Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 1

1. **Roots of unity:** Find explicit expressions of the form $a + bi$ for *all* solutions $z \in \mathbb{C}$ to the equations
$$\text{(a) } z^8 = 1, \qquad \text{(b) } z^3 = 1,$$
by using both of the following methods: **i)** Use explicit formulas for special values of sin and cos. **ii)** Use, for (a), the fact that either $z^4 = 1$ or z is a solution of the equation $z^2 = +i$ or $z^2 = -i$ — then use geometric considerations. Use, for (b), the fact that $z^3 - 1 = (z^2 + z + 1)(z - 1)$ and completing the square.

2. **Parallelogram identity:** Give a proof and geometric interpretation of the formula
$$2\left(|z_1|^2 + |z_2|^2\right) = |z_1 + z_2|^2 + |z_1 - z_2|^2 \qquad \text{for } z_1, z_2 \in \mathbb{C}.$$

3. **Complement to the triangle inequality:** Let $z_1, z_2 \in \mathbb{C}$ be both non-zero. Show that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if z_2 is a positive real multiple of z_1.

4. **Treasure quest:** Imagine an island in the South Sea. Located somewhere on the island, you will find a small tree B_1 and a big tree B_2 as well as a cross C. Starting from the cross C, go to B_1 and the same distance straight on, then, again the same distance to the left (90°). Mark this position by M_1. Now go from the tree B_2 to the cross C and then the same distance to the left. Mark this position by M_2. You'll find the treasure at half distance between M_1 and M_2. Unfortunately, arriving at the island, you realize that the cross doesn't exist anymore. Can you still find the treasure?

5. **Riemann sphere:** Consider $\mathbb{C}$ as a subset of $\mathbb{R}^3$ by mapping $z = a+bi$ to the vector $\begin{pmatrix} a & b & 0 \end{pmatrix}^T$. Consider the sphere
$$S^2 = \left\{ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, X^2 + Y^2 + Z^2 = 1 \right\}.$$
For each point $z \in \mathbb{C}$ the line through the North Pole $N := \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and z hits the sphere in exactly one other point $f(z)$.

   (a) Prove that $z \mapsto f(z)$ is a bijection of $\mathbb{C}$ with $S^2 - \{N\}$.

   (b) Prove that f and its inverse are differentiable in the sense of real analysis. (We say: f is a diffeomorphism.)

   (c) f extends to a bijection $\widehat{\mathbb{C}} \cong S^2$ by mapping ∞ to N. Prove that a Möbius transformation gives rise to a *continuous* map $S^2 \to S^2$ using this identification.

   *(d) Prove that f induces a bijection between the set of circles in $\widehat{\mathbb{C}}$ (as defined in the lecture) and the set of usual circles on S^2. To which circles on S^2 the lines in $\mathbb{C}$ do correspond?

6. **Another construction of $\mathbb{C}$:** Let $\mathbb{R}[T]$ be the ring of polynomials in one variable with coefficients in $\mathbb{R}$. Let $\langle 1+T^2 \rangle$ be the *ideal* of polynomials that can be written as $f(T) \cdot (1+T^2)$ for some polynomial $f \in \mathbb{R}[T]$. Prove that the quotient ring $\mathbb{R}[T]/\langle 1 + T^2 \rangle$ is isomorphic to the field of complex numbers.

Please hand in your solutions on Wednesday, September 15, 2010 in the lecture hall