1. **Roots of unity:** Find explicit expressions of the form $a + bi$ for all solutions $z \in \mathbb{C}$ to the equations
   
   \[(a) \ z^8 = 1, \quad (b) \ z^3 = 1, \]
   
   by using both of the following methods: 
   
   i) Use explicit formulas for special values of sin and cos. 
   
   ii) Use, for (a), the fact that either $z^4 = 1$ or $z$ is a solution of the equation $z^2 = +i$ or $z^2 = -i$ — then use geometric considerations. Use, for (b), the fact that $z^3 - 1 = (z^2 + z + 1)(z - 1)$ and completing the square.

2. **Parallelogram identity:** Give a proof and geometric interpretation of the formula
   
   \[2 \left( |z_1|^2 + |z_2|^2 \right) = |z_1 + z_2|^2 + |z_1 - z_2|^2 \quad \text{for} \ z_1, z_2 \in \mathbb{C}. \]

3. **Complement to the triangle inequality:** Let $z_1, z_2 \in \mathbb{C}$ be both non-zero. Show that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if $z_2$ is a positive real multiple of $z_1$.

4. **Treasure quest:** Imagine an island in the South Sea. Located somewhere on the island, you will find a small tree $B_1$ and a big tree $B_2$ as well as a cross $C$. Starting from the cross $C$, go to $B_1$ and the same distance straight on, then, again the same distance to the left (90°). Mark this position by $M_1$. Now go from the tree $B_2$ to the cross $C$ and then the same distance to the left. Mark this position by $M_2$. Unfortunately, arriving at the island, you realize that the cross doesn’t exist anymore. Can you still find the treasure?

5. **Riemann sphere:** Consider $\mathbb{C}$ as a subset of $\mathbb{R}^3$ by mapping $z = x + yi$ to the vector $(x \ y \ 0)^T$. Consider the sphere
   
   \[ S^2 = \left\{ \left( \begin{array}{c} X \\ Y \\ Z \end{array} \right) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1 \right\}. \]

   For each point $z \in \mathbb{C}$ the line through the North Pole $N := (0 \ 0 \ 1)^T$ and $z$ hits the sphere in exactly one other point $f(z)$.

   (a) Prove that $z \mapsto f(z)$ is a bijection of $\mathbb{C}$ with $S^2 - \{N\}$.

   (b) Prove that $f$ and its inverse are differentiable in the sense of real analysis. (We say: $f$ is a diffeomorphism.)

   (c) $f$ extends to a bijection $\hat{\mathbb{C}} \cong S^2$ by mapping $\infty$ to $N$. Prove that a Möbius transformation gives rise to a continuous map $S^2 \rightarrow S^2$ using this identification.

   *(d) Prove that $f$ induces a bijection between the set of circles in $\hat{\mathbb{C}}$ (as defined in the lecture) and the set of usual circles on $S^2$. To which circles on $S^2$ the lines in $\mathbb{C}$ do correspond?*

6. **Another construction of $\mathbb{C}$:** Let $\mathbb{R}[T]$ be the ring of polynomials in one variable with coefficients in $\mathbb{R}$. Let $\langle 1 + T^2 \rangle$ be the *ideal* of polynomials that can be written as $f(T) \cdot (1 + T^2)$ for some polynomial $f \in \mathbb{R}[T]$. Prove that the quotient ring $\mathbb{R}[T]/\langle 1 + T^2 \rangle$ is isomorphic to the field of complex numbers.

*Please hand in your solutions on Wednesday, September 15, 2010 in the lecture hall*