The ∞-categorical interpretation of Abelian and non-Abelian derived functors

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Preliminary version.
These notes explain how derived functors (also non-Abelian ones, introduced by Dold-Puppe [4, 5] and studied for instance by Quillen [15, 16], Illusie [11, 12]) are interpreted in the language of ∞-categories. We do not assume any familiarity with ∞-categories or model categories, only some familiarity with a) classical derived categories of Abelian categories and b) basic algebraic topology. Instead these notes try to give a motivation for the language of ∞-categories.

Notation

For $\mathcal{C}, I$ usual 1-categories (with I usually small, i.e. morphisms and objects form sets) we denote by $\mathcal{C}^I$ the category $\mathrm{Fun}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$ with morphisms being natural transformations. We denote by $\mathcal{SET}$ the category of sets, by $\mathcal{TOP}$ the category of topological spaces, and by Δ the simplex category, i.e. the category with objects $\Delta_n := \{0, 1, \dots, n\}$, morphisms being (non-strictly) increasing maps. For a commutative ring R, we denote by $\mathcal{A}_R$ (resp. $\mathcal{M}_R$) the categories of commutative R-algebras (resp. R-modules).

1 Preliminaries

1.1. Recall that a morphism $f : X \to Y$ of topological spaces is called a weak equivalence, if it induces isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ on all homotopy groups. There is a canonical functor "geometric realization" from simplicial sets to topological spaces $\Gamma : \mathcal{SET}^{\Delta^{\mathrm{op}}} \to \mathcal{TOP}$ which maps an abstract simplex[1] Δ_n to the corresponding standard simplex $\Delta_n^{\mathbb{R}} \subseteq \mathbb{R}^n$. In general $\Gamma(S_\bullet)$ is a disjoint union of the $\Delta_n^{\mathbb{R}}$ glued together according to the data given by the simplicial set. A map of simplicial sets $f : S_\bullet \to T_\bullet$ is called a weak equivalence, if $\Gamma(f)$ is a weak equivalence[2]. We denote by $\mathcal{W} \subset \mathrm{Mor}(\mathcal{TOP})$ resp. $\mathcal{W} \subset \mathrm{Mor}(\mathcal{SET}^{\Delta^{\mathrm{op}}})$ the classes of weak equivalences. We may define corresponding *homotopy categories* as $\mathcal{TOP}[\mathcal{W}^{-1}]$ and $\mathcal{SET}^{\Delta^{\mathrm{op}}}[\mathcal{W}^{-1}]$, formally inverting all weak equivalences.

Theorem 1.2. *There is an adjunction*

$$\mathcal{SET}^{\Delta^{\mathrm{op}}} \underset{N}{\overset{\Gamma}{\rightleftarrows}} \mathcal{TOP} \quad (\Gamma \dashv N)$$

[1] which seen as a simplicial set is the represented simplicial set $\Delta_k \mapsto \mathrm{Hom}(\Delta_k, \Delta_n)$

[2] The homotopy groups $\pi_i(\Gamma(S_\bullet), x)$ (and thus the notion of weak equivalence) can also be described purely combinatorially but this is a bit cumbersome.

where N is the functor "simplicial complex" $X \mapsto \{\Delta_k \mapsto \mathrm{Hom}(\Delta_k^{\mathbb{R}}, X)\}$. This adjunction induces an equivalence

$$\mathcal{SET}^{\Delta^{\mathrm{op}}}[\mathcal{W}^{-1}] \cong \mathcal{TOP}[\mathcal{W}^{-1}].$$

In fact much more is true: The pairs $(\mathcal{TOP}, \mathcal{W})$ and $(\mathcal{SET}^{\Delta^{\mathrm{op}}}, \mathcal{W})$ define the same *homotopy theory* altogether. It is not so easy to make this precise. It is reflected by the fact that the two functors define a Quillen equivalence of model categories or by the fact that the two pairs define equivalent ∞-categories as we will see later.

1.3. Similarly, for an Abelian category $\mathcal{A}$ and denoting by

$$C(\mathcal{A})$$

the category of chain complexes in $\mathcal{A}$, one defines the derived category as

$$D(\mathcal{A}) := C(\mathcal{A})[\mathcal{W}^{-1}]$$

formally inverting all quasi-isomorphisms between complexes ($\mathcal{W}$ here denotes the class of all quasi-isomorphisms).

1.4. For any algebraic[3] category $\mathcal{C}$ there is a non-Abelian variant of 1.3. Consider the category

$$\mathcal{C}^{\Delta^{\mathrm{op}}}$$

of simplicial objects in $\mathcal{C}$. An object $X_\bullet$ is thus a diagram

$$\cdots \quad X_3 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X_2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X_1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X_0$$

with $X_i \in \mathcal{C}$. A morphism f is called a *weak equivalence*, if applying the forgetful functor $V : \mathcal{C} \to \mathcal{SET}$ to f gives a weak equivalence. Again we have a derived category

$$D(\mathcal{C}) := \mathcal{C}^{\Delta^{\mathrm{op}}}[\mathcal{W}^{-1}]$$

where $\mathcal{W}$ denotes the class of all weak equivalences.
Applied to an Abelian category $\mathcal{A}$, we get nothing new:

Theorem 1.5 (Dold-Kan)**.** [4] *There is an equivalence of categories*

$$C_{\le 0}(\mathcal{A}) \underset{N}{\overset{\Gamma}{\rightleftarrows}} \mathcal{A}^{\Delta^{\mathrm{op}}}$$

which maps the classes $\mathcal{W}$ into each other, inducing thus an equivalence

$$\mathcal{D}_{\le 0}(\mathcal{A}) \cong \mathcal{A}^{\Delta^{\mathrm{op}}}[\mathcal{W}^{-1}].$$

Proof. By explicit calculation. N is the normalized Moore complex functor and we have

$$H^{-i}(N(S_\bullet)) \cong \pi_i(S_\bullet).$$

(For $i \ge 1$ this is even an isomorphism of groups.) □

Trivially, the pairs $(C_-(\mathcal{A}), \mathcal{W})$ and $(\mathcal{A}^{\Delta^{\mathrm{op}}}, \mathcal{W})$ will again define the same *homotopy theory* in any reasonable sense.

[3] We assume *for simplicity* that $\mathcal{C}$ is *algebraic*, i.e. has a conservative and faithful functor $V : \mathcal{C} \to \mathcal{SET}$ which has a left adjoint "free object" functor F. For example the category of (commutative) groups, (commutative) rings, (commutative) R-algebras, R-modules, etc.

[4] A faithful functor $\mathcal{A} \to \mathcal{SET}$ is not needed for this, one has to define weak-equivalences in $\mathcal{A}^{\Delta^{\mathrm{op}}}$ a bit more carefully.

2 Classical derived functors

2.1. Given a pair $(\mathcal{M}, \mathcal{W})$, like $(\mathcal{TOP}, \mathcal{W})$, $(\mathcal{C}^{\Delta^{\mathrm{op}}}, \mathcal{W})$, or $(C(\mathcal{A}), \mathcal{W})$, consisting of a category and a class of morphisms, there is an abstract notion of a total left derived functor of a functor $F : \mathcal{M} \to \mathcal{D}$ into any (homotopy) category $\mathcal{D}$. It is a functor LF

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle \iota} & \Nwarrow & \nearrow{\scriptstyle LF} \\ \mathcal{M}[\mathcal{W}^{-1}] & & \end{array}$$

together with a *universal* natural transformation $LF \circ \iota \to F$. "Universal" means that ν induces an isomorphism

$$\mathrm{Hom}(G, LF) \cong \mathrm{Hom}(\iota \circ G, F),$$

for any G, or — in other words — LF is a right Kan extension of F along ι. In particular LF is uniquely determined up to unique isomorphism.

2.2. For an object X, $LF(X)$ is computed by the following recipe: Suppose there is a resolution $P \to X$ in $\mathcal{M}$, i.e. a weak equivalence such that P lies in a specified class of *projective objects* which has the property that F maps weak equivalence between these projective objects to weak equivalences, i.e. induces a functor $\mathcal{M}^c[\mathcal{W}^{-1}] \to \mathcal{D}$, where $\mathcal{M}^c \subset \mathcal{M}$ is the full subcategory of these projective objects. Then $LF(X) := FP$ defines the (total) left derived functor.

Example 2.3. *1. In the example $(\mathcal{TOP}, \mathcal{W})$, a class of "projective objects" well-suited to derive many functors is the class of CW-complexes.*

2. In the example $(\mathcal{C}^{\Delta^{\mathrm{op}}}, \mathcal{W})$ a class of "projective objects" is given by simplicial objects consisting point-wise of projective objects[5] *in $\mathcal{C}$. A free (and hence projective) resolution can always be constructed as*

$$\cdots \quad FVFVFVX \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; FVFVX \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; FVFVX \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; FVX \tag{1}$$

using the various (co)units of the adjunction of the forgetful functor V with the free object functor F. The class can be used to derive functors $\mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{D}^{\Delta^{\mathrm{op}}}$ induced from any *functor $\mathcal{C} \to \mathcal{D}$ whatsoever*[6].

3. In the example $(C(\mathcal{A}), \mathcal{W})$ a class of "projective objects" is given by complexes consisting point-wise[7] *of projective objects in $\mathcal{A}$. It works well to derive all functors induced by* additive *functors $\mathcal{C} \to \mathcal{D}$.*

In example 2., the class of projective objects can be used to derive functors induced by *arbitrary* functors $\mathcal{C} \to \mathcal{D}$ because for those objects the notions of weak equivalence and homotopy equivalence coincide[8]. Homotopy equivalences are trivially preserved under any functor induced by some functor $\mathcal{C} \to \mathcal{D}$. The same reasoning applies to example 3. (here the additivity is used to define homotopy equivalences, hence the functor has to be additive.)

[5] An object P in an algebraic category is called projective, if for any surjection $X \to Y$, the map $\mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective. In particular, free objects are projective.

[6] Assuming that every object is fibrant in the language of model categories, i.e. here a Kan complex. This is automatic if the forgetful functor $\mathcal{C} \to \mathcal{SET}$ factors through groups.

[7] In the unbounded case one has to work with the slightly more complicated notion of dg-projective.

[8] Assuming that every object is a Kan complex as before.

2.4. For right exact functors[9] F we have in addition an isomorphism

$$\pi_0(LF(X_\bullet)) = F\pi_0(X_\bullet).$$

If F is also left exact[10] then F preserves weak equivalences[11], and hence

$$LF(X_\bullet) = F(X_\bullet).$$

Example 2.5 (Abelian case)**.** *Let R be a commutative ring. The tensor product $\otimes : \mathcal{M}_R \times \mathcal{M}_R \to \mathcal{M}_R$ induces a functor $C(\mathcal{M}_R) \times C(\mathcal{M}_R) \to C(\mathcal{M}_R)$ on complexes which can be derived* equivalently *a) as a bifunctor, b) in the first, or c) in the second variable to a functor*

$$\overset{\mathrm{L}}{\otimes} : \mathcal{D}(\mathcal{M}_R) \times \mathcal{D}(\mathcal{M}_R) \to \mathcal{D}(\mathcal{M}_R).$$

Example 2.6. *Consider the functors $\mathcal{M}_R \to \mathcal{M}_R$ given by*

$$X \mapsto T^n(X), S^n(X), \Lambda^n(X), \Gamma^n(X)$$

(tensor power, symmetric power, exterior power, divided power). These functors are non-additive but can be derived in the non-Abelian sense to a functor

$$L\cdots^n : \mathcal{D}_{\le 0}(\mathcal{M}_R) \to \mathcal{D}_{\le 0}(\mathcal{M}_R).$$

For the tensor power we have

$$LT^n(X) = X \overset{\mathrm{L}}{\otimes} \cdots \overset{\mathrm{L}}{\otimes} X,$$

in particular, $LT^n(X[1]) \cong LT^n(X)[n]$. Warning: This isomorphism anticommutes with any odd permutation of the factors. A more refined analysis shows:

$$\begin{aligned} L\Lambda^n(X[1]) &\cong L\Gamma^n(X)[n] \\ LS^n(X[1]) &\cong L\Lambda^n(X)[n] \end{aligned}$$

Example 2.7. *Let S be a commutative R-algebra and consider the derived functor*

$$- \overset{\mathrm{L}}{\otimes}_R S : \mathcal{D}(\mathcal{A}_R) \to \mathcal{D}(\mathcal{A}_S)$$

of the tensor-product (base change) $- \otimes_R S : \mathcal{A}_R \to \mathcal{A}_S$. Since the forgetful functor from R-algebras to R-modules maps free (i.e. polynomial) algebras to (infinitely generated) free modules, we see that it commutes with taking derived tensor products. In other words, on underlying R-modules, $- \overset{\mathrm{L}}{\otimes}_R S$ induces just the derived tensor product of before, justifying the use of the same symbol. An example: Using the standard free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ p\ } \mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow 0$$

[9] in the non-Abelian case: commutes with reflexive coequalizers.

[10] in the non-Abelian case: commutes with all finite limits.

[11] The property of $X_\bullet \to Y_\bullet$ being a trivial fibration (i.e. a Kan fibration and a weak equivalence) in $\mathcal{C}^{\Delta^{\mathrm{op}}}$ can equivalently be stated saying that the map $X_{n+1} \to \mathrm{cosk}_n(X_{\le n}|Y_\bullet)$ is surjective for all n, where $\mathrm{cosk}_n(X_{\le n}|Y_\bullet)$ is the relative coskeleton functor (a certain finite limit). It follows that F preserves trivial fibrations (it preserves surjections because it is right exact) and thus weak equivalences by Ken Brown's Lemma.

of $\mathbb{F}_p$ (as a module) we get that

$$\pi_0(R \overset{\mathrm{L}}{\otimes} \mathbb{F}_p) = R \otimes \mathbb{F}_p \qquad \pi_1(R \overset{\mathrm{L}}{\otimes} \mathbb{F}_p) = R[p] \ (p\text{-torsion subgroup}) \qquad \pi_i(R \overset{\mathrm{L}}{\otimes} \mathbb{F}_p) = 0 \ i \geq 2.$$

However, the non-Abelian definition shows that $R \overset{\mathrm{L}}{\otimes} \mathbb{F}_p$ carries a Frobenius. Explicitly: Take a resolution of R by free $\mathbb{Z}$-algebras $P_\bullet \to R$ (for example (5)). Then the Frobenius is just the usual Frobenius of the simplicial $\mathbb{F}_p$-algebra $(P \otimes \mathbb{F}_p)_\bullet$. An exercise shows that Frobenius induces the zero map on $\pi_1(R \overset{\mathrm{L}}{\otimes} \mathbb{F}_p)$.

Example 2.8. *An important example which motivated the study of non-Abelian derived functors is the derived functor of the functor $S \mapsto \Omega_{S|R}$ of Kähler differentials on R-algebras. Its derived functor applied to an R-algebra S*

$$L_{S|R}$$

is called the **cotangent complex** *of S over R. It can be defined as an object of $\mathcal{D}_{\leq 0}(\mathcal{M}_S)$ and agrees with $\Omega_{S|R}$ for smooth algebras, but behaves a lot better in other cases. For example for an S-algebra T we have an exact triangle*

$$L_{S|R} \overset{\mathrm{L}}{\otimes}_S T \longrightarrow L_{T|R} \longrightarrow L_{T|S} \longrightarrow (L_{S|R} \overset{\mathrm{L}}{\otimes}_S T)[1].$$

3 Derived (aka. homotopy) limits and colimits

3.1. One important class of derived functors consists of the derived limit and colimit functors. In homological algebra one works usually with the derived category $D(\mathcal{A})$ equipped with the *additional structure* given by shift functors and distinguished triangles. Similar structures (the (co)fiber sequences) can also be defined on the non-Abelian derived category $\mathcal{D}(\mathcal{C})$. A problem is that those cannot be reconstructed from (i.e. are not intrinsic to) the category $D(\mathcal{A})$ alone. In each case the additional structure is a *shadow* of the calculus of derived limits and colimits. We concentrate here on the case of colimits, the other case being dual.

3.2. Let $\mathcal{C}$ be a category. Recall that we say that $\mathcal{C}$ has all colimits, if for any diagram (i.e. small category) I there is an adjunction

$$\mathcal{C}^I \underset{c}{\overset{\text{colim}}{\rightleftarrows}} \mathcal{C}$$

in which c is the functor "constant diagram". The left adjoint "colim" applied to a diagram $X : I \to \mathcal{C}$ is, by definition, called the *colimit of X*.
In any of the examples above (Abelian or not) we may derive these functors (in the sense of the previous paragraph) to get an adjunction

$$\mathcal{D}(\mathcal{C}^I) \underset{c}{\overset{L\,\text{colim}}{\rightleftarrows}} \mathcal{D}(\mathcal{C})$$

Note that this is *not* a definition which is intrinsic to the derived category $\mathcal{D}(\mathcal{C})$ because in most cases $\mathcal{D}(\mathcal{C}^I) \ncong \mathcal{D}(\mathcal{C})^I$:

Conterexample 3.3. *Let $\mathcal{A}$ be an Abelian category and $I = \rightarrow$. Then*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})^{\rightarrow}}\left(\begin{array}{c}0\\ \downarrow\\ B\end{array}, \begin{array}{c}A[1]\\ \downarrow\\ 0\end{array}\right) = 0$$

because any morphism (from left to right)

$$\begin{array}{ccc}0 & \longrightarrow & A[1]\\ \downarrow & & \downarrow\\ B & \longrightarrow & 0\end{array}$$

in $\mathcal{D}(\mathcal{A})^{\rightarrow}$ is obviously zero. However

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A}^{\rightarrow})}\left(\begin{array}{c}0\\ \downarrow\\ B\end{array}, \begin{array}{c}A\\ \downarrow\\ 0\end{array}[1]\right) = \mathrm{Hom}_{\mathcal{A}}(A, B),$$

the morphisms corresponding to non-trivial extensions

$$\begin{array}{ccccc}A & \longrightarrow & A & \longrightarrow & 0\\ \downarrow & & \downarrow & & \downarrow\\ 0 & \longrightarrow & B & \longrightarrow & B\end{array}$$

in the Abelian category $\mathcal{A}^{\rightarrow}$.

The notion of homotopy colimit defines the additional structure on the derived categories:

Example 3.4. *In the Abelian case, it is an exercise to show that for a morphism $\alpha : X \to Y$ of complexes*

$$L\,\mathrm{coker}(\alpha) = C(\alpha)$$

is the cone of α and in particular

$$L\,\mathrm{coker}(A \to 0) = A[1].$$

Hence the shift functor is a homotopy colimit and the class of distinguished triangles is given by the sequences

$$\{\ A \xrightarrow{\alpha} B \longrightarrow L\,\mathrm{coker}(\alpha) \longrightarrow L\,\mathrm{coker}(A \to 0)\ \} \tag{2}$$

or equivalently

$$\{\ R\ker(0 \to D) \longrightarrow R\ker(\gamma) \longrightarrow C \xrightarrow{\gamma} D\ \} \tag{3}$$

Example 3.5. *In the non-Abelian case, we may still form the sequences (2) and (3)* [12]. *Those are called* cofiber *and* fiber sequences, *respectively. In contrast to the Abelian case those are not the same class!*

[12] If $\mathcal{C}$ does not have a zero object replace the 0 by the final, resp. initial object.

The equality of the classes of (2) and (3) is referred to as the **stability** of the "homotopy theory". This definition can be made precise for model categories, derivators, and infinity categories alike.

Example 3.6. *The* total complex *of a complex of complexes is a derived colimit as well. Observe that it is completely hopeless to construct it from a complex in the derived category because the cone is not functorial. By the Dold-Kan correspondence a complex (in degrees* ≤ 0*) of complexes, i.e. a double complex* $C_{\bullet,\bullet}$ *can be seen as a simplicial object* $C_\bullet := \Gamma(C_{\bullet,\bullet}) \in C(\mathcal{A})^{\Delta^{\mathrm{op}}}$ *and we have that*

$$\mathrm{tot}^{\oplus}(C_{\bullet,\bullet}) = L\,\mathrm{colim}(C_\bullet).$$

Example 3.7. *The homotopy colimit of a diagram* $X_\bullet \in \mathcal{TOP}^{\Delta^{\mathrm{op}}}$ *may be seen as a non-Abelian analogue of the total complex (where, of course, we could replace* $\mathcal{TOP}$ *by any of the categories* $\mathcal{C}^{\Delta^{\mathrm{op}}}$*). If the* X_i *are actually sets, then* $L\,\mathrm{colim}\, X_\bullet$ *is just the geometric realization* $\Gamma(X_\bullet)$ *in the sense of 1.1. Therefore a (derived) colimit over* Δ^{op} *is often called* **geometric realization**.

Example 3.8. *One feature of the (not yet precisely explained) fact that* $(\mathcal{TOP}, \mathcal{W})$ *and* $(\mathcal{SET}^{\Delta^{\mathrm{op}}}, \mathcal{W})$ *define the same "homotopy theory" is that derived limits and colimits agree*[13]*. Derived (or* homotopy*) limits in* $\mathcal{TOP}$ *can be described very explicitly. For instance the homotopy pull-back (i.e. derived limit) of a diagram of topological spaces*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

is the space of pairs $(x, z) \in X \times Z$ *together with a path from* $f(x)$ *to* $g(z)$ *in* Y*.*

4 Towards ∞-categories

Q1 Is there a variant of the concept of category that makes the derived limit and colimit the instrinsic notion of limit and colimit?

Note that this means that for such derived "categories" $\mathcal{D}(\mathcal{C})$ we should have an "equivalence" $\mathrm{Fun}(I, \mathcal{D}(\mathcal{C})) \cong \mathcal{D}(\mathcal{C}^I)$ of some sort. The notion of ∞-category provides such a variant. The decoration "∞" comes from a different point of view, though.

Recall that a (classical) n-category is recursively defined as an entity with a class of objects and $n-1$-categories of morphism between them, such that the composition is given by (strict) functors between $n-1$-categories, and which is (strictly) associative. Putting the adjective strict everywhere, this recursive definition works indeed well. Many variants have been proposed for weak variants where, e.g. composition is just pseudo-functorial or where associativity holds only up to higher isomorphisms... creating a zoo of different notions that are mostly inequivalent and difficult to work with.

Q2 Is there a workable variant of the concept of category that allows n-morphisms for all $n \in \mathbb{N}$?

We would call such a variant an ∞-category.

It is not obvious how these two questions are related. Let me give some hints in this direction. In each of the cases discussed above we can actually define something like an ∞-category:

[13] This may be expressed by saying: They define equivalent **derivators**.

4.1. In the Abelian case, one can actually define an Hom-complex of Abelian groups for each pair of chain complexes $X_\bullet, Y_\bullet$:

$$\cdots \longrightarrow \mathrm{Hom}_2(X_\bullet, Y_\bullet) \longrightarrow \mathrm{Hom}_1(X_\bullet, Y_\bullet) \longrightarrow \mathrm{Hom}_0(X_\bullet, Y_\bullet) \longrightarrow 0$$

Elements in $\mathrm{Hom}_n(X_\bullet, Y_\bullet)$ can be interpreted as $n+1$-morphisms[14]. $\mathrm{Hom}_0(X_\bullet, Y_\bullet)$ is the set of morphisms of complexes, and $\mathrm{Hom}_n(X_\bullet, Y_\bullet), n \geq 1$ consists of collections of morphisms $X_i \to Y_{i+n}$ for all i. The differential is given by $\alpha \mapsto d\alpha + \alpha d$. This gives a complex with the property that

$$H_0(\mathrm{Hom}_\bullet(X_\bullet, Y_\bullet))$$

is the set of homotopy classes of morphisms of complexes in the usual sense. If one restricts to complexes of projective objects then

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X_\bullet, Y_\bullet) \cong H_0(\mathrm{Hom}_\bullet(X_\bullet, Y_\bullet)).$$

4.2. In the non-Abelian case one can define a *simplicial set*

$$\mathrm{Hom}_\bullet(X_\bullet, Y_\bullet)$$

for any pair of simplicial objects $X_\bullet, Y_\bullet$ in $\mathcal{C}$ by the formula[15]:

$$\mathrm{Hom}_n(X_\bullet, Y_\bullet) = \mathrm{Hom}(X_\bullet \otimes \Delta_n, Y_\bullet)$$

and elements can be seen als n-morphisms. For projective objects[16] we have again that:

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(X_\bullet, Y_\bullet) \cong \pi_0(\mathrm{Hom}_\bullet(X_\bullet, Y_\bullet)).$$

It is true, but not completely obvious, that the notions of derived functors and derived (co)limits can also be defined using these collections of "higher morphisms". In fact, the notions of derived (co)limits discussed above become very similar to the notions of (1-categorical) (co)limits. How can we make the axioms for such a collection of higher morphisms precise? Note that in the non-Abelian example they are already naturally arranged in a simplicial set. This will be the key in general:

5 The homotopy hypothesis

A canonical way of producing a structure of "higher category" arises from a topological space X. Objects are the points of X. 1-Morphisms $x \to y$ are paths from x to y. 2-morphisms are homotopies between paths, and so on. Note that in this example all n-morphisms, including $n = 1$, are equivalences (i.e. invertible up to higher morphisms).
Grothendieck's [10] insight was:

> In any reasonable theory, those ∞-categories in which all n-morphisms are equivalences (also called $(\infty, 0)$-categories or ∞-groupoids) should be up to equivalence the same as topological spaces up to weak equivalence.

[14] This gives actually the structure of a *dg-category*.

[15] $X_\bullet \otimes \Delta_n$ does not need to exist as an object of $\mathcal{C}^{\Delta^{\mathrm{op}}}$. The set $\mathrm{Hom}(X_\bullet \otimes \Delta_n, Y_\bullet)$ consists of collections of morphisms $\alpha_{n,\nu} \in \mathrm{Hom}_{\mathcal{C}}(X_k, X_k)$, $n \in \mathbb{N}, \nu : \Delta_k \to \Delta_n$ which is compatible with the maps in Δ^{op} acting on both $\alpha_{n,\nu}$ and ν.

[16] Cofibrant and fibrant objects in the language of model categories. Fibrant is automatic here, if the forgetful functor factors through (Abelian) groups.

Taking this hypothesis seriously, in an ∞-category in which all n-morphisms for $n \geq 2$ are equivalences (also called $(\infty, 1)$-categories; we will discuss only those) one should have a topological space (or, equivalently, a simplicial set)

$$\mathrm{Hom}(X, Y)$$

for any two objects X, Y well-defined up to weak equivalences. Note that this matches up with the structure we got in the examples (in the Abelian case, interpret the Hom-complex as a simplicial set via the Dold-Kan correspondence).

6 Precise definitions of ∞-categories

There are many different generalized definitions of categories in which the morphisms between two objects are simplicial sets or topological spaces. It turns out that they all give (in some precise sense) equivalent theories of ∞-categories:

1. topological categories
2. simplicial categories
3. Segal categories
4. complete Segal spaces
5. quasi-categories

Since the work of Boardman-Vogt [1], Joyal [13], and Lurie [14], mostly the notion of quasi-category is used as the *definiton* of an ∞-category. Even doing so, it is convenient however, to be able to define specific ∞-categories by providing (for instance) simplicial categories and then construct an equivalent quasi-category using the abstract constructions used to see that all these models are equivalent.

7 Quasi-categories

This section gives only a very rough sketch. For a detailled overview see [9, 14].

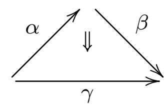
7.1. The insight of Boardman-Vogt and Joyal was that one can reasonably define an ∞-category *itself* as a simplicial set with certain properties. Observe that any usual 1-category $\mathcal{C}$ defines a simplicial set $N(\mathcal{C})$, the nerve of $\mathcal{C}$, defined by

$$N(\mathcal{C})_n := \mathrm{Ob}(\mathcal{C}^{\Delta_n}) = \{i_0 \to \cdots \to i_n \ \in \mathcal{C}\}.$$

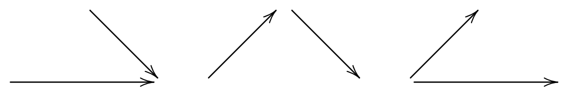
In other words, 0-simplices are the objects of $\mathcal{C}$, 1-simplicies are the morphisms in $\mathcal{C}$, and n-simplicies are the compositions of n morphisms in $\mathcal{C}$.
On the other hand, by the homotopy hypothesis, an ∞-category in which all morphisms are invertible should be represented by a simplicial set or topological space up to weak equivalence. Such can be represented by a Kan complex, i.e. a simplicial set in which all "horns can be filled". A horn is the boundary of a standard simplex with one face missing.

> Can we find a reasonable generalization of Kan complex which encompasses the nerves of usual categories?

In any case, we would like to interpret 0-simplices as objects, 1-simplices as morphisms, and 2-morphisms as "possible compositions" — or, referring to the following picture, as 2-morphisms from some fixed composition of α and β to γ:



The horn filling property of Kan complexes insists that the horns



Λ_2^0 Λ_2^1 Λ_2^2

all can be filled up to such a 2-simplex. Only for Λ_2^1 though (called *inner* horn) this corresponds to the existence of compositions. The requirement for the other horns corresponds to the existence of left and right inverses (take the long morphism to be the identity). Correspondingly, in the nerve of a usual 1-category the first two horns cannot necessarily be filled, but the inner horn can be filled, and *uniquely* so, and this gives precisely the composition in the category.
The suggests that we should only insist that the *inner* horn can be filled, but not necessarily in a unique way. What is the correct condition in higher dimensions?
An n-simplex has $n+1$ horns which are naturally ordered $\Lambda_n^0, \dots, \Lambda_n^n$. All horns except Λ_n^0 and Λ_n^n are called inner horns (Λ_2^1 is thus the first one that occurs).

Proposition 7.2. *The following are equivalent conditions on a simplicial set $X_\bullet$:*

1. *Every inner horn (in any dimension) can be filled to a simplex.*
2. *The map $\mathrm{Hom}_\bullet(\Delta_2, X_\bullet) \to \mathrm{Hom}_\bullet(\Lambda_2^1, X_\bullet)$ is a Kan fibration*[17] *and a weak equivalence.*

Let us discuss condition 2.: For any horm $h : \Lambda_2^1 \to X_\bullet$ we can define the homotopy pullback (cf. Example 3.8) of simplicial sets

$$\begin{array}{ccc} \{\text{space of possible fillings of } h\} & \longrightarrow & \mathrm{Hom}_\bullet(\Delta_2, X_\bullet) \\ \downarrow & & \downarrow \\ \{h\} & \hookrightarrow & \mathrm{Hom}_\bullet(\Lambda_2^1, X_\bullet) \end{array}$$

Condition 2. of the Theorem implies that the left vertical morphism is a weak equivalence for any h, i.e. the "space of possible fillings" is contractible. In other words: A composition exists and is "as unique as possible". We will later see that the condition of being contractible will occur frequently as the correct analogue of "uniqueness of morphisms" or of "uniqueness up to unique isomorphism of objects", respectively, in usual category theory.

Definition 7.3. *A simplicial set satisfying the equivalent conditions of Proposition 7.2 is called a* **weak Kan complex***, or* **quasi-catgeory***, or just* **∞-category***.*

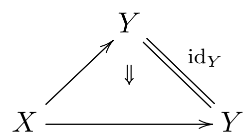
[17] a relative version of the notion of Kan complex

This definition includes usual 1-categories:

Proposition 7.4. *The following are equivalent conditions on a simplicial set*

1. *Every inner horn (in any dimension) can be filled* uniquely *to a simplex.*
2. *The simplicial set is isomorphic to the nerve of a usual 1-category.*

7.5. A quasi-category $\mathcal{C}$ gives mapping spaces (i.e. simplicial sets) $\mathrm{Hom}_\bullet(X, Y)$ for each pair of objects (i.e. 0-simplices) X, Y in which 0-simplices are just the paths (1-simplices) of $\mathcal{C}$, 1-simplicies are 2-simplices of $\mathcal{C}$ of the form



In higher dimension: Elements of $\mathrm{Hom}_n(X, Y)$ are $n+1$-simplices of $\mathcal{C}$ such that the first vertex is X and the opposite face is completely degenerate with value Y. This way, we obtain a simplicial category from a quasi-category.

7.6. There is an inverse (up to equivalence) to this construction. Starting from a simplicial category[18], one can construct a quasi-category $\mathcal{C}$ such that $\mathcal{C}_0$ becomes the set (class) of objects, and the mapping spaces above recover (up to weak equivalence) the simplicial sets of morphisms. We do not discuss the details of this construction.

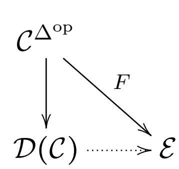
8 The derived ∞-categories

8.1. In all examples (4.1–4.2) we defined simplicial categories and thus get quasi-categories applying 7.6. It is *essential* that we restrict to full subcategories of "projective objects" in all the examples: Weak equivalences have to become actual equivalences!
We denote the ∞-categories so obtained from now on by the same letter: $\mathcal{D}(\mathcal{C}), \mathcal{D}(\mathcal{A})$, etc. The ∞-category obtained from simplicial sets (or equivalently: topological spaces) is denoted by $\mathcal{S}$.

8.2. It seems a bit unfortunate that this construction seems to depend on the particular choice of "projective objects" (usually there are many equally good choices induced by different model category structures with the same class of weak equivalences, e.g. in the Abelian case "projective" could be replaced by "injective"). However, using resolutions, one can define a functor of ∞-categories

$$\mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{D}(\mathcal{C})$$

(where the source is just the usual 1-category). This functor satisfies a universal property analogous to localizations of usual categories:



> Every functor F to an ∞-category $\mathcal{E}$ which maps weak equivalences in $\mathcal{C}^{\Delta^{\mathrm{op}}}$ to actual equivalences in $\mathcal{E}$ (see next section) factors "uniquely" through $\mathcal{D}(\mathcal{C})$.

[18] such that all Hom simplicial sets are Kan complexes

This characterizes the ∞-category $\mathcal{D}(\mathcal{C})$ up to equivalence. One could therefore define (analogously to the case of usual categories):

$$\mathcal{D}(\mathcal{C}) := \mathcal{C}^{\Delta^{\mathrm{op}}}[\mathcal{W}^{-1}]$$

as the ∞-categorical localization. Does a localization always exist? Yes, and it is best described as a simplicial category, the *simplicial localization* due to Dwyer and Kan [7, 8]. 1-morphisms are still given by zig-zags

$$X \xleftarrow{\in W} X_1 \longrightarrow X_2 \xleftarrow{\in W} \cdots \longrightarrow X_{n-2} \xleftarrow{\in W} X_{n-1} \longrightarrow Y$$

as in the usual localization. However, there are now higher morphisms.
The description as localization shows that the class of "projective objects" does not matter. However, as in the classical case, it is convenient for actual calculations.

9 Translation

Denote by $\mathcal{S}$ the ∞-category obtained from $\mathcal{SET}^{\Delta^{\mathrm{op}}}$ (or equivalently from $\mathcal{TOP}$). Objects will just be called *spaces* in the sequel.
Many familiar concepts in usual 1-category theory translate to ∞-categories in a straight-forward way [3, 14, 17]:

1-categories	**∞-categories**
$\mathcal{SET}$	$\mathcal{S}$
Hom set $\mathrm{Hom}(X,Y)$	Hom space $\mathrm{Hom}(X,Y)$
Functor category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$	Functor ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$
isomorphism $X \cong Y$	equivalence $X \cong Y$, i.e. ∃ morphisms $a: X \to Y$ and $b: Y \to X$ and 2-morphisms $ba \to \mathrm{id}_X$ and $ab \to \mathrm{id}_Y$ (necessarily invertible).
fully faithful, $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX,FY)$ isomorphism	fully faithful, $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX,FY)$ weak equivalence
representable functor $F \to \mathcal{SET}$: isomorphism $\mathrm{Hom}(X,-) \cong F(-)$ of functors	representable functor $F \to \mathcal{S}$: equivalence $\mathrm{Hom}(X,-) \cong F(-)$ of functors
adjoint functors $F \dashv G$, isomorphism $\mathrm{Hom}(F-,-) \cong \mathrm{Hom}(-,G-)$ of functors	adjoint functors $F \dashv G$, equivalence $\mathrm{Hom}(F-,-) \cong \mathrm{Hom}(-,G-)$ of functors
colimit $\mathrm{colim}\, X$, object with isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, X,-) \cong \mathrm{Hom}_{\mathcal{C}^I}(X,c-)$	colimit $\mathrm{colim}\, X$, object with equivalence $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, X,-) \cong \mathrm{Hom}_{\mathcal{C}^I}(X,c-)$
Yoneda embedding $\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}},\mathcal{SET})$ fully faithful	Yoneda embedding $\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$ fully faithful

Statements like

> representing objects, adjoint functors, and (co)limits, are defined up to unique isomorphism

translate to

> ... are defined up to a contractible choice.

Example 9.1. *The case of representable functors. The space of choices of objects X representing F is the pull-back of spaces*[19]

$$\begin{array}{ccc} \{\textit{space of } X \textit{ representing } F\} & \longrightarrow & \mathcal{C}^{\sim} \\ \downarrow & & \downarrow \\ \{F\} & \hookrightarrow & \mathrm{Fun}(\mathcal{C},\mathcal{S})^{\sim} \end{array}$$

of course to be seen in the homotopy invariant sense — i.e. as homotopy pull-back (3.8). Explicitly, this is the space of objects X of $\mathcal{C}^{\sim}$ together with an equivalence $\mathrm{Hom}(-,X) \cong F(-)$ *of functors. However, the right vertical map is a weak equivalence on connected components (this follows from the ∞-analogue of fully-faithfulness of the Yoneda embedding). Therefore the left vertical map is a weak equivalence, i.e. the space of X representing F is contractible.*

The notion of (co)limit above recovers the notion of derived (co)limit. This follows from

Proposition 9.2. *For the ∞-versions of the derived categories of a category $\mathcal{C}$ (Abelian or not), we have an equivalence*

$$\mathcal{D}(\mathcal{C}^I) \cong \mathcal{D}(\mathcal{C})^I$$

for every small category I.

10 Towards a universal property of derived categories

In the theory of ∞-categories the passage from a (usual) category $\mathcal{C}$ to the derived ∞-category $\mathcal{D}(\mathcal{C})$ enjoys an important universal property, similar to properties of presheaf categories in usual category theory. That is, not only the functors $\mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{D}(\mathcal{C})$, resp. $C(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ enjoy universal properties (as localizations), but $\mathcal{D}(\mathcal{C})$ itself, explaining thus conceptually the apperance of simplicial objects or complexes[20].

10.1. To understand those universal properties, let us review the classical universal properties of classical presheaf 1-categories. Recall the fully-faithful Yoneda embedding

$$\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C}) := \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{SET}).$$

For any cocomplete category $\mathcal{D}$, it induces an equivalence of categories[21]

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \cong \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}),\mathcal{D})$$

where Fun^L is the full subcategory of functors which commute with all colimits. Furthermore those all have automatically right adjoints just given by

$$D \mapsto \{X \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X),D)\}$$

[19]For an ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^{\sim}$ the subcategory obtained by deleting all 1-morphisms that are not equivalences. It is therefore (by the homotopy hypothesis) equivalent to a space.

[20]Keep in mind, however, citing Dan Dugger [6] in a similar context: "This is really a silly statement, as simplicial sets are in some sense built into the very fabric of what people have decided they mean by a 'homotopy theory' ". However, the homotopy theory of spaces *can* be characterized using merely abstract properties of (derived) limits and colimits [2].

[21]For this to work $\mathcal{C}$ has to be small (or $\mathcal{D}$ in a larger universe than $\mathcal{C}$).

where $F : \mathcal{C} \to \mathcal{D}$ is the functor corresponding to it.
The property is sometimes referred to by saying that $\mathcal{P}(\mathcal{C})$ is obtained from $\mathcal{C}$ by *freely adjoining all colimits*. If $\mathcal{C}$ has all colimits already then $\mathcal{C}$ becomes a reflexive subcategory of $\mathcal{P}(\mathcal{C})$.
Often one is interested in categories which are obtained from $\mathcal{C}$ by "freely adjoining" only certain colimits, say those in a subclass $\mathcal{X}$ of diagrams. Define $\mathcal{P}_{\mathcal{X}}(\mathcal{C})$ as the subcategory of presheaves in $\mathcal{P}(\mathcal{C})$ that are colimits of shape $I \in \mathcal{X}$ of representable presheaves.
Then we have analogously

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}^{\mathcal{X}}(\mathcal{P}_{\mathcal{X}}(\mathcal{C}), \mathcal{D})$$

where $\mathrm{Fun}^{\mathcal{X}}$ is the full subcategory of functors which commute with all colimits of shape $I \in \mathcal{X}$.
Reasonable classes $\mathcal{X}$ include

1. filtered diagrams
2. reflexive coequalizer diagram, i.e. $\mathcal{X} = \{\Delta^{\mathrm{op}}_{\le 1}\}$
3. sifted diagrams (in some sense the closure of 1. and 2.)

For $\mathcal{X} = \{$ filtered diagrams $\}$, the category $\mathcal{P}_{\mathcal{X}}(\mathcal{C})$ is called the Ind-category of $\mathcal{C}$.

Example 10.2. *In many cases the presheaf categories are equivalent to old friends. For example for $\mathcal{C}$ the category of finitely generated R-algebras the Ind-category is equivalent to the category of all R-algebras. Other examples (where R denotes any Noetherian ring):*

$\mathcal{P}_{\mathcal{X}}(\mathcal{C})$	$\mathcal{C} = \{$ *f.g. polynomial R-algebras* $\}$	$\mathcal{C} = \{$ *polynomial R-algebras* $\}$
$\mathcal{X} = \{$ *filtered diagrams* $\}$	$\{$ *polynomial R-algebras* $\}$	
$\mathcal{X} = \{ \Delta^{\mathrm{op}}_{\le 1} \}$	$\{$ *f.g. R-algebras* $\}$	$\{$ *R-algebras* $\}$
$\mathcal{X} = \{$ *sifted diagrams* $\}$	$\{$ *R-algebras* $\}$	

Example 10.3. *For modules (over a Noetherian ring R) there is a similar pattern:*

$\mathcal{P}_{\mathcal{X}}(\mathcal{C})$	$\mathcal{C} = \{$ *f.g. projective R-modules* $\}$	$\mathcal{C} = \{$ *projective R-modules* $\}$
$\mathcal{X} = \{$ *filtered diagrams* $\}$	$\{$ *projective R-modules* $\}$	
$\mathcal{X} = \{ \Delta^{\mathrm{op}}_{\le 1} \}$	$\{$ *f.g. R-modules* $\}$	$\{$ *R-modules* $\}$
$\mathcal{X} = \{$ *sifted diagrams* $\}$	$\{$ *R-modules* $\}$	

Remark 10.4. *Under some restrictions on $\mathcal{C}$, the presheaves in $\mathcal{P}_{\mathcal{X}}(\mathcal{C})$ can be characterized as those presheaves that transform colimits of some shape $\mathcal{X}^{\perp}$ into limits in $\mathcal{SET}$.*
The class $\mathcal{X}^{\perp}$ is precisely the class of diagrams J such that — in $\mathcal{SET}$ (!) — colimits of shape in $I \in \mathcal{X}$ commutes with limits of shape J.
Under this correspondence for instance (up to closure)

$\mathcal{X}$	$\mathcal{X}^{\perp}$
$\{$ *all diagrams* $\}$	$\{ \cdot \}$
$\{$ *filtered diagrams* $\}$	$\{$ *finite diagrams* $\}$
$\{$ *sifted diagrams* $\}$	$\{$ *finite sets* $\}$
$\{ \cdot \}$	$\{$ *all diagrams* $\}$

In each case, we have the property that *all* colimits can be computed by combining colimits of shape in $\mathcal{X}$ with colimits of shape in $\mathcal{X}^{\perp}$.

11 The universal property of derived ∞-categories

Reference [14, 5.5.8–5.5.9].

11.1. In the ∞-categorical world we have the analogous Yoneda embedding

$$\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

and define again $\mathcal{P}_{\mathcal{X}}(\mathcal{C})$ as the full subcategory of those functors which are (equivalent to) colimits of shape in $\mathcal{X}$ of representables[22]. As classes one should prefer

1. filtered diagrams
2. geometric realization, i.e. $\mathcal{X} = \{\Delta^{\mathrm{op}}\}$
3. homotopy sifted diagrams (again the closure of 1. and 2.[23])

Here a geometric realization is a colimit over Δ^{op} (cf. 3.7). Whereas in the 1-categorical world there is no difference between a colimit of shape Δ^{op} and a reflexive coequalizer (i.e. a colimit of shape $\Delta^{\mathrm{op}}_{\leq 1}$) the distinction is essential in the world of ∞-categories. Our first encounter of this essential difference have been examples 3.6 and 3.7 when discussing derived colimits.
And indeed:

Proposition 11.2. *Let $\mathcal{C}$ be a usual algebraic or Abelian category (with enough projectives) and denote by $\mathcal{C}^p$ the full subcategory of projective objects. Then we have an equivalence*

$$\mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p) \cong \mathcal{D}(\mathcal{C}).$$

For an idea of proof see 11.8 below.
In other words, the derived ∞-category fulfills the following universal property:

Proposition 11.3.

$$\mathrm{Fun}(\mathcal{C}^p, \mathcal{D}) \cong \mathrm{Fun}^{\{\Delta^{\mathrm{op}}\}}(\mathcal{D}(\mathcal{C}), \mathcal{D}), \tag{4}$$

where $\mathrm{Fun}^{\{\Delta^{\mathrm{op}}\}}$ *is the full subcategory of functors which preserve geometric realizations.*

11.4. In the examples above filtered colimits are exakt[24] so there is no difference in the ∞-categorical world and the table of Example 10.2 changes to:

$\mathcal{P}_{\mathcal{X}}(\mathcal{C})$	$\mathcal{C}$ = { f.g. polynomial R-algebras }	$\mathcal{C}$ = { polynomial R-algebras }
$\mathcal{X}$ = { filtered diagrams }	{ polynomial R-algebras }	
$\mathcal{X}$ = { Δ^{op} }	$\mathcal{D}$({ f.g. R-algebras })	$\mathcal{D}$({ R-algebras })
$\mathcal{X}$ = { homotopy sifted diagrams }	$\mathcal{D}$({ R-algebras })	

and similarly for modules.

Remark 11.5. *Under some restrictions on $\mathcal{C}$, the presheaves in $\mathcal{P}_{\mathcal{X}}(\mathcal{C})$ can (similarly as in 10.4) be characterized as those presheaves that transform colimits of some shape $\mathcal{X}^{\perp}$ into limits in $\mathcal{S}$.*

[22] here as elements of $\mathcal{X}$ ∞-categories could be allowed as well, but this is irrelevant because any colimit of shape I, where I is an ∞-category can also be described as the colimit over a classical diagram.
[23] These are just called sifted diagrams in [14].
[24] i.e. commute with finite limits as well

The class $\mathcal{X}^\perp$ is the class of diagrams J such that — now in $\mathcal{S}$ (!) — (homotopy) colimits of shape in $I \in \mathcal{X}$ commutes with (homotopy) limits of shape J.
The diagram from 10.4 changes to

$\mathcal{X}$	$\mathcal{X}^\perp$
{ *all diagrams* }	{ $\cdot$ }
{ *filtered diagrams* }	{ *finite diagrams* }
{ *homtopy sifted diagrams* }	{ *finite sets* }
{ $\cdot$ }	{ *all diagrams* }

In particular, for $\mathcal{X}$ = { homotopy sifted diagrams }, we could also define $\mathcal{P}_\mathcal{X}(\mathcal{C})$ as those presheaves in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ *that transform finite coproducts into finite products. For example $\mathcal{D}$({ R-algebras }) is the ∞-category of presheaves in* $\mathrm{Fun}(\{\text{ f.g. polynomial } R\text{-algebras }\}^{\mathrm{op}}, \mathcal{S})$ *that transform finite coproducts into finite products.*

11.6. The above universal properties do not quite describe the relation of $\mathcal{C}$ and $\mathcal{D}(\mathcal{C})$. However the inclusion $\mathcal{C}^p \hookrightarrow \mathcal{C}$ induces an adjunction

$$\mathcal{P}(\mathcal{C}^p) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C} \quad (L \dashv R)$$

and the right adjoint R is fully faithful and factors through $\mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p)$ [25].
Therefore we get abstractly a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{P}_{\Delta^{\mathrm{op}}}(\mathcal{C}^p)$, which under the above identification (Proposition 11.2) with the derived category is the canonical inclusion (constant simplicial objects) and its left adjoint is π_0.

Let $\mathcal{C}$ and $\mathcal{E}$ be algebraic or Abelian categories and let $F : \mathcal{C} \to \mathcal{E}$ be a functor. In the ∞-categorical language the left derived functor LF (as defined in Section 2) can be defined as the universal functor corresponding via (4) to the composition

$$\widetilde{F} : \mathcal{C}^p \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{E} \xrightarrow{\iota} \mathcal{D}(\mathcal{E})$$

The restriction of LF to $\mathcal{C}$ also has the following nice description:

Proposition 11.7. *The restriction of LF to $\mathcal{C}$ via the Yoneda embedding $R : \mathcal{C} \hookrightarrow \mathcal{P}_{\Delta^{\mathrm{op}}}(\mathcal{C}^p)$ is the left Kan extension (in the ∞-categorical sense) of $\widetilde{F}$ along $\mathcal{C}^p \hookrightarrow \mathcal{C}$.*

Proof. Consider the following diagram

$$\begin{array}{ccccc} & & \mathcal{C}^p & \hookrightarrow & \mathcal{C} \\ & & \downarrow & \underset{R}{\overset{\pi_0}{\rightleftarrows}} & \downarrow \iota\circ F \\ \mathcal{D}(C) & = & \mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p) & \xrightarrow[LF]{} & \mathcal{D}(\mathcal{E}) \end{array}$$

The functor LF itself is the Kan extension of $\widetilde{F}$ along the embedding $\mathcal{C}^p \hookrightarrow \mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p)$ as follows from the universal property (4). Therefore it suffices to see that the Kan extension of LF along

[25] $R(C)$ is the simplicial presheaf $P \mapsto \mathrm{Hom}(P, C)$ (with values in $\mathcal{SET} \subset \mathcal{SET}^{\Delta^{\mathrm{op}}}$). A projective resolution $P_\bullet \to C$ defines an equivalent presheaf given by $P \mapsto \mathrm{Hom}(P, P_\bullet)$. This is the same as the geometric realization of $P_\bullet$ (considered as diagram of represented presheaves). See also the end of 11.8.

π_0 is the same as the restriction via R. This is, however, a general non-sense fact: A left Kan extension along a left adjoint is composition with the right adjoint (this holds true in 1-category theory and ∞-category theory alike[26]). $\square$

11.8. *Sketch of proof of Proposition 11.2.* The existence of an equivalence

$$\mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p) \cong \mathcal{D}(\mathcal{C})$$

may be seen as follows. Both categories are localizations (in the ∞-categorical sense) of classical categories along a class of weak equivalences, namely $\mathcal{D}(\mathcal{C})$ is by definition

$$\mathcal{C}^{\Delta^{\mathrm{op}}}$$

localized at those morphisms which become weak equivalences in $\mathcal{SET}^{\Delta^{\mathrm{op}}}$ applying the forgetful functor. The full presheaf category $\mathcal{P}(\mathcal{C}^p)$ is the localization of

$$\mathrm{Fun}(\mathcal{C}^p, \mathcal{SET}^{\Delta^{\mathrm{op}}})$$

at those morphisms which at any object of $\mathcal{C}^p$ evaluate to weak equivalences.
The classical (1-categorical) theory gives an adjunction

$$\mathrm{Fun}((\mathcal{C}^p)^{\mathrm{op}}, \mathcal{SET}) \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$$

where R is just the Yoneda functor and L is produced by the universal property (using that $\mathcal{C}$ is cocomplete). This induces an adjunction on simplical objects:

$$\mathrm{Fun}((\mathcal{C}^p)^{\mathrm{op}}, \mathcal{SET}^{\Delta^{\mathrm{op}}}) \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}^{\Delta^{\mathrm{op}}}$$

R is fully faithful and a morphism α is a weak equivalence if and only if $R(\alpha)$ is a weak equivalence. For this, it is essential that we took presheaves on $\mathcal{C}^p$ — only those faithfully reflect the properties of the underlying set of the objects in $\mathcal{C}$ used to define the weak equivalences in $\mathcal{C}^{\Delta^{\mathrm{op}}}$.
A slightly more refined argument shows that L and R define in fact a Quillen adjunction w.r.t. appropriate model category structures. From this it follows that we have an adjunction

$$\mathcal{P}(\mathcal{C}^p) \underset{R_\infty}{\overset{L_\infty}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}(\mathcal{C})$$

of ∞-categories with R_∞ still fully faithful. It remains to see that the essential image of R_∞ is $\mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p)$. But every object X of $\mathcal{P}_{\{\Delta^{\mathrm{op}}\}}(\mathcal{C}^p)$ is a colimit of representables of shape Δ^{op}. This gives rise to a diagram $P_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}^p$ with $X \cong \operatorname{colim} P_\bullet$. A variant of Example 3.7 shows that a representative of the geometric realization of $P_\bullet$ is given by the simplicial presheaf

$$\cdots \quad \mathrm{Hom}(-, P_3) \substack{\longrightarrow \\ \leftarrow \\ \longrightarrow \\ \leftarrow \\ \longrightarrow \\ \leftarrow \\ \longrightarrow} \mathrm{Hom}(-, P_2) \substack{\longrightarrow \\ \leftarrow \\ \longrightarrow \\ \leftarrow \\ \longrightarrow} \mathrm{Hom}(-, P_1) \substack{\longrightarrow \\ \leftarrow \\ \longrightarrow} \mathrm{Hom}(-, P_0) \tag{5}$$

This is R applied to a projective object (more precisely, it is the image of $P_\bullet$ considered as object in $\mathcal{C}^{\Delta^{\mathrm{op}}}$) and hence in the essential image of R_∞. $\square$

[26]Proof: The functors "composition with π_0" and "composition with R" inherit unit and counit from the adjunction of π_0 and R which satisfy the unit-counit equations (up to 2-isomorphisms in the ∞-categorical language, which is enough) again. Therefore they are adjoint as well. By definition the left Kan extension along R ist the left adjoint of "composition with R", hence it is "composition with L".

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