1. **Elementary domains:** Which of the following subsets \( U \subseteq \mathbb{C} \) have the property that for every holomorphic function \( f : U \to \mathbb{C} \) and every closed path \( \varphi : [a, b] \to U \), we have
\[
\int_{\varphi} f(z) \, dz = 0.
\]
(a) \( U = \mathbb{C} \),
(b) \( U = B_1(0) = \{ z \in \mathbb{C} \mid |z| < 1 \} \),
(c) \( U = B_1(0) \setminus \{0\} \),
(d) \( U = \{ z \in \mathbb{C} \mid -1 < \text{Re}(z) < 1, -1 < \text{Im}(z) < 1 \} \),
(e) \( U = \mathbb{C}^* \setminus \{ z \in \mathbb{C}^* \mid \arg z = \pi \} \).

Give a counterexample, if you answer that a set \( U \) above does not have this property.

*Hint: Examine whether the set is star-like or not. If it’s not, try to find a counterexample.*

2. **Doubly periodic functions:** Let \( f : \mathbb{C} \to \mathbb{C} \) be a holomorphic function with is doubly periodic in the following way:
\[
f(z + 1) = f(z) \quad \text{and} \quad f(z + i) = f(z) \quad \forall z \in \mathbb{C}
\]
Prove that \( f \) is constant.

*Hint: Liouville’s theorem.*

3. **Using Cauchy’s estimate I:** Let \( f : B_r(0) \to \mathbb{C} \) be a holomorphic function, defined in a small disc around 0. Prove that \( f \) cannot satisfy the inequality \( |f^{(n)}(0)| \geq n! \cdot n^n \) for sufficiently big \( n \).

4. **Using Cauchy’s estimate II:** Let \( n \geq 0 \) be an integer and let \( f : \mathbb{C} \to \mathbb{C} \) be a holomorphic function satisfying
\[
|f(z)| < |z|^n,
\]
for all \( z \in \mathbb{C} \) with \( |z| > R \) for some \( R > 0 \). Prove that \( f \) is a polynomial.

5. **Analytic functions:** Determine the convergence radius of the power series expansion of the function \( f : \mathbb{C} \setminus \{-i, +i\} \to \mathbb{C} \), given by \( f(z) = \frac{1}{z^2 + 1} \) at any point \( z_0 \).

*Hint: Do not calculate! Use Corollary 4.3.6.*

Please hand in your solutions on Wednesday, October 13, 2010 in the lecture hall