Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 8

1. **Real integrals I:** Calculate with the aid of the residue theorem:

   (a) $\int_0^\pi \frac{\mathrm{d}\theta}{a+\cos(\theta)}$, $a \in \mathbb{R}, a > 1$,

   (b) $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{a+\sin^2(\theta)}$, $a \in \mathbb{R}, a > 1$,

   (c) $\int_0^{2\pi} \frac{\cos(3\theta)\mathrm{d}\theta}{5-4\cos(\theta)}$.

   Hint: Bring the integrals first into a form involving the range $[0, 2\pi]$*, by exploiting a suitable periodicity of* $\sin, \cos, \sin^2$*, ... Then use the reduction to a path integral and to the residue theorem as given in the lecture. For (c) do not try to find the function* R *explicitly, but mimic the transformation given in the lecture directly.*

2. **Real integrals II:** Calculate with the aid of the residue theorem:

   (a) $\int_0^\infty \frac{x^2}{x^4+5x^2+6}\mathrm{d}x$,

   (b) $\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\mathrm{d}x$, $a \in \mathbb{R}, a > 0$,

   (c) $\int_0^\infty \frac{\cos(x)}{x^2+a^2}\mathrm{d}x$, $a \in \mathbb{R}, a > 0$.

   Hint: Bring the integrals first into a form involving the range $(-\infty, \infty)$ *using parity of the function involved. Then use the theorems provided in the lecture. Be careful with (c). First write* $\cos(x) = \frac{\exp(ix)+\exp(-ix)}{2}$*. Then make a change of variables* $x \mapsto -x$ *for the summand involving* $\exp(-ix)$ *to bring it in the form of theorem 6.6.2.*

3. **Using Rouché's theorem I:** How many zeros (counted with multiplicity) has

   $$g(z) = z^7 - 2z^5 + 6z^3 - z + 1$$

   in $B_1(0)$?

   Hint: Choose a suitable among the monomials z^7*,* $-2z^5$*,* $6z^3$*,* $-z$*, resp.* 1 *as the function* f *in Rouché's theorem.*

4. **A fixed point:** Let h be a holomorphic function on $B_{1+\varepsilon}(0)$ and assume $|h(z)| < |z|$ for all z with $|z| = 1$. Show that there is exactly one $z \in B_1(0)$ with $h(z) = z$.

5. **Using Rouché's theorem II:** How many zeros (counted with multiplicity) has

   $$g(z) = z^4 - 6z + 3$$

   on the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$?

   Hint: First show, that g *has no zeros on the circles of radius 1 and 2 respectively. Then apply Rouché's theorem twice.*

Please hand in your solutions on Friday, November 19, 2010 in the lecture hall