Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Solutions to exercise sheet 2

1. **Möbius transformations:** Prove that

   (a) the group of Möbius transformations $\mathrm{Aut}(\widehat{\mathbb{C}})$ acts transitively on the set of circles in $\widehat{\mathbb{C}}$ (as defined in the lecture),

   (b) four points $z_0, z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ lie on a circle, if and only if the cross-ratio (z_0, z_1, z_2, z_3) is in $\widehat{\mathbb{R}}$, that is, either real or equal to infinity.

   Hint: Do not calculate! For a), use the fact that 3 different points in $\widehat{\mathbb{C}}$ determine a unique circle. Reduce b) (using the invariance of the cross-ratio under Möbius transformations) to the case $z_1 = 1$, $z_2 = 0$, $z_3 = \infty$.

(a) We learned during the course that for pair of triples of disjoint numbers $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ and $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$, there is a Möbius transformation f_M with $f(z_i) = w_i$ for $i = 1, 2, 3$. It is obvious that 3 disjoint points determine a unique circle in $\widehat{\mathbb{C}}$ (either they lie on a line or determine a classical circle in $\mathbb{C}$). We learned also that f_M transforms circles in $\widehat{\mathbb{C}}$ into circles in $\widehat{\mathbb{C}}$. (Observe that this means that 1. a classical circle is transformed either into a line or into a classical circle and that 2. a line is also transformed either into a classical circle or into a line.)
Let now circles C, C' in $\widehat{\mathbb{C}}$ be given and choose 3 different points $z_1, z_2, z_3 \in C$ and 3 different points $w_1, w_2, w_3 \in C'$. Consider the Möbius transformation f_M as above. It has to send the circle C to the circle C' because it sends C to *some* circle but there is only one circle going through w_1, w_2, w_3.
(b) First assume that z_0, z_1, z_2, z_3 are disjoint. Choose a Möbius transformation f_M sending the triple z_1, z_2, z_3 to the triple $1, 0, \infty$. We have

$$(z_0, z_1, z_2, z_3) = (f_M(z_0), f_M(z_1), f_M(z_2), f_M(z_3)) = (f_M(z_0), 1, 0, \infty) = \frac{f_M(z_0) - 0}{1 - 0} = f_M(z_0)$$

(here the invariance of the cross-ratio under Möbius transformations, proven in the lecture, is used). Therefore the cross-ratio is real, if and only if $f_M(z_0)$ is real. Since the circle in $\widehat{\mathbb{C}}$ going through the 3 points $1, 0, \infty \in \widehat{\mathbb{C}}$ is $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we can formulate this as follows. The cross-ratio is real, if and only if $f_M(z_0)$ lies on the circle through $1, 0, \infty$. By the argument from (a), this is equivalent to z_0 lying on the circle through $f_M^{-1}(1), f_M^{-1}(0), f_M^{-1}(\infty)$, which are the points z_1, z_2, z_3. Hence the cross-ratio is real, if and only if z_0 lies on the circle determined by z_1, z_2, z_3, that is, all 4 points lie on one circle.
If 2 or more of the z_i coincide, the cross-ratio is ∞ but the 4 points then obviously lie on a circle, too.

2. **Complex differentiability:** Decide for the following functions $f : \mathbb{C} \to \mathbb{C}$ whether they are

   (1) continuous at 0,

   (2) partially differentiable at 0 in x and y direction (identifying $\mathbb{C}$ with $\mathbb{R}^2$ as usual),

   (3) real differentiable at 0 (identifying $\mathbb{C}$ with $\mathbb{R}^2$ as usual),

   (4) complex differentiable at 0,

   (5) holomorphic in a neighborhood of 0 (for example a small disc around).

   (a) $f(z) := \overline{z}$

(b) $f(z) := |z|$

(c) $f(z) := |z|^2$

(d) $f(z) := 0$ if $x \neq y$ or $z = 0$, $f(z) := 1$, otherwise

(e) $f(z) := \frac{y^3}{x^2+y^2}$ for $z \neq 0$ and $f(0) := 0$

(f) $f(z) := u(z) + iv(z)$, where $u(z) := \exp(x)\cos(y)$ and $v(z) := \exp(x)\sin(y)$

(g) $f(z) := u(z) + iv(z)$, where $u(z) := x^3 - 3xy^2$ and $v(z) := 3x^2y - y^3$

(h) $f(z) := \overline{g(\overline{z})}$ for any holomorphic function g

where we wrote $z = x + yi$.

Summarize your answers in a table! You do not have to give proofs.

We use freely the following fact from real analysis: If we have given any function

$$f : \mathbb{R}^2 = \mathbb{C} \to \mathbb{C} = \mathbb{R}^2$$

which is of the form

$$f(z) = \begin{pmatrix} f_x(z) \\ f_y(z) \end{pmatrix} = f_x(z) + if_y(z) \qquad z = x + iy$$

for $f_x(z)$ and $f_y(z)$ are *any* composition of algebraic expressions and/or smooth functions in x and y, then f is real differentiable. These are: addition, subtraction, multiplication, divison (with non-zero denominator!), sin, cos, exp, etc.

Observe also that for the above conditions (5) $\Rightarrow$ (4) $\Rightarrow$(3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) but not necessarily (2) $\Rightarrow$ (1). This exercise shows in particular that no other implications are possible.

(a) $f(z) = \overline{z}$. The function is obviously real-differentiable at $z = 0$, by what was said in the beginning. For complex differentiability, we have to see whether the Jacobian matrix at $z = 0$ is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(which means that it is $\mathbb{C}$-linear — the same as multiplication by $m = a + bi$) or, in other words, whether the Cauchy-Riemann equations are satisfied there.

We have here $f_x(x + iy) = x$ and $f_y(x + iy) = -y$. Therefore

$$\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

even for all $z \in \mathbb{C}$, which is *not* of the form required for complex differentiability.

(b) $f(z) = |z|$ is obviously continuous, but already its restriction to the real line, which coincides with the usual real absolute value, is not differentiable at $x = 0$. In other words, it is not partially differentiable in x direction, hence neither real- nor complex differentiable.

(c) $f(x + iy) = f_x(x + iy) = x^2 + y^2$ is real-differentiable (at any point) by what was said in the beginning. Its Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(z) = \begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix},$$

which is of the form required for complex differentiability *only* at $z = 0$. This is kind of an accident and the function is not holomorphic (on any neighborhood of $z = 0$).

(d) The function first of all is obviously not continuous at $z = 0$: For the sequence $z_k = x_k + ix_k$ with any real zero-sequence $x_1, x_2, \ldots$ (with non-zero terms) we have

$$\lim_{k\to\infty} z_k = 0 \qquad \lim_{k\to\infty} f(z_k) = 1,$$

but for the sequence $z_k = x_k$, we have

$$\lim_{k\to\infty} z_k = 0 \qquad \lim_{k\to\infty} f(z_k) = 0.$$

Nevertheless, its restrictions to the x and y-axes are both identically zero, hence f is partially differentiable in these directions.

(e) First f is continuous at 0 because we have

$$|\frac{y^3}{x^2+y^2}| = |\frac{y^3}{|z|^2}| \le \frac{|z|^3}{|z|^2} = |z|.$$

Furthermore, the partial derivatives in x and y direction exist and we get the following potential (!) Jacobian matrix:

$$J = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now remember: Real differentiability means that we can write f in a neighborhood of $z = 0$ as follows

$$f(z) = f(0) + J \cdot \begin{pmatrix} x \\ y \end{pmatrix} + |z| r(z),$$

where r is continuous at 0 with $r(0) = 0$. Hence we have to investigate, whether

$$r(z) = \begin{cases} \frac{f(z)-f(0)-J\cdot\begin{pmatrix} x \\ y \end{pmatrix}}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous at 0. We get for $z \neq 0$:

$$r(z) = \frac{y^3}{|z|^3} - \frac{y}{|z|} = \frac{y}{|z|}(\frac{y^2}{|z|^2} - 1)$$

The sequence $z_k = x_k + ix_k$ for any real zero-sequence $x_1, x_2, \ldots$ leads to

$$r(z_k) = \frac{x_k}{2x_k}(\frac{x_k^2}{4x_k^2} - 1) = -\frac{3}{8}$$

which is not zero. Hence f is not real-differentiable at 0 even though it is partially differentiable at $z = 0$ (even in *any* direction).

(f) f is obviously real-differentiable by what was said in the beginning. We have to check the Cauchy-Riemann conditions. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(z) = \begin{pmatrix} \exp(x)\cos(y) & -\exp(x)\sin(y) \\ \exp(x)\sin(y) & \exp(x)\cos(y) \end{pmatrix}.$$

This is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for all z. Hence f is holomorphic on the whole complex plane.
Remark: We see in addition: $f'(z) = f(z)$ *for all* z*. Meanwhile, we learned* $f(z) = \exp(z)$ *is the complex exponential function.*
(g) f is obviously real-differentiable by what was said in the beginning. We have to check the Cauchy-Riemann conditions. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(z) = \begin{pmatrix} 3x^2 - 3y^2 & -3xy \\ 3xy & 3x^2 - 3y^2 \end{pmatrix}.$$

This is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for all z. Hence f is holomorphic on the whole complex plane.
Remark: We see in addition: $f'(z) = 3(x^2 - y^2 + ixy)$ *for all* z*, which you may recognize as* $f'(z) = 3z^2$ *and indeed* $f(z) = z^3$*, written out in* x, y *coordinates.*
(h) Obviously the function f is real-differentiable because it is a composition of real-differentiable functions. Remember that we saw in (a) that $h : z \mapsto \overline{z}$ is everywhere real-differentiable with Jacobian matrix given by

$$J(h, z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence the Jacobian matrix of the function $f = h \circ g \circ h$ at z is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} J(g, \overline{z}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by the real-analytic chain rule. Here $J(g, \overline{z})$ is the Jacobian matrix of g at $\overline{z}$. It is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for any z because g *is* holomorphic. The calculation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

shows that also f is holomorphic everywhere with $f'(z) = \overline{g'(\overline{z})}$.

Summarized:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
(1)	x	x	x		x	x	x	x
(2)	x		x	x	x	x	x	x
(3)	x		x			x	x	x
(4)			x			x	x	x
(5)						x	x	x

3. **Cauchy-Riemann operator:** Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a real differentiable function (at every point in U). Show that f is holomorphic, if and only if $\frac{\partial f}{\partial \overline{z}} = 0$, where $\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$. Show furthermore that in this case $f'(z) = \frac{\partial f}{\partial z}$, where $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.

Since $f(z) = f_x(z) + if_y(z)$ is assumed to be real-differentiable everywhere, being holomorphic is equivalent to the validity of the Cauchy-Riemann equations everywhere. They express the condition that the Jacobian matrix

$$J(f,z) = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}(z)$$

is of the form

$$\begin{pmatrix} a(z) & -b(z) \\ b(z) & a(z) \end{pmatrix}.$$

We have then $f'(z) = a(z) + b(z)i$.
Let us check what $\frac{\partial f}{\partial \overline{z}} = 0$ means:

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} + i(\frac{\partial f_x}{\partial y} + i\frac{\partial f_y}{\partial y})\right) \\
&= \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} + i(\frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x})\right).
\end{aligned}$$

This is zero obviously, if and only if we have

$$\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y}$$

and

$$\frac{\partial f_x}{\partial y} = -\frac{\partial f_y}{\partial x}.$$

Furthermore, if f is holomorphic we have

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} - i(\frac{\partial f_x}{\partial y} + i\frac{\partial f_y}{\partial y})\right) \\
&= \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + i(\frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x})\right) \\
&= a(z) + b(z)i = f'(z).
\end{aligned}$$