Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 2

1. **Möbius transformations:** Prove that

   (a) the group of Möbius transformations $\mathrm{Aut}(\widehat{\mathbb{C}})$ acts transitively on the set of circles in $\widehat{\mathbb{C}}$ (as defined in the lecture),

   (b) four points $z_0, z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ lie on a circle, if and only if the cross-ratio (z_0, z_1, z_2, z_3) is in $\widehat{\mathbb{R}}$, that is, either real or equal to infinity.

   Hint: Do not calculate! For a), use the fact that 3 different points in $\widehat{\mathbb{C}}$ determine a unique circle. Reduce b) (using the invariance of the cross-ratio under Möbius transformations) to the case $z_1 = 1$, $z_2 = 0$, $z_3 = \infty$.

2. **Complex differentiability:** Decide for the following functions $f : \mathbb{C} \to \mathbb{C}$ whether they are

   (1) continuous at 0,

   (2) partially differentiable at 0 in x and y direction (identifying $\mathbb{C}$ with $\mathbb{R}^2$ as usual),

   (3) real differentiable at 0 (identifying $\mathbb{C}$ with $\mathbb{R}^2$ as usual),

   (4) complex differentiable at 0,

   (5) holomorphic in a neighborhood of 0 (for example a small disc around).

   (a) $f(z) := \overline{z}$

   (b) $f(z) := |z|$

   (c) $f(z) := |z|^2$

   (d) $f(z) := 0$ if $x \neq y$ or $z = 0$, $f(z) := 1$, otherwise

   (e) $f(z) := \frac{y^3}{x^2+y^2}$ for $z \neq 0$ and $f(0) := 0$

   (f) $f(z) := u(z) + iv(z)$, where $u(z) := \exp(x)\cos(y)$ and $v(z) := \exp(x)\sin(y)$

   (g) $f(z) := u(z) + iv(z)$, where $u(z) := x^3 - 3xy^2$ and $v(z) := 3x^2y - y^3$

   (h) $f(z) := \overline{g(\overline{z})}$ for any holomorphic function g

   where we wrote $z = x + yi$.

   Summarize your answers in a table! You do not have to give proofs.

3. **Cauchy-Riemann operator:** Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a real differentiable function (at every point in U). Show that f is holomorphic, if and only if $\frac{\partial f}{\partial \overline{z}} = 0$, where $\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$. Show furthermore that in this case $f'(z) = \frac{\partial f}{\partial z}$, where $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.

Please hand in your solutions on Monday, September 20, 2010 in the lecture hall