Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 6

1. **Laurent series I:** Determine Laurent series expansions of the function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$

$$f(z) = \frac{1}{z-1}$$

in the following annuli with center 0:

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} \mid 0 < |z| < 1\} \\ U_2 &= \{z \in \mathbb{C} \mid 1 < |z| < \infty\} \end{aligned}$$

(the first one is just a power series expansion!)

We use the formula (geometric series):

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

valid if $|z| < 1$.
It already gives the power series expansion valid on U_1:

$$f(z) = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k.$$

For the expansion on U_2, we expand the fraction with z^{-1} to get:

$$f(z) = \frac{z^{-1}}{1-z^{-1}} = z^{-1} \sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty} z^{-k-1}.$$

This expansion converges if $|z^{-1}| < 1$, that is, on U_2.

2. **Laurent series II:** Determine Laurent series expansions of the function $f : \mathbb{C} \setminus \{0, +1, -1\} \to \mathbb{C}$

$$f(z) = \frac{1}{z^3 - z}$$

in the following annuli with center 0:

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} \mid 0 < |z| < 1\} \\ U_2 &= \{z \in \mathbb{C} \mid 1 < |z| < \infty\} \end{aligned}$$

We apply the geometric series with z replaced by z^2 to get:

$$\frac{1}{z^3 - z} = -z^{-1} \frac{1}{1-z^2} = -z^{-1} \sum_{k=0}^{\infty} (z^2)^k = -\sum_{k=0}^{\infty} z^{2k-1}$$

This converges if $0 < |z^2| < 1$ hence if $0 < |z| < 1$, that is, on U_1.
To get the other Laurent series expansion, we apply the geometric series with z replaced by z^{-2} and get:

$$\frac{1}{z^3 - z} = z^{-3} \frac{1}{1-z^{-2}} = z^{-3} \sum_{k=0}^{\infty} (z^{-2})^k = \sum_{k=0}^{\infty} z^{-2k-3}$$

This converges if $1 < |z^2|$ hence if $1 < |z|$, that is, on U_2.

3. **Laurent series III:** Determine all possible different Laurent series expansions of the function $f : \mathbb{C} \setminus S \to \mathbb{C}$

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

with center 0. Determine also the (minimal) set S of points, where f is singular.

Hint: In this case there are 3 different annuli to be considered.

We can write

$$f(z) = \frac{1}{(z-1)(z-2)}$$

In this case, we have singularities at $z = 1$ and $z = 2$. The maximal annuli with center at 0 contained in $\mathbb{C} \setminus \{1, 2\}$ are therefore:

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} \mid |z| < 1\} \\ U_2 &= \{z \in \mathbb{C} \mid 1 < |z| < 2\} \\ U_3 &= \{z \in \mathbb{C} \mid 2 < |z|\} \end{aligned}$$

We use partial fractional decomposition to get:

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

For $-\frac{1}{z-1}$ we got two different expansions in exercise 1:

$$A : -\frac{1}{z-1} = \sum_{k=0}^{\infty} z^k$$

valid if $|z| < 1$ and

$$B : -\frac{z^{-1}}{1 - z^{-1}} = -\sum_{k=0}^{\infty} z^{-k-1}.$$

valid if $|z| > 1$.

For $\frac{1}{z-2}$, we repeat the calculations of exercise 1 We get

$$C : \frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}(\frac{z}{2})^k$$

valid if $|z| < 2$ and

$$D : \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = z^{-1}\sum_{k=0}^{\infty}(\frac{z}{2})^{-k}$$

valid if $|z| > 2$.

We get the following Laurent expansions of $f(z)$:

on U_1: $A + C$
on U_2: $B + C$
on U_3: $B + D$

The fourth combination $A + D$ converges nowhere.

4. **Singularities:** Consider the following holomorphic functions $f : B_\varepsilon(0) \setminus \{0\} \to \mathbb{C}$ and determine the type of the singularity at 0 (removable, pole of order k, or essential).

$$\begin{aligned}
(1) \quad f(z) &= \frac{1}{z^3} + z^2 \\
(2) \quad f(z) &= \frac{1}{z(z-1)(z-2)} \\
(3) \quad f(z) &= \frac{\exp(z) - 1}{z} \\
(4) \quad f(z) &= \sin(\frac{1}{z})
\end{aligned}$$

(1) The function $f(z) = \frac{1}{z^3} + z^2$ is already given as a Laurent series (a Laurent polynomial) at $z = 0$. The lowest power of z occurring is z^{-3}, hence f has a pole of order 3.
(2) The function $f(z) = \frac{1}{z(z-1)(z-2)}$ can be written as

$$f(z) = \frac{1}{z}(a_{-1} + a_0 z + \cdots)$$

with $a_{-1} = \frac{1}{(0-1)(0-2)} \neq 0$. The order is therefore -1, hence f has a pole of order 1.
(3) The function $f(z) = \frac{\exp(z)-1}{z}$ has the expansion

$$f(z) = \frac{\sum_{k=0}^{\infty} \frac{z^k}{k!} - 1}{z} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}$$

This is a power series (without negative powers of z), hence 0 is removable for f.
(4) The function $f(z) = \sin(\frac{1}{z})$ has the expansion

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{-2k-1}}{(2k+1)!}.$$

Here infinitely many negative powers of z occur, hence f has an essential singularity at 0.

5. **Conformal automorphisms:** Let $f : \mathbb{H} \to \mathbb{H}$ a bijective holomorphic function, where

$$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$$

is the upper half plane.

Prove that

$$f(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{R}$ and $\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \neq 0$.

Hint: Use Theorem 5.3.5 and a suitable Möbius transformation (Cayley transform — end of section 2.1) mapping $\mathbb{H}$ bijectively to $B_1(0)$.

We know from section 2.1, that the Möbius transformation g associated with the matrix

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

is a bijective holomorphic map from $\mathbb{H}$ to $B_1(0)$. Its inverse g^{-1} is given by the Möbius transformation associated with the inverse matrix

$$\frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$$

Here we can omit the factor $\frac{1}{2i}$ because a Möbius transformation does depend on its matrix only up to scalar (that is: if I multiply the underlying matrix with a constant, it doesn't change the resulting function — this is obvious from the formula for a Möbius transformation).
We have the following maps (all bijective holomorphic maps):

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] \arrow[d, "g"] & \mathbb{H} \arrow[d, "g"] \\ B_1(0) \arrow[r, "g\circ f\circ g^{-1}"] & B_1(0) \end{tikzcd}$$

By Theorem 5.3.5 we know that $g \circ f \circ g^{-1}$ has to be a Möbius transformation associated with the matrix

$$\begin{pmatrix} \alpha & -\alpha z_0 \\ \overline{z_0} & -1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -z_0 \\ \overline{z_0} & -1 \end{pmatrix}$$

where $a \in \mathbb{C}, |a| = 1$ and $z_0 \in \mathbb{C}$ with $|z_0| < 1$.
Denote by h_α and h_0 the respective Möbius transformations associated with the matrices on the right.
We have $g \circ f \circ g^{-1} = h_\alpha \circ h_0$. Or $f = (g^{-1} \circ h_\alpha \circ g) \circ (g^{-1} \circ h_0 \circ g)$. It suffices to show, that each of the Möbius transformations $g^{-1} \circ h_\alpha \circ g$ and $g^{-1} \circ h_0 \circ g$ can be described by a real matrix.
1st case: $g^{-1} \circ h_\alpha \circ g$. This transformation is associated with the matrix

$$\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} i(\alpha+1) & \alpha-1 \\ 1-\alpha & i(\alpha+1) \end{pmatrix}$$

It is also associated (see the remark above) with any complex nonzero multiple of this matrix. So we have to find a complex number such that its product with this matrix is a real matrix. Looking at the upper right entry, we see that the only possibility for such a complex number is $\overline{\alpha - 1}$ (or a real multiple of it). We have

$$\overline{\alpha-1}\begin{pmatrix} i(\alpha+1) & \alpha-1 \\ 1-\alpha & i(\alpha+1) \end{pmatrix} = \begin{pmatrix} i(\alpha\overline{\alpha}-1)+i(\overline{\alpha}-\alpha) & |\alpha-1|^2 \\ -|\alpha-1|^2 & i(\alpha\overline{\alpha}-1)+i(\overline{\alpha}-\alpha) \end{pmatrix}$$

Since $\overline{\alpha}\alpha = 1$, this matrix is indeed real.
2nd case: $g^{-1} \circ h_0 \circ g$. This transformation is associated with the matrix

$$\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -z_0 \\ \overline{z_0} & -1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} i(\overline{z_0}-z_0) & 2+z_0+\overline{z_0} \\ -2+z_0+\overline{z_0} & i(z_0-\overline{z_0}) \end{pmatrix}$$

This matrix is already real.
The determinant has to be nonzero because otherwise the Möbius transformation would not be bijective. One can see, that the determinant has even to be positive in this case, because otherwise it would map $\mathbb{H}$ to $-\mathbb{H}$. Conversely any real matrix with positive determinant describes a Möbius transformation f which is a bijection $\mathbb{H} \to \mathbb{H}$.

6. **Schwarz-Pick theorem:** Let $f : B_1(0) \to B_1(0)$ be a holomorphic function: Prove that for all $z_1, z_2 \in B_1(0)$:

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right|$$

and for all $z \in B_1(0)$:

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Hint: Consider the Möbius transformations $g(z) = \frac{z_1 - z}{1 - \overline{z_1}z}$ *and* $h(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)}z}$. *Apply the Schwarz Lemma to the composition* $h \circ f \circ g^{-1}$. *Why can you apply it?*

We can apply Schwarz Lemma to the composition $h \circ f \circ g^{-1}$ because, by construction, it has the property of mapping 0 to 0 and it is still a map $B_1(0) \to B_1(0)$ because the functions g and h are bijective holomorphic maps $B_1(0) \to B_1(0)$. This was shown in the proof of theorem 5.3.5 in the lecture. Hence we get

$$|h(f(g^{-1}(z)))| \leq |z|.$$

Substituting $g(z_2)$ for z, we get

$$|h(f(z_2))| \leq |g(z_2)|$$

or written out, using the definitions of h and g:

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right|,$$

q.e.d.

Dividing both sides by $|z_1 - z_2|$, in the limit $z_2 \to z_1$ we get

$$\frac{|f'(z_1)|}{1 - |f(z_1)|^2} \leq \frac{1}{1 - |z_1|^2}.$$

Remark: The following distance function on $B_1(0)$*:*

$$d(z_1, z_2) = \tanh^{-1}\left(\frac{|z_1 - z_2|}{|1 - \overline{z_1}z_2|}\right)$$

is called the Poincaré metric and renders $B_1(0)$ *into a model for a hyperbolic geometry in dimension 2. The statement above implies, that holomorphic functions from* $B_1(0)$ *to itself are necessarily contractions w.r.t. this hyperbolic metric.*