1. **Using the residue theorem:** Let \( \varphi(t) = 3 \exp(2\pi it), \ t \in [0,1] \). Compute the following integral using the residue theorem:

(a) \( \int_{\varphi} \frac{\exp(az)}{z^2(z^2+2z+2)} \, dz \),

where \( a \in \mathbb{C} \).

Let \( \varphi(t) = 5 \exp(2\pi it), \ t \in [0,1] \). Compute the following integral using the residue theorem:

(b) \( \int_{\varphi} \frac{\exp(z)}{\cosh(z)} \, dz \).

Let \( \varphi \) be a path describing a rectangle with vertices \( (3+3i, -3 + 3i, -3 - 3i, 3 - 3i) \) in this order. Compute the following integral using the residue theorem:

(c) \( \int_{\varphi} \frac{2+3 \sin(\pi z)}{z(z-1)^2} \, dz \).

2. **Winding number:** Let \( \zeta = \exp(2\pi i/5) \), a 5th root of unity. Let \( \varphi \) be a path describing the polygon with vertices \( 1, \zeta^2, \zeta^4, \zeta, \zeta^3, 1 \) in this order. Compute the following integral using the residue theorem:

3. **Residues and primitives:** Let \( U \) be an elementary domain, \( S \subset U \) a finite set of singularities and \( f : U \setminus S \to \mathbb{C} \) a holomorphic function.

Prove that \( f \) has a primitive on \( U \setminus S \) if and only if \( \text{Res}_s(f) = 0 \) for all \( s \in S \).

**Hint:** Use exercise 4 on assignment 4 and the residue theorem. Remember also to explicitly show the only if direction!

4. **Existence of logarithms of holomorphic functions:** Let \( U \) be an elementary domain, \( S \subset U \) a finite set of singularities and \( f : U \setminus S \to \mathbb{C} \) a holomorphic function with \( f(z) \neq 0 \) for all \( z \in U \setminus S \). Assume no \( s \in S \) is an essential singularity.

Prove: There exists a holomorphic function \( g : U \setminus S \to \mathbb{C} \) with

\[ f(z) = \exp(g(z)) \]

if and only if \( \text{ord}_s(f) = 0 \) for all \( z \in S \) (in particular all \( s \in S \) are removable).

**Hint:** Use exercise 3 applied to \( \frac{f'(z)}{f(z)} \) and the argument principle.

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*Please turn over*
Existence of roots of holomorphic functions: Let $U$ be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function with $f(z) \neq 0$ for all $z \in U \setminus S$. Assume no $s \in S$ is an essential singularity. Let $n$ be a positive integer.

Prove: There exists a holomorphic function $g : U \setminus S \to \mathbb{C}$ with

$$f(z) = g(z)^n$$

if and only if $n|\text{ord}_s(f)$ for all $s \in S$.

Choose some point $z_0 \in U \setminus S$ and a root $(w_0)^n = f(z_0)$. Consider the function $g(z) = \exp\left(\frac{1}{n} \int_{\varphi_z} \frac{f'(z)}{f(z)} \, dz\right)w_0$, where $\varphi_z : [a, b] \to U \setminus S$ is some fixed path with $\varphi_z(a) = z_0$ and $\varphi_z(b) = z$. Use the argument principle to see that $g(z)$ is independent of the choice of path. Then vary $z$ in small discs $D \subset U \setminus S$, such that on $D$ a primitive of $\frac{f'(z)}{f(z)}$ exists.

Please hand in your solutions on Wednesday, November 10, 2010 in the lecture hall.