Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 4

1.-3. **Using Cauchy's integral formula:** Calculate:

$$\int_\varphi \exp(z) z^k \mathrm{d}z \qquad k \in \mathbb{Z} \tag{1}$$

$$\int_\varphi \frac{1}{z^2+1} \mathrm{d}z \tag{2}$$

$$\int_\varphi z^m (1-z)^n \mathrm{d}z \qquad m, n \in \mathbb{Z} \tag{3}$$

for the path $\varphi(t) = 2\exp(2\pi i t)$, $t \in [0, 1]$.

Hint: Use the following general version of Cauchy's integral formula for $m \geq 0$:

$$\frac{f^{(m)}(z_0)}{m!} = \frac{1}{2\pi i} \int_\varphi \frac{f(z)}{(z-z_0)^{m+1}} dz$$

where $\varphi(t) = r\exp(2\pi i t), t \in [0,1]$ and $|z_0| < r$. Here $f^{(m)}$ is the m-th derivative of f.

(1) has already the form of Cauchy's integral formula for $k < 0$. For (2) use partial fraction decomposition. For (3), if $n < 0$, use the binomial series for $(1 - \frac{1}{z})^n$.

4. **Existence of primitives:** Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Prove that f has a primitive (that is, there exists a holomophic $F : U \to \mathbb{C}$ with $F' = f$), if and only if

$$\int_\varphi f(z) \mathrm{d}z = 0,$$

for every closed, piecewise smooth path $\varphi : [a, b] \to U$.

Hint: This is a small adaption of the proofs (E2) ⇒ (E3) and (E3) ⇒ (E4) given in the lecture.

5. Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic function, with $f(z) \neq 0$ and $\arg(f(z)) \neq \pi$ for all $z \in U$. Prove

$$\int_\varphi \frac{f'(z)}{f(z)} \mathrm{d}z = 0,$$

for any closed path $\varphi : [a, b] \to U$.

Please hand in your solutions on Monday, October 4, 2010 in the lecture hall