition implies in view of Lemma 8.2 that a right adjoint must be $\mathbb{D}$-coacyclic. But for a right adjoint $\alpha : I \to J$ (with left adjoint β) actually the category $j \times_{/J} I$ has the final object $(\beta(j), u : j \to \alpha(\beta(j)))$ hence $j \times_{/J} I \to \{\cdot\}$ is a $\mathbb{D}$-equivalence.*

8.8. Obviously the notion of $\mathbb{D}$-equivalence and hence $\mathbb{D}$-acyclic morphisms depends on $\mathbb{D}$. But we might ask about the class of $\mathbb{D}$-equivalences that are $\mathbb{D}$-equivalences for any $\mathbb{D}$? The answer is given by the following Theorem of Cisinski:

Theorem 8.9. *The class of functors $\alpha : I \to J$ between small categories which are $\mathbb{D}$-equivalences for any derivator $\mathbb{D}$ are precisely those which induce a weak equivalence of nerves $N(I) \to N(J)$.*

Idea of proof. A class of functors W between small categories is called a **Grothendieck localizer** if

1. W is weakly saturated
2. If I has a final object then $p_I : I \to \{\cdot\}$ is in W.
3. If

$$\begin{array}{ccc} I & \longrightarrow & J \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

   is a commutative diagram and all induced functors $I \times_{/K} k \to J \times_{/K} k$ are in W then so is α.

It is actually an exercise to show that $\mathbb{D}$-equivalences form a localizer. Cisinski proves that W_∞ is the smallest localizer. They form a localizer because they are the $\mathbb{D}_{\mathcal{SCAT}}$ equivalences (ref.). ☐

Corollary 8.10. *The class of functors $\alpha : I \to J$ which preserve homotopy (co)limits in any derivator are precisely those such that $N(I \times_{/J} j)$ (resp. $N(j \times_{/J} I)$) is contractible for all j.*

Remark 8.11. *This is really a statement about* homotopy *limits and colimits. For example it is easy to show that* $\operatorname{colim}_{\Delta^{op}} X$ *for a simplicial object X is the same as the coequalizer*

$$X_1 \rightrightarrows X_0$$

This is not at all true for homotopy colimits (see for example 5.16).

8.12. In 6.8 we have seen that:

$$\mathbb{H} := \mathbb{D}_{\mathcal{SET}^{op}} = \mathbb{D}_{\mathcal{CAT}} = \mathbb{D}_{\mathcal{TOP}}$$

because the corresponding model categories are all Quillen equivalent.
This may be used to obtain an action of simplicial sets on any derivator (as we have seen concretely in the application ...), however only "up to homotopy", in other words there is a morphism of derivators

$$\begin{array}{rcl} \otimes : \mathbb{H} \times \mathbb{D} & \to & \mathbb{D} \\ I, X & \mapsto & p_{I,!} p_I^* X \end{array}$$

where $I \in \mathbb{D}_{\mathcal{CAT}}(\cdot)$.
Its adjoint w.r.t. the second variable is given by

$$\begin{array}{rcl} \underline{\operatorname{Hom}} : \mathbb{H}^{op} \times \mathbb{D} & \to & \mathbb{D} \\ I, X & \mapsto & p_{I,*} p_I^* X \end{array}$$

and Cisinski shows [6] that at least for derivators that come from model categories they have an adjoint in the first variable:

$$\begin{array}{rcl} R\operatorname{Hom} : \mathbb{D}^{op} \times \mathbb{D} & \to & \mathbb{H} \\ X, Y & \mapsto & R\operatorname{Hom}(X, Y) \end{array}$$

such that $\operatorname{Hom}(X,Y) = \pi_0(R\operatorname{Hom}(X,Y))$.
From this it follows that that $\otimes$ preserves homotopy left Kan extensions in both variables.

Exercises

Exercise 8.1. *Show that $\mathbb{D}$-equivalences form a localizer.*

Exercise* 8.2. *Let $\mathcal{C}$ be $\mathcal{TOP}$ or $C(\mathcal{A})$ and $\mathbb{D}$ the associated derivator. Prove that the adjunction of 5.7 is compatible with the one in 8.12. Here you have to regard $\mathbb{H}$ as $\mathbb{D}_{\mathcal{SET}^{\Delta^{op}}}$.*

Exercise* 8.3. *Show directly (without using the existence of $R\operatorname{Hom}$) that $\otimes$ preserves homotopy left Kan extensions in both variables.*

Exercise* 8.4. *Prove that ι (cf. 6.15 and 5.15)*

$$(\Delta^\circ)^{op}/\mathbf{N}^\circ(I) \xrightarrow{\iota} I$$

is $\mathbb{D}$-coacyclic for any $\mathbb{D}$.

A Some facts from category theory

References: [22]

A.1 Adjoints

Lemma A.1. *For two functors*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

The following data are in bijection:

1. *Natural isomorphisms of functors* $\mathcal{C}^{op} \times \mathcal{D} \to \mathcal{SET}$*:*

$$h_{X,Y} : \mathrm{Hom}(FX, Y) \to \mathrm{Hom}(X, GY) \tag{12}$$

2. *Natural transformations*

$$\varepsilon : FG \to \mathrm{id}_{\mathcal{C}} \qquad \textbf{(counit)}$$
$$\eta : \mathrm{id}_{\mathcal{D}} \to GF \qquad \textbf{(unit)}$$

such that the compositions

$$FX \xrightarrow{F\eta_X} FGFX \xrightarrow{\varepsilon_{FX}} FX$$

$$GY \xrightarrow{\eta_{GY}} GFGY \xrightarrow{G\varepsilon_Y} GY$$

are identities for all $X \in \mathcal{C}$ *and* $Y \in \mathcal{D}$.

Note that the second characterization makes sense in any 2-category.
An adjoint (or adjunction) is always understood together with the bijection (12) or equivalently together with either counit or unit. Either of these data characterizes an adjoint up to a *unique* natural isomorphism.

A.2 Grothendieck (op)fibrations

A.2. Grothendieck fibration. Let $p : \mathcal{D} \to \mathcal{S}$ be a morphism of categories, $f : S \to T$ be a morphism in $\mathcal{S}$. A morphism $\xi : \mathcal{E}' \to \mathcal{E}$ over f with the property that the composition with ξ induces an isomorphism

$$\mathrm{Hom}_g(\mathcal{F}, \mathcal{E}') \cong \mathrm{Hom}_{f \circ g}(\mathcal{F}, \mathcal{E})$$

for any $g : R \to S$ and $\mathcal{F} \in \mathcal{D}_R$ is called **cartesian**.
p is called a **Grothendieck fibration**, if for any $f : S \to T$ and $\mathcal{E}$ an object in $\mathcal{D}_T$ there exists a cartesian $\mathcal{E}' \to \mathcal{E}$.

A.3. Grothendieck opfibration. This is the dual notion. Let $p : \mathcal{D} \to \mathcal{S}$ be a morphism of categories, $f : S \to T$ be a morphism in $\mathcal{S}$. A morphism $\xi : \mathcal{E} \to \mathcal{E}'$ over f with the property that the composition with ξ induces an isomorphism

$$\mathrm{Hom}_g(\mathcal{E}', \mathcal{F}) \cong \mathrm{Hom}_{g \circ f}(\mathcal{E}, \mathcal{F})$$

for any $g : T \to U$ and $\mathcal{F} \in \mathcal{D}_U$ is called **cartesian**.
p is called **a Grothendieck opfibration**, if for any $f : S \to T$ and $\mathcal{E}$ an object in $\mathcal{D}_S$ (i.e. such that $p(\mathcal{E}) = S$) there exists a cocartesian $\mathcal{E} \to \mathcal{E}'$
$p : \mathcal{D} \to \mathcal{S}$ is a Grothendieck opfibration, iff $p^{op} : \mathcal{D}^{op} \to \mathcal{S}^{op}$ is a Grothendieck fibration. We say that p is a **bifibration** if is a fibration and opfibration at the same time.

A.4. There is actually an equivalence of concepts: Grothendieck fibrations $\mathcal{D} \to \mathcal{S}$ and pseudo-functors $\mathcal{S} \to \mathcal{CAT}$.

A.3 Comma categories

For a diagram of categories and functors

$$\begin{array}{ccc} & & I \\ & & \downarrow{\scriptstyle \alpha} \\ K & \xrightarrow{\beta} & J \end{array}$$

the comma category $I \times_{/J} K$ (often denoted (α/β) or I/j if $K = \{j\}$ for an object j of J) is defined by

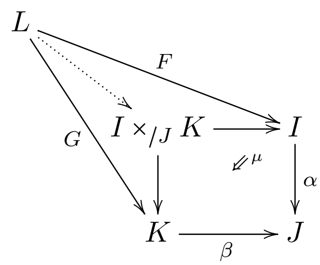
$$\begin{aligned} \mathrm{Ob}(I \times_{/J} K) &= \{i \in I, j \in J, \mu : F(i) \to G(j)\} \\ \mathrm{Hom}_{I \times_{/J} K}((i, \mu, j), (i', \mu', j')) &= \left\{f \in \mathrm{Hom}_I(i, i'), g \in \mathrm{Hom}_J(j, j') \mid \alpha(f)\mu = \mu' \beta(g)\right\} \end{aligned}$$

It sits in an obvious 2-commutative diagram:

$$\begin{array}{ccc} I \times_{/J} K & \longrightarrow & I \\ \downarrow & \swarrow^{\mu} & \downarrow{\scriptstyle \alpha} \\ K & \xrightarrow{\beta} & J \end{array}$$

Note that, in general, $I \times_{/J} K$ is *not* equivalent to $K \times_{/J} I$ (unless J is a groupoid).

A.5. $(I \times_{/J} K, \mu)$ is a 2-pullback, that is, it has the following universal property: Given any diagram

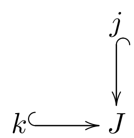


together with a 2-morphism (natural transformation)

$$\nu : \alpha F \Longrightarrow \beta G,$$

there is a unique functor $H : L \to I \times_{/J} K$ making the triangles commute and such that $\nu = \mu * H$.

Example A.6. *If J is a category and $j, k \in \mathrm{Ob}(J)$*



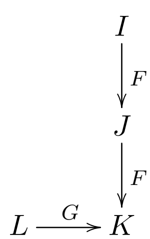
then $j \times_{/J} k = \mathrm{Hom}_J(j, k)$.

Exercises

Exercise A.1. *Prove that for for I and $i \in I$, there is an adjunction*

$$I \times_{/I} i \rightleftarrows i$$

Exercise A.2. *For*



there is an adjunction

$$L \times_{/K} J \times_{/J} I \rightleftarrows L \times_{/K} I$$

with counit the identity.

Exercise A.3. *Prove that $I \times_{/J} K \to I$ is a Grothendieck fibration and that $I \times_{/J} K \to K$ is a Grothendieck opfibration.*

Exercise A.4. *For a Grothendieck fibration $T \to K$ and $k \in K$ there is an adjunction*

$$T \times_{/K} k \rightleftarrows T \times_K k$$

A.4 Abelian categories

Definition A.7. *A category $\mathcal{A}$ is abelian, if the following axioms hold:*

(AB0) $\mathcal{A}$ has a zero object.

(AB1) $\mathcal{A}$ has binary products, binary coproducts, kernels and cokernels.

(AB2) Any monomorphism in $\mathcal{A}$ is a kernel and any epimorphism is a cokernel.

Definition A.8. *We will frequently use the following additional axioms for an abelian category $\mathcal{A}$:*

(AB3) Arbitrary coproducts exist in $\mathcal{A}$ (it follows that $\mathcal{A}$ is cocomplete).

(AB4) $\mathcal{A}$ satisfies (AB3) and arbitrary coproducts are exact.

and dually:

(AB3) Arbitrary products exist in $\mathcal{A}$ (it follows that $\mathcal{A}$ is complete).*

(AB4) $\mathcal{A}$ satisfies (AB3*) and arbitrary products are exact.*

A.5 (Co)limits and Kan extensions

A.9. Let $\alpha : I \to J$ be a (in general small) categories. There is a functor

$$\alpha^* : \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$$

given by composition with α. A particular case is the functor $p : I \to \{\cdot\}$.

$$p^* : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$$

A left adjoint of α^*, denoted $\alpha_!$ if it exists, is called a **left Kan extension** of α. A right adjoint of α^*, denoted α_* if it exists, is called a **right Kan extension** of α. $p_! X$ is also called the **colimit** of X: $\mathrm{colim}_I X$, and $p_* X$ is also called the **limit** of X: $\lim_I X$. Note that an adjoint is, if it exists, uniquely determined up to a unique natural isomorphism. And this is a point-wise statement, in the sense that even if it does not exist for all objects X then the value for those for which it does

exist is uniquely determined up to a unique isomorphism. In the case of the limit and colimit we say that the limit and colimit exist for an X.
For $\alpha : I \to J$ and an object $j \in \mathrm{Ob}(J)$ we consider the 2 commutative diagram (cf. A.3)

$$\begin{array}{ccc} I \times_{/J} j & \xrightarrow{\iota_j} & I \\ \downarrow{\scriptstyle p} & \swarrow & \downarrow{\scriptstyle \alpha} \\ j & \xhookrightarrow{j} & J \end{array}$$

Proposition A.10 (Kan)**.** *Let $\alpha : I \to J$ be a functor of diagram (i.e. small categories). If for all the categories $I \times_{/J} j$ the colimit of $\iota_j^* X$ exists then $\alpha_! X$ exists and is given by*

$$\alpha_! X := (j \mapsto \mathrm{colim}_{I \times_{/J} j}\, \iota_j^* X)$$

Proof. □

The formula might be expressed by saying that the natural morphism (exchange morphism associated with the canonical morphism $\iota^* \alpha^* \to p^* j^*$)

$$p_! \iota_j^* \to j^* \alpha_!$$

is an isomorphism.
Recall that the colimit $\mathrm{colim}_I X$ may be computed as the cokernel (or coequalizer) of the following pair of maps

$$\coprod_{\mu : \Delta_1 \to I} X(s(\mu)) \underset{d}{\overset{s}{\rightrightarrows}} \coprod_{i : \Delta_0 \to I} X(i)$$

A.6 Dinatural transformations and (Co)ends

Let $\alpha, \beta : I^{op} \times I \to \mathcal{C}$ be a functor. A dinatural transformation

$$\alpha \Rightarrow \beta$$

is a morphism for each $i \in I$

$$\alpha(i,i) \to \beta(i,i)$$

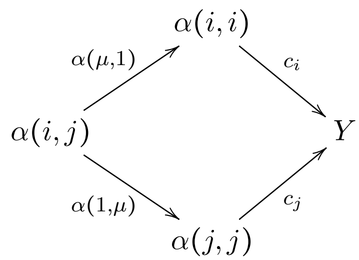
such that for all $\mu : i \to j$ the following diagram commutes:

$$\begin{array}{ccccc} & \alpha(i,i) & \longrightarrow & \beta(i,i) & \\ {\scriptstyle \alpha(\mu,1)} \nearrow & & & & \searrow {\scriptstyle \beta(1,\mu)} \\ \alpha(i,j) & & & & \beta(j,i) \\ {\scriptstyle \alpha(1,\mu)} \searrow & & & & \nearrow {\scriptstyle \beta(\mu,1)} \\ & \alpha(j,j) & \longrightarrow & \beta(j,j) & \end{array}$$

Be aware that dinatural transformations cannot be composed!
For a functor $\alpha : I^{op}\times I \to \mathcal{C}$ we denote by $\int^i \alpha(i,i)$ an object characterised up to unique isomorphism by

$$\mathrm{Hom}(\int^i \alpha(i,i), Y) = \{ \text{ dinatural transformations } \alpha \to Y \ \}$$

is called the **coend** of α. Here Y is considered to be the constant functor $I^{op} \times I \to \mathcal{C}$ with image Y.
Note that dinatural transformations $\alpha \to Y$ are actually collections $c_i : \alpha(i,i) \to Y$ such that



commutes.
For a functor $\alpha : I^{op} \times I \to \mathcal{C}$ which is actually constant in I^{op}, we get

$$\int^i \alpha(i,i) = \mathrm{colim}_I\, \alpha(i,-)$$

where $i \in I^{op}$ is any object.

A.11. If coproducts and cokernels exists in $\mathcal{C}$ then also coends exists and is the cokernel (or coequalizer)

$$\coprod_{\mu:\Delta_1\to I} \alpha(s(\mu), d(\mu)) \underset{d}{\overset{s}{\rightrightarrows}} \coprod_{i:\Delta_0\to I} \alpha(i,i)$$

which generalizes the formula for the cokernel.

There is an obvious dual construction which is called the **end**.

Exercises

The following will be our main use for (co)ends:

Exercise A.5. *If there is an adjunction of two variables*

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \quad \mathrm{Hom}_l : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{C} \quad \mathrm{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{D},$$

we have the following adjunction of two variables

$$\begin{array}{rcl} \mathrm{Fun}(I^{op}\times J,\mathcal{C})\times \mathrm{Fun}(I,\mathcal{D}) & \to & \mathrm{Fun}(J,\mathcal{E}) \\ X,Y & \mapsto & (j\mapsto \int^{i} X(i,j)\otimes Y(i)) \end{array}$$

$$\begin{array}{rcl} \mathrm{Fun}(I^{op}\times J,\mathcal{C})^{op}\times \mathcal{E} & \to & \mathrm{Fun}(I,\mathcal{D}) \\ X,Z & \mapsto & (i\mapsto \int_{j} \mathrm{Hom}_l(X(i,j),Z(j))) \end{array}$$

$$\begin{array}{rcl} \mathrm{Fun}(I,\mathcal{D})^{op}\times \mathrm{Fun}(J,\mathcal{E}) & \to & \mathrm{Fun}(I^{op}\times J,\mathcal{C}) \\ Y,Z & \mapsto & (i,j\mapsto \mathrm{Hom}_r(Y(i),Z(j))) \end{array}$$

Exercise A.6. *In the situation of exercise A.5 prove that if a Kan extension along $\alpha: I\to J$ exists, then*

$$\int^{i} X(i)\otimes(\alpha^{*}Y)(i) = \int^{j} ((\alpha^{op})_{!}X)(j)\otimes Y(j)$$

Exercise A.7 (A variant of Yoneda's Lemma)**.** *Let $\mathcal{SET}\times\mathcal{C}\to\mathcal{C}$ be the canonical adjunction in two variables given by $S\otimes X=\coprod_{s\in S}X$. The adjoints are $\mathrm{Hom}_l(S,Y)=\prod_{s\in S}Y$ and $\mathrm{Hom}_r=\mathrm{Hom}_{\mathcal{C}}$. Prove that for all $A\in[I,\mathcal{C}]$ and $j\in I$:*

$$\int^{i}\mathrm{Hom}(i,j)\otimes A(i) = A(j) \qquad \int_{j}\mathrm{Hom}_l(\mathrm{Hom}(i,j),A(j)) = A(i)$$

Exercise A.8. *In the situation of exercise A.7 prove that the following formula defines a left Kan extension for $\alpha: I\to J$:*

$$(\alpha_{!}A)(j) := \int^{i}\mathrm{Hom}(\alpha(i),j)\otimes A(i).$$

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