Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Solutions to exercise sheet 1

1. **Roots of unity:** Find explicit expressions of the form $a + bi$ for *all* solutions $z \in \mathbb{C}$ to the equations

$$\text{(a) } z^8 = 1, \qquad \text{(b) } z^3 = 1,$$

by using both of the following methods: **i)** Use explicit formulas for special values of sin and cos. **ii)** Use, for (a), the fact that either $z^4 = 1$ or z is a solution of the equation $z^2 = +i$ or $z^2 = -i$ — then use geometric considerations. Use, for (b), the fact that $z^3 - 1 = (z^2 + z + 1)(z - 1)$ and completing the square.

In the lecture the formula

$$z_k = e(\frac{2\pi k}{n}) = \cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n}) \qquad k = 0, \ldots, n-1,$$

for the n different n-th roots of unity has been given. (Meanwhile we learned, that $e(y) = \exp(iy)$ for the complex exponential function). Using tables for the special values of sin and cos at $2\pi\frac{1}{3}$ and $2\pi\frac{2}{3}$, we get

$$z_0 = 1, \quad z_1 = \frac{1}{2}(-1 + \sqrt{3}i), \quad z_2 = \frac{1}{2}(-1 - \sqrt{3}i),$$

for the 3 third roots of unity.
Similarly using the values for $2\pi\frac{1}{8}$, etc., we get

$$z_0 = 1, \quad z_1 = \frac{1}{\sqrt{2}}(1 + i), \quad z_2 = i, \quad z_3 = \frac{1}{\sqrt{2}}(-1 + i),$$

$$z_4 = -1, \quad z_5 = \frac{1}{\sqrt{2}}(-1 - i), \quad z_6 = -i, \quad z_7 = \frac{1}{\sqrt{2}}(1 - i),$$

for the 8 eighth root of unity.
However using calculations with complex numbers, we get determine these values ourselves, thereby proving the formulas for these special values of sin and cos.
For the third roots of unity observe that they are zeros of the equation

$$z^3 - 1 = (z - 1)(z^2 + z + 1) = (z - 1)((z + \frac{1}{2})^2 + \frac{3}{4}).$$

Hence we get, that z is either 1 or $z + \frac{1}{2} = \pm\sqrt{-\frac{3}{4}} = \pm\frac{1}{2}\sqrt{3}i$. These are the expressions obtained before.
For the eighth roots of unity observe, that they are zeros of the equation

$$(z^8 - 1) = (z^4 - 1)(z^4 + 1).$$

Hence z is either a fourth root of unity, that is, equal to 1, -1, i or $-i$, or it is a zero of $z^4 + 1 = (z^2 + i)(z^2 - i)$. In the latter case, we get, setting $z = a + bi$:

$$(a + bi)^2 = \pm i \quad \Rightarrow \quad a^2 - b^2 = 0, 2ab = \pm 1.$$

Hence $b = \pm a$ and $a^2 = \frac{1}{2}$. This gives the expressions obtained before.
Formulas involving only square roots of rational numbers exist only for n-th roots of unity with $n|24$. However, try to determine $Re(z)$ for z being a (primitive) fifth root of unity!

2. **Parallelogram identity:** Give a proof and geometric interpretation of the formula

$$2\left(|z_1|^2 + |z_2|^2\right) = |z_1 + z_2|^2 + |z_1 - z_2|^2 \qquad \text{for } z_1, z_2 \in \mathbb{C}.$$

We calculate:

$$\begin{aligned} & (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)} \\ = \; & z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\ & + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\ = \; & 2z_1\overline{z_1} + 2z_2\overline{z_2} \end{aligned}$$

Geometric interpretation: Consider the parallelogram spanned by z_1 and z_2. The sides of it have length $|z_1|$, resp. $|z_2|$ and its diagonals have length $|z_1 + z_2|$, resp. $|z_1 - z_2|$.

3. **Complement to the triangle inequality:** Let $z_1, z_2 \in \mathbb{C}$ be both non-zero. Show that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if z_2 is a positive real multiple of z_1.

The "if" direction is obvious because $|\alpha z_1| = \alpha |z_1|$ for a positive real α. For the "only if" direction: By multiplying the equation with $\frac{1}{|z_1|}$, resp. z_1, z_2 by $\frac{1}{z_1}$ we may without loss of generality assume that $z_1 = 1$. Then, writing $z_2 = a + bi$, we have

$$\begin{aligned} (1+a)^2 + b^2 &= (1 + \sqrt{a^2+b^2})^2 \\ 1 + 2a + a^2 + b^2 &= 1 + 2\sqrt{a^2+b^2} + a^2 + b^2 \\ a &= \sqrt{a^2+b^2}. \end{aligned}$$

From this it follows $b = 0$ and $a > 0$.

4. **Treasure quest:** Imagine an island in the South Sea. Located somewhere on the island, you will find a small tree B_1 and a big tree B_2 as well as a cross C. Starting from the cross C, go to B_1 and the same distance straight on, then, again the same distance to the left (90°). Mark this position by M_1. Now go from the tree B_2 to the cross C and then the same distance to the left. Mark this position by M_2. You'll find the treasure at half distance between M_1 and M_2. Unfortunately, arriving at the island, you realize that the cross doesn't exist anymore. Can you still find the treasure?

We consider the map of the island as a subset of $\mathbb{C}$. Without loss of generality, we can assume that $B_1 = 0$. Call $z := B_2$. We get for the position of M_1

$$(-1-i)C$$

(since multiplication by i has the effect of rotation by 90° to the left) and for the position of M_2, we get

$$C + i(C - z)$$

The position at half distance between M_1 and M_2 is hence given by

$$\frac{1}{2}(M_1 + M_2) = \frac{1}{2}((-1-i)C + C + i(C-z)) = -\frac{1}{2}iz$$

which is independent of C. Therefore the position of the cross does not matter for finding the treasure.

5. **Riemann sphere:** Consider $\mathbb{C}$ as a subset of $\mathbb{R}^3$ by mapping $z = a+bi$ to the vector $\begin{pmatrix} a & b & 0 \end{pmatrix}^T$. Consider the sphere

$$S^2 = \left\{ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3 \middle| X^2 + Y^2 + Z^2 = 1 \right\}.$$

For each point $z \in \mathbb{C}$ the line through the North Pole $N := \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and z hits the sphere in exactly one other point $f(z)$.

(a) Prove that $z \mapsto f(z)$ is a bijection of $\mathbb{C}$ with $S^2 - \{N\}$.

(b) Prove that f and its inverse are differentiable in the sense of real analysis. (We say: f is a diffeomorphism.)

(c) f extends to a bijection $\widehat{\mathbb{C}} \cong S^2$ by mapping ∞ to N. Prove that a Möbius transformation gives rise to a *continuous* map $S^2 \to S^2$ using this identification.

*(d) Prove that f induces a bijection between the set of circles in $\widehat{\mathbb{C}}$ (as defined in the lecture) and the set of usual circles on S^2. To which circles on S^2 the lines in $\mathbb{C}$ do correspond?

(a) and (b): We first try to find the coordinates for $f(z)$. The line through $N = (0\ 0\ 1)^T$ and $z = (a\ b\ 0)^T$ may be parametrized by

$$\begin{pmatrix} ta \\ tb \\ 1-t \end{pmatrix} \qquad t \in \mathbb{R}.$$

Let us investigate its intersection points with the sphere S. They satisfy

$$t^2a^2 + t^2b^2 + (1-t)^2 = 1$$

or in other form:

$$t^2(a^2 + b^2 + 1) = 2t.$$

The solutions of this equation are $t = 0$ and $t = \frac{2}{a^2+b^2+1}$. The first obviously corresponds to N itself and the other yields the coordinates of $f(z)$:

$$f(z) = \begin{pmatrix} \frac{2a}{a^2+b^2+1} \\ \frac{2b}{a^2+b^2+1} \\ 1 - \frac{2}{a^2+b^2+1} \end{pmatrix}.$$

An inverse of this is obviously:

$$z = \frac{X}{1-Z} + \frac{Y}{1-Z}i.$$

(This formula is not uniquely determined, since for points on S, we have a relation between X, Y and Z.)
Therefore f is bijective and obviously f and its inverse are continuous and differentiable because they are compositions of algebraic operations (observe that the denominator $a^2 + b^2 + 1$ is never 0 and $1 - Z$ is never 0 on $S - \{N\}$).
(c) Let g be a Möbius transformation obtained by extending the map $z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$ to $\widehat{\mathbb{C}}$, as explained in the lecture. Since f defines a bijection of $S - \{N\}$ with $\mathbb{C}$, such that f and f^{-1} are continuous,

the only points in question are 1. $z = \infty$ and 2. the point z such that $g(z) = \infty$. We treat the case $\gamma = 0$ and $\gamma \neq 0$ separately.
First let $\gamma \neq 0$. 1. We have by definition $g(\infty) = \frac{\alpha}{\gamma}$. We have to prove, that for any sequence of numbers $z_1, z_2, \ldots$ with

$$\lim_{n\to\infty} f(z_n) = N, \tag{1}$$

we have

$$\lim_{n\to\infty} g(z_n) = \frac{\alpha}{\gamma}. \tag{2}$$

Now (1) is equivalent to

$$\lim_{n\to\infty} 1 - \frac{2}{a_n^2 + b_n^2 + 1} = 1$$

which is equivalent to

$$\lim_{n\to\infty} |z_n| = \infty.$$

On the other hand:

$$\begin{aligned} g(z_n) &= \frac{\alpha z_n + \beta}{\gamma z_n + \delta} \\ &= \frac{\alpha + \frac{\beta}{z_n}}{\gamma + \frac{\delta}{z_n}}. \end{aligned}$$

Now, of course, $\lim_{n\to\infty} \frac{1}{z_n} = 0$ and since algebraic operations in $\mathbb{C}$ are continuous (see section 2.2 of the lecture), we get (2).
2. We have by definition $g(-\frac{\delta}{\gamma}) = \infty$, therefore we have to prove, that for any sequence of numbers $z_1, z_2, \ldots$ with

$$\lim_{n\to\infty} z_n = -\frac{\delta}{\gamma}, \tag{3}$$

we have

$$\lim_{n\to\infty} |g(z_n)| = \infty \tag{4}$$

(using what we obtained before). We calculate

$$\begin{aligned} |g(z_n)| &= \frac{|\alpha z_n + \beta|}{|\gamma z_n + \delta|} \\ &= \frac{|\frac{\alpha}{\gamma} z_n + \frac{\beta}{\gamma}|}{|z_n + \frac{\delta}{\gamma}|}. \end{aligned}$$

Now we have $\lim_{n\to\infty} |z_n + \frac{\delta}{\gamma}| = 0$ and $\lim_{n\to\infty} |\frac{\alpha}{\gamma} z_n + \frac{\beta}{\gamma}| = |\frac{\alpha\delta - \beta\gamma}{\gamma^2}|$ which is non-zero because the matrix describing the Möbius transformation has non-zero determinant. Therefore, we get (4).
The case $\gamma = 0$ is easy.
(d) In the lecture, it was shown that any circle in $\widehat{\mathbb{C}}$ can be described by the equation

$$\begin{pmatrix} \overline{z} & 1 \end{pmatrix} H \begin{pmatrix} z \\ 1 \end{pmatrix} = 0$$

where $H = \begin{pmatrix} \alpha & w \\ \overline{w} & \delta \end{pmatrix}$, $w \in \mathbb{C}$, $\alpha, \delta \in \mathbb{R}$ is an arbitrary indefinite Hermitian matrix. In other words

$$\alpha z\overline{z} + w\overline{z} + z\overline{w} + \delta = 0 \tag{5}$$

$$\alpha(a^2 + b^2) + 2w_1 a + 2w_2 b + \delta = 0 \qquad w = w_1 + iw_2 \tag{6}$$

with $\alpha\delta - w\overline{w} < 0$. A circle on S is described by the equation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \cdot \begin{pmatrix} r \\ s \\ t \end{pmatrix} = u$$

with

$$0 \leq u < \left| \begin{pmatrix} r \\ s \\ t \end{pmatrix} \right| \tag{7}$$

(constance of scalar product) or written out:

$$rX + sY + tZ = u.$$

Inserting $f(z)$ in this equation and multiplying by $a^2 + b^2 + 1$, we get

$$\begin{array}{rcl} 2ra + 2sb + t(a^2 + b^2 - 1) & = & u(a^2 + b^2 + 1) \\ (t-u)(a^2 + b^2) + 2ra + 2sb - u - t & = & 0. \end{array}$$

This equation is of the same form as (6), if and only if

$$(t-u)(-u-t) - r^2 - s^2 < 0,$$

that is

$$u^2 < t^2 + r^2 + s^2.$$

This is just condition (7).

The lines in $\mathbb{C}$, which we considered as special "circles" in $\widehat{\mathbb{C}}$ are characterized by $\alpha = 0$. They correspond to circles on S satisfying $u = t$, hence (by multiplying with a scalar) to

$$rX + sY + Z = 1 \quad \text{or} \quad rX + sY = 0.$$

These are those circles $C \subset S$ with $N \in C$ (the first equation describes those circles which do not contain the South Pole, whereas the second equation describes the meridians).

6. **Another construction of $\mathbb{C}$:** Let $\mathbb{R}[T]$ be the ring of polynomials in one variable with coefficients in $\mathbb{R}$. Let $\langle 1+T^2 \rangle$ be the *ideal* of polynomials that can be written as $f(T) \cdot (1+T^2)$ for some polynomial $f \in \mathbb{R}[T]$. Prove that the quotient ring $\mathbb{R}[T]/\langle 1+T^2 \rangle$ is isomorphic to the field of complex numbers.

First we define a ring homomorphism $g : \mathbb{R}[T] \to \mathbb{C}$ by mapping the coefficients in $\mathbb{R}$ to $\mathbb{R} \subset \mathbb{C}$ and sending T to i. Remember: To give a homomorphism of $\mathbb{R}[T]$ into any ring X is the same as giving a homomorphism of $\mathbb{R}$ to X and an element in X which becomes the image of T (universal property of the polynomial ring).

This homomorphism g factors into a homomorphism

$$\widetilde{g} : \mathbb{R}[T]/\langle 1+T^2 \rangle \to \mathbb{C}$$

if and only if $g(1+T^2) = 0$. This condition, however, is satisfied because i is a solution of the equation $1 + T^2 = 0$.

We have to show that $\widetilde{g}$ is an isomorphism. For this note that

$$\{f \in \mathbb{R}[X] \mid g(f) = 0\}$$

is an ideal of $\mathbb{R}[T]$ and hence generated by a polynomial h ($\mathbb{R}[X]$ is a principal ideal domain). Of course h has degree ≥ 1 because $\mathbb{C} \neq \{0\}$. By the fundamental theorem of homomorphisms, we have $\mathbb{R}[X]/\langle h \rangle \cong \mathbb{C}$. We have to see, that $\langle h \rangle = \langle 1 + T^2 \rangle$. Now obviously, we have $\langle h \rangle \supseteq \langle 1 + T^2 \rangle$, hence

$$1 + T^2 = p \cdot h$$

for some other polynomial p. If p has degree 0 and h has degree 2, that is: $p \in \mathbb{R}$, we are done because the 2 ideals will be the same. If p and h have degree 1, we get

$$1 + T^2 = (\alpha T + \beta)(\gamma T + \delta)$$

which implies that $1 + T^2 = 0$ has a solution in $\mathbb{R}$ — a contradiction!

Remark: If you don't want to use the fact, that $\mathbb{R}[X]$ is a principal ideal domain, you can argue as follows: Let f be a polynomial with $g(f) = 0$, that is, satisfying $f(i) = 0$. We have to show that f lies in $\langle 1 + T^2 \rangle$. By polynomial division, we may write

$$f = (1 + T^2) \cdot h + p$$

where $\deg(p) \leq 1$. Inserting i for T in this equation (that is: applying g to it), we get $p(i) = 0$. If p has degree 0, then $p = 0$, and so $f \in \langle 1 + T^2 \rangle$. If p has degree 1, we have $\alpha i + \beta = 0$ for $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ — a contradiction. (Actually a similar kind of argument is used to prove that $\mathbb{R}[X]$ is a principal ideal domain).