1. **Real integrals I**: Calculate with the aid of the residue theorem:

(a) \( \int_0^\pi \frac{d\theta}{a + \cos(\theta)} \), \( a \in \mathbb{R}, a > 1 \),

(b) \( \int_0^{\pi/2} \frac{d\theta}{a + \sin^2(\theta)} \), \( a \in \mathbb{R}, a > 1 \),

(c) \( \int_0^{2\pi} \frac{\cos(3\theta)d\theta}{5-4\cos(\theta)} \)

**Hint**: Bring the integrals first into a form involving the range \([0, 2\pi]\), by exploiting a suitable periodicity of \(\sin, \cos, \sin^2, \ldots\) Then use the reduction to a path integral and to the residue theorem as given in the lecture. For (c) do not try to find the function \(R\) explicitly, but mimic the transformation given in the lecture directly.

(a) We have \( \cos(x) = \cos(2\pi - x) \), hence

\[
\int_0^\pi \frac{d\theta}{a + \cos(\theta)} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)}.
\]

The second integral may be computed as explained in the lecture by finding a function \(R(z_1, z_2)\) such that

\[
R(\cos(\theta), \sin(\theta)) = \frac{1}{a + \cos(\theta)}.
\]

Such a function is obviously provided by \(R(z_1, z_2) = \frac{1}{a + z_1}\). Then it was shown in the lecture that

\[
\int_0^{2\pi} R(\cos(\theta), \sin(\theta))d\theta = 2\pi \sum_{z \in B_1(0)} \text{Res}_z(f),
\]

where

\[
f(z) = z^{-1}R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right).
\]

Here, we get

\[
f(z) = z^{-1} \frac{1}{a + \frac{z + z^{-1}}{2}} = \frac{2}{z^2 + 2az + 1} = \frac{2}{(z + a + \sqrt{a^2 - 1})(z + a - \sqrt{a^2 - 1})}.
\]

Since \(a > 1\), only the point \(z = -a + \sqrt{a^2 - 1}\) lies in the unit disc. It is a **simple** pole. For this point, we get

\[
\text{Res}_z\left(\frac{2}{(z + a + \sqrt{a^2 - 1})(z + a - \sqrt{a^2 - 1})}\right) = \frac{2}{-a + \sqrt{a^2 - 1} + a + \sqrt{a^2 - 1}} = \frac{1}{\sqrt{a^2 - 1}}.
\]

Putting everything together, we get

\[
\int_0^\pi \frac{d\theta}{a + \cos(\theta)} = \frac{\pi}{\sqrt{a^2 - 1}}.
\]

(b) Since \(\sin^2\) is periodic with period \(\pi\) and \(\sin^2(\pi - x) = \sin^2(\pi - x)\), we get

\[
\int_0^{\pi/2} \frac{d\theta}{a + \sin^2(\theta)} = \frac{1}{4} \int_0^{2\pi} \frac{d\theta}{a + \sin^2(\theta)}
\]
A function $R$ is given by

$$R(z_1, z_2) = \frac{1}{a + z_1^2}$$

and we get

$$f(z) = z^{-1} \frac{1}{a + (\frac{z-z_1}{2a})^2} = \frac{-4z}{z^4 - (4a + 2)z^2 + 1} = \frac{-4z}{(z^2 - 2a - 1 + 2\sqrt{a^2 + a})(z^2 - 2a - 1 - 2\sqrt{a^2 + a})}.$$  

Again, only $2a + 1 - 2\sqrt{a^2 + a}$ lies in $B_1(0)$, hence $z_1 = \sqrt{2a + 1 - 2\sqrt{a^2 + a}}$ and $z_2 = -\sqrt{2a + 1 - 2\sqrt{a^2 + a}}$ are the singularities of our function in $B_1(0)$. They are simple poles. Therefore, we get, letting $p(z) = z^4 - (4a + 2)z^2 + 1$.

$$\int_0^{2\pi} \frac{d\theta}{a + \sin^2(\theta)} = 2\pi \text{Res}_{z_1} \left(\frac{-4z}{p(z)}\right) + 2\pi \text{Res}_{z_2} \left(\frac{-4z}{p(z)}\right) = 2\pi \frac{-4z_1}{p'(z_1)} + 2\pi \frac{-4z_2}{p'(z_2)}.$$  

We have $\frac{-4z_1}{p'(z_1)} = \frac{z_1 - 1}{z_1 - 2a - 1}$. Therefore this expression is equal to:

$$2\pi \left(\frac{1}{2\sqrt{a^2 + a}} + \frac{1}{2\sqrt{a^2 + a}}\right).$$

Putting everything together, we get

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{a + \sin^2(\theta)} = \frac{\pi}{2\sqrt{a^2 + a}}.$$  

(c) In this case, we could find a function $R$ as before because $\cos(3\theta)$ may be expressed as a polynomial in $\sin(\theta)$ and $\cos(\theta)$. This is a little bit cumbersome, however. Its better to mimic the transformation given in the lecture.

$$\int_0^{2\pi} \frac{\cos(3\theta)d\theta}{5 - 4\cos(\theta)} = \int_0^{2\pi} \frac{e^{3i\theta} + e^{-3i\theta}}{5 - 4e^{i\theta} + 4e^{-i\theta}}d\theta = \frac{1}{i} \int_0^{2\pi} \frac{(e^{i\theta} + 1) + \frac{1}{10 - 4e^{i\theta} - 4e^{-i\theta}}d\theta}{10e^{i\theta} - 4e^{3i\theta} - 4e^{3i\theta}i e^{i\theta}}.$$  

And introducing the path $\varphi(t) = e^{it}$, where $t \in [0, 2\pi]$, we have $\varphi'(t) = i \varphi(t)$, and the integral may be written as the path-integral:

$$= \frac{1}{i} \int_\varphi \frac{z^6 + 1}{10z^4 - 4z^2 + 4z^4}dz = -\frac{1}{2i} \int_\varphi \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}dz.$$  

Only the singularities 0 and $\frac{1}{2}$ do lie in the unit disc, hence we get:

$$= -\pi \text{Res}_0 \left(\frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\right) - \pi \text{Res}_{\frac{1}{2}} \left(\frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\right)$$

$$= -\pi \left(\frac{z^6 + 1}{(2z - 1)(z - 2)}\right)|_{z=0} = -\pi \frac{(\frac{1}{2})^6 + 1}{2! (\frac{1}{2})^3 (\frac{1}{2} - 2)} = -\pi \frac{21}{8} + \pi \frac{65}{24} = \frac{\pi}{12}.$$
2. Real integrals II: Calculate with the aid of the residue theorem:

(a) \( \int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx \),

(b) \( \int_0^\infty \frac{x^2}{(x^2 + a^2)^2} \, dx, \ a \in \mathbb{R}, \ a > 0 \),

(c) \( \int_0^\infty \frac{\cos(x)}{x^2 + a^2} \, dx, \ a \in \mathbb{R}, \ a > 0 \).

Hint: Bring the integrals first into a form involving the range \((-\infty, \infty)\) using parity of the function involved. Then use the theorems provided in the lecture. Be careful with (c). First write \( \cos(x) = \frac{\exp(ix) + \exp(-ix)}{2} \). Then make a change of variables \( x \mapsto -x \) for the summand involving \( \exp(-ix) \) to bring it in the form of theorem 6.6.2.

(a) Since the integrand is an even function of \( x \), we get

\[
\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx.
\]

Since the degree of \( x^4 + 5x^2 + 6 \) exceeds the degree of \( x^2 \) by two, we may apply the residue theorem to calculate this integral. The singularities of the integrand lie at the zeros of \( p(z) = z^4 + 5z^2 + 6 = (z^2 + 2)(z^2 + 3) \). Only the ones in the upper half plane are counted, namely \( \sqrt{2}i \) and \( \sqrt{3}i \). These are simple poles, hence we get:

\[
2\pi i \text{Res}_{\sqrt{2}i} \left( \frac{z^2}{p(z)} \right) + 2\pi i \text{Res}_{\sqrt{3}i} \left( \frac{z^2}{p(z)} \right) = 2\pi i \left( \frac{2i}{p'(\sqrt{2}i)} + \frac{3i}{p'(\sqrt{3}i)} \right) = 2\pi \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \right).
\]

using \( p'(z) = 4z^3 + 10z \). Putting everything together, we get:

\[
\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx = \pi \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \right).
\]

(b) Since the integrand is an even function of \( x \), we get

\[
\int_0^\infty \frac{x^2}{(x^2 + a^2)^3} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{(x^2 + a^2)^3} \, dx.
\]

Since the degree of \( (x^2 + a^2)^3 \) exceeds the degree of \( x^2 \) by more than two, we may apply the residue theorem to calculate this integral. The singularities of the integrand are at \( \pm ia \), only \( ia \) in the upper half plane is counted. Hence

\[
\int_{-\infty}^\infty \frac{x^2}{(x^2 + a^2)^3} \, dx = 2\pi i \text{Res}_{ia} \left( \frac{x^2}{(x - ia)^3(x + ai)^3} \right) = 2\pi i \left( \frac{\frac{x^2}{(x + ai)^3}'|_{x=ai}}{2!} \right) = \frac{\pi}{8a^3},
\]

Putting everything together, we get:

\[
\int_0^\infty \frac{x^2}{(x^2 + a^2)^3} \, dx = \frac{\pi}{16a^3}.
\]

(c) Since the integrand is an even function of \( x \), we get

\[
\int_0^\infty \frac{\cos(x)}{x^2 + a^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos(x)}{x^2 + a^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix} + e^{-ix}}{x^2 + a^2} \, dx = \frac{1}{4} \int_{-\infty}^\infty \frac{e^{ix} + e^{-ix}}{x^2 + a^2} \, dx = \frac{1}{4} \left( \int_{-\infty}^\infty \frac{e^{ix}}{x^2 + a^2} \, dx + \int_{-\infty}^\infty \frac{e^{-ix}}{x^2 + a^2} \, dx \right)
\]
We transform the second integral by $x \mapsto -x$ and get:
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx + \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx
\]
This integral is of the form considered in the lecture because the degree of $x^2 + a^2$ is bigger than 1. Hence we can apply the residue theorem. The singularities are at $z = \pm ai$, but only $ia$ in the upper half plane is to be considered. Hence we get
\[
\pi i \text{Res}_{ia}(\frac{e^{ix}}{(x - ai)(x + ai)}) = \pi i \frac{e^{-a}}{2ai} = \frac{\pi}{2ae^a}.
\]

3. Using Rouché’s theorem I: How many zeros (counted with multiplicity) has
\[
g(z) = z^7 - 2z^5 + 6z^3 - z + 1
\]
in $B_1(0)$?

*Hint: Choose a suitable among the monomials $z^7$, $-2z^5$, $6z^3$, $-z$, resp. 1 as the function $f$ in Rouché’s theorem.*

We choose $f(z) = 6z^3$ and get the following estimate for $z \in \mathbb{C}$ with $|z| = 1$:
\[
|g(z) - f(z)| = |z^7 - 2z^5 - z + 1| \leq |z^7| + |z| - 2|z|^5 + |z| + 1 = 1 + 2 + 1 + 1 = 5 < |6z^2| = |f(z)|.
\]

Hence $g(z)$ is not zero for $z$ with $|z| = 1$ and Rouché’s theorem tells us, that $f$ and $g$ have the same number of zeros (counted with multiplicity) in $B_1(0)$. $f$, however, has one zero at $z = 0$ with multiplicity 3. Therefore also $g$ has 3 zeros (counted with multiplicity) in $B_1(0)$.

4. A fixed point: Let $h$ be a holomorphic function on $B_{1+\epsilon}(0)$ and assume $|h(z)| < |z|$ for all $z$ with $|z| = 1$. Show that there is exactly one $z \in B_1(0)$ with $h(z) = z$.

A fixed point $h(z) = z$ is a zero of the function $g(z) := h(z) - z$. Taking $f(z) = -z$ we get the estimate
\[
|g(z) - f(z)| = |h(z)| < |z| = |f(z)|.
\]

Hence $g(z)$ is not zero for $z$ with $|z| = 1$ and has precisely 1 zero in $B_1(0)$ because $f$ obviously has one. This is the required fixed point of $h$.

5. Using Rouché’s theorem II: How many zeros (counted with multiplicity) has
\[
g(z) = z^4 - 6z + 3
\]
on the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$?

*Hint: First show, that $g$ has no zeros on the circles of radius 1 and 2 respectively. Then apply Rouché’s theorem twice.*

First let $f(z) = z^4$. For $z \in \mathbb{C}$ with $|z| = 2$, we get the estimate
\[
|g(z) - f(z)| = |-6z + 3| \leq |6z| + |3| = 15 < 16 = 2^4 = |f(z)|
\]
Therefore $g(z)$ has no zero in the circle with radius 2 around 0 and precisely 4 zeros (counted with multiplicity) in $B_2(0)$. 

Now let $f(z) := -6z$. For $z \in \mathbb{C}$ with $|z| = 1$, we get the estimate

$$|g(z) - f(z)| = |z^4 + 3| \leq |z^4| + |3| = 4 < 6 = | - 6z| = |f(z)|$$

Therefore $g(z)$ has no zero in the circle with radius 1 around 0 either and precisely 1 simple zero in $B_1(0)$.

The other 3 zeros (counted with multiplicity), therefore, must lie in the open annulus

$$\{z \in \mathbb{C} | 1 < |z| < 2\}.$$