Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 3

1. **Basic path integrals:** Consider the closed path $\varphi : [0,1] \to \mathbb{C}^*$ given by $\varphi(t) = -r\exp(2\pi i t)$ for some $r \in \mathbb{R}_{>0}$ and the family of holomorphic functions $\mathbb{C}^* \to \mathbb{C}$, given by $z \mapsto z^k$ for $k \in \mathbb{Z}$.

   (a) Calculate
   $$\int_\varphi z^k \mathrm{d}z$$
   depending on k (and r?) using *only* the definition of path integral and elementary formulas for real integrals.

   (b) Assume now $k \neq -1$. Prove that your calculation was correct by constructing a primitive for $z \mapsto z^k$ on $\mathbb{C}^*$.

   (c) Assume now $k = -1$. Redo your calculation in the following way. Shorten the path a little bit, going from $[\varepsilon, 1-\varepsilon] \to \mathbb{C}^*$ with the same φ. It has now values in the open set $U = \{z \in \mathbb{C}^* \mid \arg(z) \neq \pi\}$ (here $-\pi < \arg(z) \leq \pi$ is determined by $z = |z|\exp(i\arg(z))$). Construct a primitive for $z \mapsto z^{-1}$ on U as follows. Recall the following fact (Lemma 2.3.3 in the lecture): *If U, U' are open subsets $\subseteq \mathbb{C}$, $f : U \to U'$ is continuous and $g : U' \to \mathbb{C}$ is holomorphic with $g \circ f = id$ and $g'(z) \neq 0$ for all $z \in U'$ then f is holomorphic on U and*
   $$f'(z) = \frac{1}{g'(f(z))}$$
   for all $z \in U$.

   Apply this Lemma to $g(z) = \exp(z)$, U as above, f a suitable inverse of exp on U. Lastly take the limit $\varepsilon \to 0$.

2. **The complex logarithm:** Any function $f : U \to \mathbb{C}$ constructed as in exercise (1.c) is called a *branch of the complex logarithm.* Determine a power series representation for it around $1 \in \mathbb{C}$, that is, determine coefficients $a_0, a_1, \ldots$ such that
   $$f(z) = \sum_{k=0}^{\infty} a_k (z-1)^k$$
   for z in some disc around 1. What is its convergence radius?

   Hint: Use from exercise (1.c) that f is a primitive for the function $z \mapsto z^{-1}$ on U.

3. **Local injectivity:** Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic, $z_0 \in U$ and $f'(z_0) \neq 0$. Assume f' continuous on U (we will later see that this is automatically satisfied).

   Prove the following. There exists an $r > 0$ such that f, restricted to $B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$, is injective.

   Hint: Integrate f' over a small (non-closed) path around z_0!

— *Please turn over!* —

4. **Transforming path integrals:** Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic. Assume f' continuous on U (we will later see that this is automatically satisfied). Let $\varphi : [a, b] \to \mathbb{C}$ be any closed smooth path. Prove that

$$\int_{\varphi} \overline{f(z)} f'(z) dz$$

is purely imaginary.

Hint: Prove and use the transformation formula:

$$\int_{\varphi} g(f(z)) f'(z) dz = \int_{f \circ \varphi} g(z) dz$$

for $g : Im(f) \to \mathbb{C}$ continuous.

*5. **Reparameterization of paths:** Let $U \subseteq \mathbb{C}$ be open. Let φ be a piecewise smooth path $[a, b] \to U$. Show that there exists a continuously differentiable function $\psi : [0, 1] \to [a, b]$ such that $\varphi \circ \psi$ is a *smooth* path and

$$\int_{\varphi} f(z) \mathrm{d}z = \int_{\varphi \circ \psi} f(z) \mathrm{d}z$$

for every continuous function $f : U \to \mathbb{C}$.

Please hand in your solutions on Monday, September 27, 2010 in the lecture hall