Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 6

1. **Laurent series I:** Determine Laurent series expansions of the function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$

$$f(z) = \frac{1}{z-1}$$

in the following annuli with center 0:

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} \mid 0 < |z| < 1\} \\ U_2 &= \{z \in \mathbb{C} \mid 1 < |z| < \infty\} \end{aligned}$$

(the first one is just a power series expansion!)

2. **Laurent series II:** Determine Laurent series expansions of the function $f : \mathbb{C} \setminus \{0, +1, -1\} \to \mathbb{C}$

$$f(z) = \frac{1}{z^3 - z}$$

in the following annuli with center 0:

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} \mid 0 < |z| < 1\} \\ U_2 &= \{z \in \mathbb{C} \mid 1 < |z| < \infty\} \end{aligned}$$

3. **Laurent series III:** Determine all possible different Laurent series expansions of the function $f : \mathbb{C} \setminus S \to \mathbb{C}$

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

with center 0. Determine also the (minimal) set S of points, where f is singular.

Hint: In this case there are 3 different annuli to be considered.

4. **Singularities:** Consider the following holomorphic functions $f : B_\varepsilon(0) \setminus \{0\} \to \mathbb{C}$ and determine the type of the singularity at 0 (removable, pole of order k, or essential).

$$\begin{aligned} &(1) \quad f(z) = \frac{1}{z^3} + z^2 \\ &(2) \quad f(z) = \frac{1}{z(z-1)(z-2)} \\ &(3) \quad f(z) = \frac{\exp(z) - 1}{z} \\ &(4) \quad f(z) = \sin(\frac{1}{z}) \end{aligned}$$

— Please turn over —

5. **Conformal automorphisms:** Let $f : \mathbb{H} \to \mathbb{H}$ a bijective holomorphic function, where

$$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$$

is the upper half plane.

Prove that

$$f(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{R}$ and $\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \neq 0$.

Hint: Use Theorem 5.3.5 and a suitable Möbius transformation (Cayley transform — end of section 2.1) mapping $\mathbb{H}$ bijectively to $B_1(0)$.

6. **Schwarz-Pick theorem:** Let $f : B_1(0) \to B_1(0)$ be a holomorphic function: Prove that for all $z_1, z_2 \in B_1(0)$:

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|$$

and for all $z \in B_1(0)$:

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Hint: Consider the Möbius transformations $g(z) = \frac{z_1 - z}{1 - \overline{z_1} z}$ and $h(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)} z}$. Apply the Schwarz Lemma to the composition $h \circ f \circ g^{-1}$. Why can you apply it?

Please hand in your solutions on Wednesday, November 3, 2010 in the lecture hall