Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 7

1. **Using the residue theorem (3 points):** Let $\varphi(t) = 3\exp(2\pi i t)$, $t \in [0,1]$. Compute the following integral using the residue theorem:

   (a) $\int_\varphi \frac{\exp(az)}{z^2(z^2+2z+2)} \mathrm{d}z$.

   where $a \in \mathbb{C}$.

   Let $\varphi(t) = 5\exp(2\pi i t)$, $t \in [0,1]$. Compute the following integral using the residue theorem:

   (b) $\int_\varphi \frac{\exp(z)}{\cosh(z)} \mathrm{d}z$.

   Let φ be a path describing a rectangle with vertices $(3+3i, -3+3i, -3-3i, 3-3i)$ in this order. Compute the following integral using the residue theorem:

   (c) $\int_\varphi \frac{2+3\sin(\pi z)}{z(z-1)^2} \mathrm{d}z$.

(a) We can write (completing the square):

$$f(z) := \frac{\exp(az)}{z^2(z^2+2z+2)} = \frac{\exp(az)}{z^2(z+i+1)(z-i+1)}.$$

This implies that the function f has singularities at 0, $-i-1$ and $i-1$. By the residue theorem, we get

$$\int_\varphi f(z)\mathrm{d}z = 2\pi i(\mathrm{Res}_0(f) + \mathrm{Res}_{-i-1}(f) + \mathrm{Res}_{i-1}(f)),$$

because the winding number is 1 for all of the singularities (they all lie in the interior of the circle of integration).
Let us compute the residue at 0. We can write $f(z) = \frac{g(z)}{z^2}$, where $g(z) = \frac{\exp(az)}{z^2+2z+2}$ is holomorphic at 0. We get $\mathrm{Res}_0(f) = \frac{g'(0)}{1!} = g'(0)$. We have

$$g'(z) = \frac{a\exp(az)(z^2+2z+2) - \exp(az)(2z+2)}{(z^2+2z+2)^2}.$$

hence

$$g'(0) = \frac{2a-2}{4} = \frac{1}{2}(a-1)$$

and

$$\mathrm{Res}_0(f) = \frac{1}{2}(a-1).$$

Let us compute the residue at $-i-1$. We can write $f(z) = \frac{g(z)}{z+i+1}$, where $g(z) = \frac{\exp(az)}{z^2(z-i+1)}$ is holomorphic at $-i-1$. We get

$$\mathrm{Res}_{-i-1}(f) = \frac{g(0)}{0!} = g(0) = \frac{\exp((-1-i)a)}{(-1-i)^2(-2i)} = \frac{\exp((-1-i)a)}{4}.$$

Similarly

$$\mathrm{Res}_{i-1}(f) = \frac{\exp((-1+i)a)}{(-1+i)^2(2i)} = \frac{\exp((-1+i)a)}{4}.$$

Putting everything together and using the formula for $\cos(z)$, we get

$$\int_\varphi f(z)\mathrm{d}z = 2\pi i \left(\frac{a-1}{2} + \frac{1}{2}\exp(-a)\cos(a)\right).$$

(b) The function $\cosh(z) = \cos(iz)$ has a zero, if and only if iz is of the form $\frac{1}{2}\pi + k\pi$ with $k \in \mathbb{Z}$. All these zeros are simple because $\sin(iz) = \pm 1$ at these points. In the circle of radius 5, there are therefore 4 singularities, namely at the points $\pm\frac{1}{2}\pi i$ and $\pm\frac{3}{2}\pi i$. Let s be one of these zeros. We can expand $\cosh(z) = (z-s)(a_1 + a_2(z-s) + ...)$. This shows: $\mathrm{Res}_s(\frac{1}{\cosh(z)}) = a_1^{-1}$ (property ii of the residue). But observe that $a_1 = \cosh'(s) = \sinh(s)$.
Applying property ii again, we get $\mathrm{Res}_s(\frac{\exp(z)}{\cosh(z)}) = \frac{\exp(s)}{\sinh(s)}$. Inserting the 4 values in this expression, we see that the residue is always equal to 1. Hence

$$\int_\varphi f(z)\mathrm{d}z = 2\pi i 4 = 8\pi i.$$

(c) The function

$$f(z) := \frac{2+3\sin(\pi z)}{z(z-1)^2}$$

has singularities at 0 and 1. Both lies inside the rectangle around which we integrate and have winding number 1. Hence applying the residue theorem, we get:

$$\int_\varphi f(z)\mathrm{d}z = 2\pi i(\mathrm{Res}_0(\frac{2+3\sin(\pi z)}{z(z-1)^2}) + \mathrm{Res}_1(\frac{2+3\sin(\pi z)}{z(z-1)^2})).$$

We have (property iii of residues):

$$\mathrm{Res}_0(\frac{2+3\sin(\pi z)}{z(z-1)^2}) = \left.\frac{2+3\sin(\pi z)}{(z-1)^2}\right|_{z=0} = 2$$

and

$$\mathrm{Res}_1(\frac{2+3\sin(\pi z)}{z(z-1)^2}) = \left(\frac{2+3\sin(\pi z)}{z}\right)'_{z=1} = -2-3\pi.$$

Everything put together:

$$\int_\varphi \frac{2+3\sin(\pi z)}{z(z-1)^2}\mathrm{d}z = -6\pi^2 i.$$

2. **Winding number:** Let $\zeta = \exp(2\pi i/5)$, a 5th root of unity. Let φ be a path describing the polygon with vertices $1, \zeta^2, \zeta^4, \zeta, \zeta^3, 1$ in this order. $\mathbb{C} \setminus \mathrm{image}(\varphi)$ decomposes into several regions with different winding numbers. Draw a picture of the path, indicating these different regions and their winding numbers.

Yields a pentagram, where the points outside have winding number 0, the points in the middle pentagon have winding number 2, and all other points (those in the little triangles) have winding number 1.

3. **Residues and primitives:** Let U be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function.

   Prove that f has a primitive on $U \setminus S$ if and only if $\mathrm{Res}_s(f) = 0$ for all $s \in S$.

   Hint: Use exercise 4 on assignment 4 and the residue theorem. Remember also to explicitly show the only if direction!

Exercise 4 on assignment 4 states that on any open connected subset (domain) V of $\mathbb{C}$ the following are equivalent:

- f has a primitive on V.
- For any *closed* path $\varphi : [a, b] \to V$, we have $\int_\varphi f(z)\mathrm{d}z = 0$.

Applying this to $V := U \setminus S$, we are left to show the following two statements:

$\Rightarrow$ If $\mathrm{Res}_s(f) = 0$ for all $s \in S$ then for any *closed* path $\varphi : [a, b] \to U \setminus S$, we have $\int_\varphi f(z)\mathrm{d}z = 0$.

$\Leftarrow$ If for any *closed* path $\varphi : [a, b] \to U \setminus S$, we have $\int_\varphi f(z)\mathrm{d}z = 0$ then $\mathrm{Res}_s(f) = 0$ for all $s \in S$.

Proof of $\Rightarrow$:

$$\int_\varphi f(z)\mathrm{d}z = 2\pi i \sum_{s\in S} N(\varphi, s)\mathrm{Res}_s(f),$$

by the residue theorem. If all residues are zero, this expression is 0.
Proof of $\Leftarrow$: Assume that $\mathrm{Res}_s(f) \neq 0$ for one $s \in S$. Then consider the path $\varphi(t) = s + \varepsilon \exp(2\pi i t)$, $t \in [0, 1]$ (a small circle around s). If ε is small enough, because S is finite, we get $\int_\varphi f(z)\mathrm{d}z = 2\pi i \mathrm{Res}_s(f)$ for this φ, which is not zero. A contradiction.

4. **Existence of logarithms of holomorphic functions:** Let U be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function with $f(z) \neq 0$ for all $z \in U \setminus S$. Assume no $s \in S$ is an essential singularity.

   Prove: There exists a holomorphic function $g : U \setminus S \to \mathbb{C}$ with

   $$f(z) = \exp(g(z))$$

   if and only if $\mathrm{ord}_s(f) = 0$ for all $z \in S$ (in particular all $s \in S$ are removable).

   Hint: Use exercise 3 applied to $\frac{f'(z)}{f(z)}$ *and the argument principle.*

The construction of a function $g(z)$ with $f(z) = \exp(g(z))$ is equivalent to the task of finding a primitive for $\frac{f'(z)}{f(z)}$. This is seen as follows: Let G be a primitive for $\frac{f'(z)}{f(z)}$. We have then

$$\left(\frac{\exp(G(z))}{f(z)}\right)' = \frac{\exp(G(z))G'(z)f(z) - \exp(G(z))f'(z)}{f(z)^2} = 0$$

using $G'(z) = \frac{f'(z)}{f(z)}$. Hence $\exp(G(z)) = \alpha f(z)$ for an $\alpha \in \mathbb{C}^*$. Choosing a β with $\exp(\beta) = \alpha$, we see that the function $g(z) = G(z) - \beta$ is indeed a logarithm of f because $\exp(G(z) - \beta) = \exp(G(z))/\exp(\beta) = \alpha f(z)/\alpha = f(z)$.
Conversely, if $f(z) = \exp(g(z))$, the same calculation shows, that $g'(z) = \frac{f'(z)}{f(z)}$.
Now by exercise 3, a primitive of $\frac{f'(z)}{f(z)}$ exists, if and only if all residues of $\frac{f'(z)}{f(z)}$ vanish. But we have (property iv of residues or argument principle):

$$\mathrm{Res}_s(\frac{f'(z)}{f(z)}) = \mathrm{ord}_s(f).$$

Therefore a primitive of $\frac{f'}{f}$, and accordingly a logarithm of f exists, if and only if $\mathrm{ord}_s(f) = 0$ for all $s \in S$.

*5. **Existence of roots of holomorphic functions:** Let U be an elementary domain, $S \subset U$ a finite set of singularities and $f : U \setminus S \to \mathbb{C}$ a holomorphic function with $f(z) \neq 0$ for all $z \in U \setminus S$. Assume no $s \in S$ is an essential singularity. Let n be a positive integer.

Prove: There exists a holomorphic function $g : U \setminus S \to \mathbb{C}$ with

$$f(z) = g(z)^n$$

if and only if $n|\mathrm{ord}_s(f)$ for all $s \in S$.

Choose some point $z_0 \in U \setminus S$ *and a root* $(w_0)^n = f(z_0)$. *Consider the function* $g(z) = \exp(\frac{1}{n}\int_{\varphi_z} \frac{f'(z)}{f(z)} dz)w_0$, *where* $\varphi_z : [a,b] \to U \setminus S$ *is some fixed path with* $\varphi_z(a) = z_0$ *and* $\varphi_z(b) = z$. *Use the argument principle to see that* $g(z)$ *is independent of the choice of path. Then vary* z *in small discs* $D \subset U \setminus S$, *such that on* D *a primitive of* $\frac{f'(z)}{f(z)}$ *exists.*

We show first that a root function exists, if $n|\mathrm{ord}_s(f)$ for all $s \in S$. Consider the function

$$g(z) := \exp(\frac{1}{n}\int_{\varphi_z} \frac{f'(z)}{f(z)}\mathrm{d}z)w_0$$

as given in the hint. Why is it independent of the path chosen?
Let us define

$$\widetilde{g}(z) := \exp(\frac{1}{n}\int_{\widetilde{\varphi}_z} \frac{f'(z)}{f(z)}\mathrm{d}z)w_0$$

using a different path $\widetilde{\varphi}_z$. We get

$$\frac{g(z)}{\widetilde{g}(z)} := \exp(\frac{1}{n}(\int_{\varphi_z} \frac{f'(z)}{f(z)}\mathrm{d}z - \int_{\widetilde{\varphi}_z} \frac{f'(z)}{f(z)}\mathrm{d}z))$$

We have however

$$\int_{\varphi_z} \frac{f'(z)}{f(z)}\mathrm{d}z - \int_{\widetilde{\varphi}_z} \frac{f'(z)}{f(z)}\mathrm{d}z = \int_{\varphi} \frac{f'(z)}{f(z)}\mathrm{d}z$$

where φ is a closed path. (It is the path following φ_z and then $\widetilde{\varphi}_z$ back in the opposite direction.) By the argument principle, we get:

$$\int_{\varphi} \frac{f'(z)}{f(z)}\mathrm{d}z = 2\pi i \sum_{s \in S} N(\varphi, s)\mathrm{ord}_s(f)$$

Since $N(\varphi, s)$ is an integer and $\mathrm{ord}_s(f)$ is divisible by n by assumption, this is an *integral* multiple of $2\pi i n$. Therefore

$$\frac{g(z)}{\widetilde{g}(z)} = \exp(\frac{1}{n}\int_{\varphi} \frac{f'(z)}{f(z)}\mathrm{d}z) = 1.$$

$g(z)$ is therefore independent of the path.
Now we have to see that indeed $f(z) = g(z)^n$. Let us look at a small disc $B_\varepsilon(z_1) \subset U \setminus S$ around a point z_1 and take a point $z \in B_\varepsilon(z_1)$. Since $f(z)$ is non-zero everywhere on the disc, we get a $F(z)$ with $f(z) = \exp(F(z))$ (last exercise) and know:

$$F'(z) = \frac{f'(z)}{f(z)}.$$

Therefore writing φ_z as a composition of a path φ_{z_1} and a linear path from z_1 to z, which lies entirely inside $B_\varepsilon(z_1)$, we get (independence of path!):

$$g(z)^n = \exp(\frac{1}{n}(\int_{\varphi_{z_1}} \frac{f'(z)}{f(z)}\mathrm{d}z + F(z) - F(z_1)))^n w_0^n$$

$$= \exp(\int_{\varphi_{z_1}} \frac{f'(z)}{f(z)}\mathrm{d}z - F(z_1)) \exp(F(z)) f(z_0) = \exp(\int_{\varphi_{z_1}} \frac{f'(z)}{f(z)}\mathrm{d}z - F(z_1)) f(z) f(z_0)$$

This shows that $\frac{f(z)}{g(z)^n}$ is locally constant on $U \setminus S$, hence constant because U and also $U \setminus S$ are connected. But obviously $f(z_0) = g(z_0)^n$, hence $f(z) = g(z)^n$ everywhere.
Conversely assume that $f(z) = g(z)^n$. It is easy to see that g cannot have essential singularities because otherwise also f would have. At each pole or zero s of g, g has an expansion:

$$(z - s)^k (a_k + a_{k+1}(z - s) + \dots)$$

with $a_k \neq 0$. Therefore f has an expansion

$$(z - s)^{kn}((a_k)^n + b_{kn+1}(z - s) + \dots),$$

that is, $\mathrm{ord}_s(f)$ is divisible by n.