Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 3

1. **Basic path integrals:** Consider the closed path $\varphi : [0,1] \to \mathbb{C}^*$ given by $\varphi(t) = -r\exp(2\pi i t)$ for some $r \in \mathbb{R}_{>0}$ and the family of holomorphic functions $\mathbb{C}^* \to \mathbb{C}$, given by $z \mapsto z^k$ for $k \in \mathbb{Z}$.

   (a) Calculate
   $$\int_\varphi z^k \mathrm{d}z$$
   depending on k (and r?) using *only* the definition of path integral and elementary formulas for real integrals.

   (b) Assume now $k \neq -1$. Prove that your calculation was correct by constructing a primitive for $z \mapsto z^k$ on $\mathbb{C}^*$.

   (c) Assume now $k = -1$. Redo your calculation in the following way. Shorten the path a little bit, going from $[\varepsilon, 1-\varepsilon] \to \mathbb{C}^*$ with the same φ. It has now values in the open set $U = \{z \in \mathbb{C}^* \mid \arg(z) \neq \pi\}$ (here $-\pi < \arg(z) \leq \pi$ is determined by $z = |z|\exp(i\arg(z))$). Construct a primitive for $z \mapsto z^{-1}$ on U as follows. Recall the following fact (Lemma 2.3.3 in the lecture): *If U, U' are open subsets $\subseteq \mathbb{C}$, $f : U \to U'$ is continuous and $g : U' \to \mathbb{C}$ is holomorphic with $g \circ f = id$ and $g'(z) \neq 0$ for all $z \in U'$ then f is holomorphic on U and*
   $$f'(z) = \frac{1}{g'(f(z))}$$
   for all $z \in U$.

   Apply this Lemma to $g(z) = \exp(z)$, U as above, f a suitable inverse of exp on U. Lastly take the limit $\varepsilon \to 0$.

(a) We have

$$\begin{aligned}
\varphi(t) &= -r\exp(2\pi i t) \qquad t \in [0,1] \\
\varphi'(t) &= -r2\pi i \exp(2\pi i t) \\
\int_\varphi z^k \mathrm{d}z &= \int_0^1 \varphi(t)^k \varphi'(t)\mathrm{d}t \\
&= 2\pi i \int_0^1 (-r)^{k+1} \exp(2\pi i(k+1)t)\mathrm{d}t \\
&= 2\pi i \left(\int_0^1 (-r)^{k+1} \cos(2\pi(k+1)t)\mathrm{d}t + i \int_0^1 (-r)^{k+1} \sin(2\pi(k+1)t)\mathrm{d}t \right)
\end{aligned}$$

If $k \neq -1$ this is equal to

$$\begin{aligned}
&= \frac{2\pi i(-r)^{k+1}}{2\pi(k+1)} \left([\sin(2\pi(k+1)t)]_{t=0}^{t=1} + i\,[-\cos(2\pi(k+1)t)]_{t=0}^{t=1} \right) \\
&= 0
\end{aligned}$$

independently of r.

If $k = -1$ it is equal to

$$= \quad 2\pi i \int_0^1 1\mathrm{d}t = 2\pi i,$$

independently of r.
(b) Let $k \neq -1$. Consider the function $F_k : \mathbb{C}^* \to \mathbb{C}$ given by

$$F_k(z) := \frac{1}{k+1} z^{k+1}$$

It satisfies

$$F_k'(z) = z^k.$$

Its existence shows that

$$\int_\varphi z^k \mathrm{d}z = 0,$$

for *any* closed path $\varphi : [a,b] \to \mathbb{C}^*$.
(c) Let $k = -1$. Consider the following open subsets of $\mathbb{C}$:

$$\begin{aligned} U &:= \{z \in \mathbb{C}^* \mid \arg(z) \neq \pi\} \\ U' &:= \{z \in \mathbb{C} \mid -\pi < \operatorname{Im}(z) < \pi\} \end{aligned}$$

Here $\arg(z)$ is the argument (angle) of the complex number $z \neq 0$, such that

$$z = |z| \exp(\arg(z) i) = |z|(\cos(\arg(z)) + i \sin(\arg(z)))$$

and $-\pi < \arg(z) \leq \pi$. We have seen in the lecture that $\arg(z)$ is well-defined with this property. Therefore, we get an inverse of exp on U as follows: Define

$$\begin{aligned} f : U &\to U' \\ f(z) &:= \log(|z|) + i \arg(z). \end{aligned}$$

We have then $f(\exp(z)) = z$ for $z \in U'$ and $\exp(f(z)) = z$ for $z \in U$.
We now apply Lemma 2.3.3 to this f, as well as $g = \exp$, U and U' as given above. Since $g'(z) = \exp(z)$ it is never 0 and therefore f is holomorphic with

$$f'(z) = \frac{1}{g'(f(z))} = \frac{1}{\exp(f(z))} = \frac{1}{z}.$$

Consider the path φ_ε given by restriction of φ to the interval $[\varepsilon, 1-\varepsilon]$. The arguments $\arg(\varphi(t))$ vary now between $-\pi + \varepsilon$ and $\pi - \varepsilon$, hence φ_ε has images in U.
We may approximate the integral now as follows:

$$\begin{aligned} \int_\varphi \frac{1}{z} \mathrm{d}z &= \lim_{\varepsilon \to 0} \int_{\varphi_\varepsilon} \frac{1}{z} \mathrm{d}z \\ &= \lim_{\varepsilon \to 0} (f(-r \exp(2\pi i \varepsilon)) - f(-r \exp(2\pi i (1-\varepsilon)))) \\ &= \lim_{\varepsilon \to 0} (\log(r) + i(\pi - \varepsilon) - (\log(r) + i(-\pi + \varepsilon))) \\ &= i(2\pi - \lim_{\varepsilon \to 0} 2\varepsilon) = 2\pi i. \end{aligned}$$

Remark: This gives an "explanation" of the result in terms of the non-uniqueness of the argument (angle) of a complex number. The calculation suggests that the value is the same for any *closed path going around 0 once (it is however not so easy to make this statement mathematically precise!).*

2. **The complex logarithm:** Any function $f : U \to \mathbb{C}$ constructed as in exercise (1.c) is called a *branch of the complex logarithm*. Determine a power series representation for it around $1 \in \mathbb{C}$, that is, determine coefficients $a_0, a_1, \dots$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z-1)^k$$

for z in some disc around 1. What is its convergence radius?

Hint: Use from exercise (1.c) that f is a primitive for the function $z \mapsto z^{-1}$ on U.

We first establish a power series expansion for $\frac{1}{z}$ around $z = 1$, that is, a representation

$$\frac{1}{z} = \sum_{k=0}^{\infty} a_k (z-1)^k.$$

Substituting $z = (1-w)$, we get

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} a_k (-w)^k.$$

We saw in the lecture that

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k$$

(geometric series). Therefore $a_k = (-1)^k$. The convergence radius is given by the formula

$$\frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{1}} = 1.$$

Note that a bigger convergence radius cannot be expected because $\frac{1}{z}$ is singular at 0.
Any primitive for $\frac{1}{z}$ on $B_1(1) = \{z \in \mathbb{C} \mid |z-1| < 1\}$ is therefore given by termwise integrating the above series:

$$C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (z-1)^{k+1}$$

(it is easy to see that two primitives differ by a constant, like in real analysis.)
Now the function given by $f(z) = \log(|z|) + i \arg(z)$ is holomorphic and a primitive for $\frac{1}{z}$ by (1.c). It satisfies $f(1) = 0$ — this determines $C = 0$ in the series above — and we get

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (z-1)^{k+1}$$

on $B_1(1)$.
Note that, restricted to the real line, this coincides with the familiar expansion of the real analytic logarithm.

3. **Local injectivity:** Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic, $z_0 \in U$ and $f'(z_0) \neq 0$. Assume f' continuous on U (we will later see that this is automatically satisfied).

   Prove the following. There exists an $r > 0$ such that f, restricted to $B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$, is injective.

   Hint: Integrate f' over a small (non-closed) path around z_0!

We integrate f' over a linear path from z_1 to z_2, where z_1 and z_2 are both "close to" z_0. It may be parametrized by $\varphi(t) = z_1 + t(z_2 - z_1)$, $t \in [0, 1]$. We get $\varphi'(t) = z_2 - z_1$ and therefore:

$$\int_\varphi f'(w)\mathrm{d}w = (z_2 - z_1) \int_0^1 f'(\varphi(t))dt = (z_2 - z_1) \int_0^1 (f'(z_0) + \delta(\varphi(t))\mathrm{d}t$$

with $\delta(z) = f'(z) - f'(z_0)$. We can assume that $|\delta(z)|$ is smaller then $|f'(z_0)|$ in a suitable small disc around z_0 because f' is continuous. We then get the estimate

$$\left| \int_0^1 \delta(\varphi(t))\mathrm{d}t \right| < |f'(z_0)|.$$

Therefore if $z_2 \neq z_1$:

$$\left| \int_\varphi f'(w)\mathrm{d}w \right| \geq |z_2 - z_1| \left(|f'(z_0)| - \left| \int_0^1 \delta(\varphi(t))\mathrm{d}t \right| \right) > |z_2 - z_1|(|f'(z_0)| - |f'(z_0)|) = 0$$

(Here $|A + B| \geq |A| - |B|$ is used, which is equivalent to the triangle inequality.)
On the other hand, we have

$$\int_\varphi f'(w)\mathrm{d}w = f(z_2) - f(z_1).$$

Together: $f(z_2) \neq f(z_1)$, that is, f is injective in this small disc.

4. **Transforming path integrals:** Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic. Assume f' continuous on U (we will later see that this is automatically satisfied). Let $\varphi : [a, b] \to U$ be any closed smooth path. Prove that

$$\int_\varphi \overline{f(z)} f'(z)dz$$

is purely imaginary.

Hint: Prove and use the transformation formula:

$$\int_\varphi g(f(z))f'(z)dz = \int_{f\circ\varphi} g(z)dz$$

for $g : Im(f) \to \mathbb{C}$ *continuous.*

We first prove the transformation formula:

$$\begin{aligned} \int_{f\circ\varphi} g(z)\mathrm{d}z &= \int_a^b g(f(\varphi(t)))(f \circ \varphi)'(t)\mathrm{d}z \\ &= \int_a^b g(f(\varphi(t)))f'(\varphi(t))\varphi'(t)\mathrm{d}z \\ &= \int_\varphi g(f(z)f'(z)\mathrm{d}z. \end{aligned}$$

If we apply it with $g(z) = \overline{z}$, we get

$$\int_\varphi \overline{f(z)} f'(z)dz = \int_{f\circ\varphi} \overline{z}\mathrm{d}z.$$

Now the path $f \circ \varphi$ is obviously closed, too.
Therefore it suffices to show that

$$\operatorname{Re} \int_\psi \overline{z} \mathrm{d}z = 0$$

for every closed path $\psi : [a, b] \to \mathbb{C}$.
We insert the definition of path integral:

$$\operatorname{Re} \int_\psi \overline{z} \mathrm{d}z = \operatorname{Re} \int_a^b \overline{\psi(t)} \psi'(t) \mathrm{d}t.$$

We now write $\psi(t) = \psi_x(t) + i\psi_y(t)$ and get

$$\begin{aligned} \overline{\psi(t)}\psi'(t) &= (\psi_x(t) - i\psi_y(t))(\psi'_x(t) + i\psi'_y(t)) \\ &= \psi_x(t)\psi'_x(t) + \psi_y(t)\psi'_y(t) + i(-\psi_y(t)\psi'_x(t) + \psi_y(t)\psi'_x(t)). \end{aligned}$$

Because, by definition, integration of complex functions is defined by integration over real and imaginary part separately, we have:

$$\begin{aligned} \operatorname{Re} \int_a^b \overline{\psi(t)}\psi'(t)\mathrm{d}t &= \int_a^b \operatorname{Re}(\overline{\psi(t)}\psi'(t))\mathrm{d}t \\ &= \int_a^b (\psi_x(t)\psi'_x(t) + \psi_y(t)\psi'_y(t))\mathrm{d}t \end{aligned}$$

Now $\psi_x(t)\psi'_x(t)$ has the (real) primitive: $\frac{1}{2}\psi_x^2(t)$. Therefore

$$\int_a^b \psi_x(t)\psi'_x(t)\mathrm{d}t = \frac{1}{2}(\psi_x^2(b) - \psi_x^2(a)).$$

Since $\psi_x(a) = \psi_x(b)$ (the path is closed!) this is zero. Similarly $\int_a^b \psi_y(t)\psi'_y(t)\mathrm{d}t = \frac{1}{2}(\psi_y^2(b) - \psi_y^2(a)) = 0$.
Putting everything together, we get:

$$\operatorname{Re} \int_\psi \overline{z} \mathrm{d}z = 0.$$

*5. **Reparameterization of paths:** Let $U \subseteq \mathbb{C}$ be open. Let φ be a piecewise smooth path $[a, b] \to U$. Show that there exists a continuously differentiable function $\psi : [0, 1] \to [a, b]$ such that $\varphi \circ \psi$ is a *smooth* path and

$$\int_\varphi f(z)\mathrm{d}z = \int_{\varphi \circ \psi} f(z)\mathrm{d}z$$

for every continuous function $f : U \to \mathbb{C}$.

Sketch: If we are given two paths $\varphi_1 : [a, b] \to U$ and $\varphi_2 : [b, c] \to U$ with $a < b < c$ and $\varphi_1(b) = \varphi_2(b)$ we may concatenate these paths to get a new path

$$\varphi_1\varphi_2 : [a, c] \to U.$$

Obviously this path is piecewise smooth if the φ_i's are. We have in addition

$$\int_{\varphi_1\varphi_2} f(z)\mathrm{d}z = \int_{\varphi_1} f(z)\mathrm{d}z + \int_{\varphi_2} f(z)\mathrm{d}z.$$

for every continuous function $f : U \to \mathbb{C}$. Thus, the definition of piecewise smooth may be reformulated as follows. A path is piecewise smooth, if and only if it is obtained from finitely many smooth paths by concatenation.
When is the concatenation of 2 smooth paths φ_1, φ_2 smooth again? It is not difficult to see that this is true if and only if $\varphi_1'(b) = \varphi_2'(b)$.
The strategy now is as follows:

1. Decompose the given piecewise smooth path into smooth paths $\varphi_k : [a_k, a_{k+1}] \to U$, $k = 0, \ldots, n-1$.
2. Transform each path by composition with a certain continously differentiable map $\psi : [\frac{k}{n}, \frac{k+1}{n}] \to [a_k, a_{k+1}]$, mapping endpoints to endpoints, into a path $\widetilde{\varphi}_k = \varphi_k \circ \psi$ with satisfies $\widetilde{\varphi}_k'(\frac{k}{n}) = \widetilde{\varphi}_k'(\frac{k+1}{n}) = 0$.
3. Concatenate the paths $\widetilde{\varphi}_k$ to a *smooth* path $\widetilde{\varphi} : [0,1] \to U$.

Let's investigate the second, crucial step a little bit further. We get for a ψ as above: $\widetilde{\varphi}'(\frac{k}{n}) = \psi'(\frac{k}{n})\varphi'(a_k)$ and $\widetilde{\varphi}'(\frac{k+1}{n}) = \psi'(\frac{k+1}{n})\varphi'(a_{k+1})$. Hence we may archive our goal by choosing an arbitrary continuously differentiable ψ satisfying

$$\psi(\frac{k}{n}) = a_k, \qquad \psi(\frac{k+1}{n}) = a_{k+1}, \qquad \psi'(\frac{k}{n}) = \psi'(\frac{k+1}{n}) = 0.$$

Using the elementary reparametrization

$$\psi(t) = \widetilde{\psi}(nt - k)(a_{k+1} - a_k) + a_k,$$

this boils down to finding a continuously differentiable $\widetilde{\psi} : [0,1] \to [0,1]$ satisfying:

$$\widetilde{\psi}(0) = 0, \qquad \widetilde{\psi}(1) = 1, \qquad \widetilde{\psi}'(0) = \widetilde{\psi}'(1) = 0.$$

Such a function is, for example, given by $\widetilde{\psi}(t) = 3t^2 - 2t^3$.
Remark: This exercise shows first of all that it would lead basically to the same theory, if we would have insisted in paths being smooth instead of piecewise smooth. It also shows, that the image of *a smooth path need not to be smooth in the colloquial sense, that is, it may contain salient points. Visualize for example the path $\varphi(t) = t^2 + it^3$, $t \in [-1,1]$.*