Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 5

1. **Elementary domains:** Which of the following subsets $U \subseteq \mathbb{C}$ have the property that *for every* holomorphic function $f : U \to \mathbb{C}$ and *every* closed path $\varphi : [a, b] \to U$, we have

$$\int_\varphi f(z)\mathrm{d}z = 0.$$

   (a) $U = \mathbb{C}$,

   (b) $U = B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\}$,

   (c) $U = B_1(0) \setminus \{0\}$,

   (d) $U = \{z \in \mathbb{C} \mid -1 < \mathrm{Re}(z) < 1,\ -1 < \mathrm{Im}(z) < 1\}$,

   (e) $U = \mathbb{C}^* \setminus \{z \in \mathbb{C}^* \mid \arg z = \pi\}$.

   Give a counterexample, if you answer that a set U above does not have this property.

   Hint: Examine whether the set is star-like or not. If it's not, try to find a counterexample.

2. **Doubly periodic functions:** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with is doubly periodic in the following way:

$$f(z + 1) = f(z) \text{ and } f(z + i) = f(z) \qquad \forall z \in \mathbb{C}$$

   Prove that f is constant.

   Hint: Lioville's theorem.

3. **Using Cauchy's estimate I:** Let $f : B_r(0) \to \mathbb{C}$ be a holomorphic function, defined in a small disc around 0. Prove that f cannot satisfy the inequality $|f^{(n)}(0)| \geq n! \cdot n^n$ for sufficiently big n.

4. **Using Cauchy's estimate II:** Let $n \geq 0$ be an integer and let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying

$$|f(z)| < |z|^n,$$

   for all $z \in \mathbb{C}$ with $|z| > R$ for some $R > 0$. Prove that f is a polynomial.

5. **Analytic functions:** Determine the convergence radius of the power series expansion of the function $f : \mathbb{C} \setminus \{-i, +i\} \to \mathbb{C}$, given by $f(z) = \frac{1}{z^2+1}$ at any point z_0.

   Hint: Do not calculate! Use Corollary 4.3.6.

Please hand in your solutions on Wednesday, October 13, 2010 in the lecture hall