1. Möbius transformations: Prove that

(a) the group of Möbius transformations \( \text{Aut}(\hat{\mathbb{C}}) \) acts transitively on the set of circles in \( \hat{\mathbb{C}} \) (as defined in the lecture),
(b) four points \( z_0, z_1, z_2, z_3 \in \hat{\mathbb{C}} \) lie on a circle, if and only if the cross-ratio \( (z_0, z_1, z_2, z_3) \) is

in \( \mathbb{R} \), that is, either real or equal to infinity.

Hint: Do not calculate! For a), use the fact that 3 different points in \( \hat{\mathbb{C}} \) determine a unique circle. Reduce b) (using the invariance of the cross-ratio under Möbius transformations) to the case \( z_1 = 1, z_2 = 0, z_3 = \infty \).

(a) We learned during the course that for pair of triples of disjoint numbers \( z_1, z_2, z_3 \in \hat{\mathbb{C}} \) and \( w_1, w_2, w_3 \in \hat{\mathbb{C}} \), there is a Möbius transformation \( f_M \) with \( f(z_i) = w_i \) for \( i = 1, 2, 3 \). It is obvious that 3 disjoint points determine a unique circle in \( \mathbb{C} \) (either they lie on a line or determine a classical circle in \( \mathbb{C} \)). We learned also that \( f_M \) transforms circles in \( \mathbb{C} \) into circles in \( \hat{\mathbb{C}} \). (Observe that this means that 1. a classical circle is transformed either into a line or into a classical circle and that 2. a line is also transformed either into a classical circle or into a line.)

Let now circles \( C, C' \) in \( \hat{\mathbb{C}} \) be given and choose 3 different points \( z_1, z_2, z_3 \in C \) and 3 different points \( w_1, w_2, w_3 \in C' \). Consider the Möbius transformation \( f_M \) as above. It has to send the circle \( C \) to the circle \( C' \) because it sends \( C \) to some circle but there is only one circle going through \( w_1, w_2, w_3 \).

(b) First assume that \( z_0, z_1, z_2, z_3 \) are disjoint. Choose a Möbius transformation \( f_M \) sending the triple \( z_1, z_2, z_3 \) to the triple \( 1, 0, \infty \). We have

\[
(z_0, z_1, z_2, z_3) = (f_M(z_0), f_M(z_1), f_M(z_2), f_M(z_3)) = (f_M(z_0), 1, 0, \infty) = \frac{f_M(z_0) - 0}{1 - 0} = f_M(z_0)
\]

(here the invariance of the cross-ratio under Möbius transformations, proven in the lecture, is used). Therefore the cross-ratio is real, if and only if \( f_M(z_0) \) is real. Since the circle in \( \hat{\mathbb{C}} \) going through the 3 points \( 1, 0, \infty \in \hat{\mathbb{C}} \) is \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \), we can formulate this as follows. The cross-ratio is real, if and only if \( f_M(z_0) \) lies on the circle through \( 1, 0, \infty \). By the argument from (a), this is equivalent to \( z_0 \) lying on the circle through \( f_M^{-1}(1), f_M^{-1}(0), f_M^{-1}(\infty) \), which are the points \( z_1, z_2, z_3 \). Hence the cross-ratio is real, if and only if \( z_0 \) lies on the circle determined by \( z_1, z_2, z_3 \), that is, all 4 points lie on one circle.

If 2 or more of the \( z_i \) coincide, the cross-ratio is \( \infty \) but the 4 points then obviously lie on a circle, too.

2. Complex differentiability: Decide for the following functions \( f : \mathbb{C} \to \mathbb{C} \) whether they are

(1) continuous at 0,
(2) partially differentiable at 0 in \( x \) and \( y \) direction (identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \) as usual),
(3) real differentiable at 0 (identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \) as usual),
(4) complex differentiable at 0,
(5) holomorphic in a neighborhood of 0 (for example a small disc around).

(a) \( f(z) := z \)
We use freely the following fact from real analysis: If we have given any function
\[ f : \mathbb{R}^2 = \mathbb{C} \rightarrow \mathbb{C} = \mathbb{R}^2 \]
which is of the form
\[ f(z) = \begin{pmatrix} f_x(z) \\ f_y(z) \end{pmatrix} = f_x(z) + if_y(z) \]
for \( f_x(z) \) and \( f_y(z) \) are any composition of algebraic expressions and/or smooth functions in \( x \) and \( y \), then \( f \) is real differentiable. These are: addition, subtraction, multiplication, division (with non-zero denominator!), \( \sin \), \( \cos \), \( \exp \), etc. 

Observe also that for the above conditions (5) \( \Rightarrow \) (4) \( \Rightarrow \) (3) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (1) but not necessarily (2) \( \Rightarrow \) (1). This exercise shows in particular that no other implications are possible.

(a) \( f(z) = \overline{z} \). The function is obviously real-differentiable at \( z = 0 \), by what was said in the beginning. For complex differentiability, we have to see whether the Jacobian matrix at \( z = 0 \) is of the form
\[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \]
(which means that it is \( \mathbb{C} \)-linear — the same as multiplication by \( m = a + bi \)) or, in other words, whether the Cauchy-Riemann equations are satisfied there.

We have here
\[ f_x(x + iy) = x \quad \text{and} \quad f_y(x + iy) = -y. \]
Therefore
\[ \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix} (z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
even for all \( z \in \mathbb{C} \), which is not of the form required for complex differentiability.

(b) \( f(z) = |z| \) is obviously continuous, but already its restriction to the real line, which coincides with the usual real absolute value, is not differentiable at \( x = 0 \). In other words, it is not partially differentiable in \( x \) direction, hence neither real- nor complex differentiable.

(c) \( f(x + iy) = f_x(x + iy) = x^2 + y^2 \) is real-differentiable (at any point) by what was said in the beginning. Its Jacobian matrix is given by
\[ \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix} (z) = \begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix}, \]
which is of the form required for complex differentiability only at \( z = 0 \). This is kind of an accident and the function is not holomorphic (on any neighborhood of \( z = 0 \)).
(d) The function first of all is obviously not continuous at \( z = 0 \): For the sequence \( z_k = x_k + ix_k \) with any real zero-sequence \( x_1, x_2, \ldots \) (with non-zero terms) we have
\[
\lim_{k \to \infty} z_k = 0 \quad \lim_{k \to \infty} f(z_k) = 1,
\]
but for the sequence \( z_k = x_k \), we have
\[
\lim_{k \to \infty} z_k = 0 \quad \lim_{k \to \infty} f(z_k) = 0.
\]
Nevertheless, its restrictions to the \( x \) and \( y \)-axes are both identically zero, hence \( f \) is partially differentiable in these directions.

(e) First \( f \) is continuous at \( 0 \) because we have
\[
\left| \frac{y^3}{x^2 + y^2} \right| = \left| \frac{y^3}{|z|^2} \right| \leq \left| \frac{|z|^3}{|z|^2} \right| = |z|.
\]
Furthermore, the partial derivatives in \( x \) and \( y \) direction exist and we get the following potential (!) Jacobian matrix:
\[
J = \begin{pmatrix}
\frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\
\frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y}
\end{pmatrix}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Now remember: Real differentiability means that we can write \( f \) in a neighborhood of \( z = 0 \) as follows
\[
f(z) = f(0) + J \cdot (x, y) + |z|r(z),
\]
where \( r \) is continuous at \( 0 \) with \( r(0) = 0 \). Hence we have to investigate, whether
\[
r(z) = \begin{cases}
\frac{f(z) - f(0) - J \cdot (x, y)}{|z|} & \text{if } z \neq 0 \\
0 & \text{if } z = 0
\end{cases}
\]
is continuous at \( 0 \). We get for \( z \neq 0 \):
\[
r(z) = \frac{y^3}{|z|^3} - \frac{y}{|z|} = \frac{y}{|z|} \left( \frac{y^2}{|z|^2} - 1 \right)
\]
The sequence \( z_k = x_k + ix_k \) for any real zero-sequence \( x_1, x_2, \ldots \) leads to
\[
r(z_k) = \frac{x_k}{2x_k} \left( \frac{x_k^2}{4x_k^2} - 1 \right) = -\frac{3}{8}
\]
which is not zero. Hence \( f \) is not real-differentiable at \( 0 \) even though it is partially differentiable at \( z = 0 \) (even in any direction).

(f) \( f \) is obviously real-differentiable by what was said in the beginning. We have to check the Cauchy-Riemann conditions. The Jacobian matrix is given by
\[
\begin{pmatrix}
\frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\
\frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y}
\end{pmatrix}(z) = \begin{pmatrix}
\exp(x) \cos(y) & -\exp(x) \sin(y) \\
\exp(x) \sin(y) & \exp(x) \cos(y)
\end{pmatrix}.
\]
This is of the form \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) for all \( z \). Hence \( f \) is holomorphic on the whole complex plane.

**Remark:** We see in addition: \( f'(z) = f(z) \) for all \( z \). Meanwhile, we learned \( f(z) = \exp(z) \) is the complex exponential function.

(g) \( f \) is obviously real-differentiable by what was said in the beginning. We have to check the Cauchy-Riemann conditions. The Jacobian matrix is given by

\[
\begin{pmatrix}
\frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\
\frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y}
\end{pmatrix}(z) = \begin{pmatrix} 3x^2 - 3y^2 & -3xy \\ 3xy & 3x^2 - 3y^2 \end{pmatrix}.
\]

This is of the form \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) for all \( z \). Hence \( f \) is holomorphic on the whole complex plane.

**Remark:** We see in addition: \( f'(z) = 3(x^2 - y^2 + ixy) \) for all \( z \), which you may recognize as \( f'(z) = 3z^2 \) and indeed \( f(z) = z^3 \), written out in \( x, y \) coordinates.

(h) Obviously the function \( f \) is real-differentiable because it is a composition of real-differentiable functions. Remember that we saw in (a) that \( h: z \mapsto \overline{z} \) is everywhere real-differentiable with Jacobian matrix given by

\[
J(h, z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Hence the Jacobian matrix of the function \( f = h \circ g \circ h \) at \( z \) is given by

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} J(g, \overline{z}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

by the real-analytic chain rule. Here \( J(g, \overline{z}) \) is the Jacobian matrix of \( g \) at \( \overline{z} \). It is of the form

\[
\begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]

for any \( z \) because \( g \) is holomorphic. The calculation

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}
\]

shows that also \( f \) is holomorphic everywhere with \( f'(z) = g'(\overline{z}) \).

**Summarized:**

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3. **Cauchy-Riemann operator:** Let \( U \subseteq \mathbb{C} \) be open and \( f: U \to \mathbb{C} \) be a real differentiable function (at every point in \( U \)). Show that \( f \) is holomorphic, if and only if \( \frac{\partial f}{\partial \overline{z}} = 0 \), where \( \frac{\partial f}{\partial \overline{z}} := \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \). Show furthermore that in this case \( f'(z) = \frac{\partial f}{\partial \overline{z}} \), where \( \frac{\partial f}{\partial \overline{z}} := \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \).
Since $f(z) = f_x(z) + if_y(z)$ is assumed to be real-differentiable everywhere, being holomorphic is equivalent to the validity of the Cauchy-Riemann equations everywhere. They express the condition that the Jacobian matrix

$$J(f, z) = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(z)$$

is of the form

$$\begin{pmatrix} a(z) & -b(z) \\ b(z) & a(z) \end{pmatrix}.$$ 

We have then $f'(z) = a(z) + b(z)i$.

Let us check what $\frac{\partial f}{\partial z} = 0$ means:

\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} + i \frac{\partial f_y}{\partial y} + i \frac{\partial f_x}{\partial x} + i \frac{\partial f_y}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} + i \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} \right).
\]

This is zero obviously, if and only if we have

$$\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y}$$

and

$$\frac{\partial f_x}{\partial y} = -\frac{\partial f_y}{\partial x}.$$ 

Furthermore, if $f$ is holomorphic we have

\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} + i \frac{\partial f_y}{\partial y} - i \frac{\partial f_x}{\partial x} + i \frac{\partial f_y}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + i \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} \right) = a(z) + b(z)i = f'(z).
\]