Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 7

1. **Using the residue theorem:** Let $\varphi(t) = 3\exp(2\pi i t)$, $t \in [0,1]$. Compute the following integral using the residue theorem:

   (a) $\int_\varphi \frac{\exp(az)}{z^2(z^2+2z+2)} \mathrm{d}z.$

   where $a \in \mathbb{C}$.

   Let $\varphi(t) = 5\exp(2\pi i t)$, $t \in [0,1]$. Compute the following integral using the residue theorem:

   (b) $\int_\varphi \frac{\exp(z)}{\cosh(z)} \mathrm{d}z.$

   Let φ be a path describing a rectangle with vertices $(3+3i, -3+3i, -3-3i, 3-3i)$ in this order. Compute the following integral using the residue theorem:

   (c) $\int_\varphi \frac{2+3\sin(\pi z)}{z(z-1)^2} \mathrm{d}z.$

2. **Winding number:** Let $\zeta = \exp(2\pi i/5)$, a 5th root of unity. Let φ be a path describing the polygon with vertices $1, \zeta^2, \zeta^4, \zeta, \zeta^3, 1$ in this order. $\mathbb{C} \setminus \mathrm{image}(\varphi)$ decomposes into several regions with different winding numbers. Draw a picture of the path, indicating these different regions and their winding numbers.

3. **Residues and primitives:** Let U be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function.

   Prove that f has a primitive on $U \setminus S$ if and only if $\mathrm{Res}_s(f) = 0$ for all $s \in S$.

   Hint: Use exercise 4 on assignment 4 and the residue theorem. Remember also to explicitly show the only if direction!

4. **Existence of logarithms of holomorphic functions:** Let U be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function with $f(z) \neq 0$ for all $z \in U \setminus S$. Assume no $s \in S$ is an essential singularity.

   Prove: There exists a holomorphic function $g : U \setminus S \to \mathbb{C}$ with

   $$f(z) = \exp(g(z))$$

   if and only if $\mathrm{ord}_s(f) = 0$ for all $z \in S$ (in particular all $s \in S$ are removable).

   Hint: Use exercise 3 applied to $\frac{f'(z)}{f(z)}$ *and the argument principle.*

— Please turn over —

*5 **Existence of roots of holomorphic functions:** Let U be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function with $f(z) \neq 0$ for all $z \in U \setminus S$. Assume no $s \in S$ is an essential singularity. Let n be a positive integer.

Prove: There exists a holomorphic function $g : U \setminus S \to \mathbb{C}$ with

$$f(z) = g(z)^n$$

if and only if $n | \mathrm{ord}_s(f)$ for all $s \in S$.

Choose some point $z_0 \in U \setminus S$ and a root $(w_0)^n = f(z_0)$. Consider the function $g(z) = \exp(\frac{1}{n}\int_{\varphi_z} \frac{f'(z)}{f(z)} dz)w_0$, where $\varphi_z : [a,b] \to U \setminus S$ is some fixed path with $\varphi_z(a) = z_0$ and $\varphi_z(b) = z$. Use the argument principle to see that $g(z)$ is independent of the choice of path. Then vary z in small discs $D \subset U \setminus S$, such that on D a primitive of $\frac{f'(z)}{f(z)}$ exists.

Please hand in your solutions on Wednesday, November 10, 2010 in the lecture hall