Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 8

1. **Real integrals I:** Calculate with the aid of the residue theorem:

   (a) $\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos(\theta)}$, $a \in \mathbb{R}, a > 1$,

   (b) $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{a+\sin^2(\theta)}$, $a \in \mathbb{R}, a > 1$,

   (c) $\int_0^{2\pi} \frac{\cos(3\theta)\mathrm{d}\theta}{5-4\cos(\theta)}$.

   Hint: Bring the integrals first into a form involving the range $[0, 2\pi]$*, by exploiting a suitable periodicity of* $\sin, \cos, \sin^2$*, ... Then use the reduction to a path integral and to the residue theorem as given in the lecture. For (c) do not try to find the function* R *explicitly, but mimic the transformation given in the lecture directly.*

(a) We have $\cos(x) = \cos(2\pi - x)$, hence

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos(\theta)} = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos(\theta)}.$$

The second integral may be computed as explained in the lecture by finding a function $R(z_1, z_2)$ such that

$$R(\cos(\theta), \sin(\theta)) = \frac{1}{a+\cos(\theta)}.$$

Such a function is obviously provided by $R(z_1, z_2) = \frac{1}{a+z_1}$. Then it was shown in the lecture that

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\mathrm{d}\theta = 2\pi \sum_{z \in B_1(0)} \mathrm{Res}_z(f),$$

where

$$f(z) = z^{-1} R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}).$$

Here, we get

$$f(z) = z^{-1}\frac{1}{a+\frac{z+z^{-1}}{2}} = \frac{2}{z^2+2az+1} = \frac{2}{(z+a+\sqrt{a^2-1})(z+a-\sqrt{a^2-1})}.$$

Since $a > 1$, only the point $z = -a + \sqrt{a^2-1}$ lies in the unit disc. It is a **simple** pole. For this point, we get

$$\mathrm{Res}_z(\frac{2}{(z+a+\sqrt{a^2-1})(z+a-\sqrt{a^2-1})}) = \frac{2}{-a+\sqrt{a^2-1}+a+\sqrt{a^2-1}} = \frac{1}{\sqrt{a^2-1}}.$$

Putting everything together, we get

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos(\theta)} = \frac{\pi}{\sqrt{a^2-1}}.$$

(b) Since $\sin^2$ is periodic with period π and $\sin^2(x) = \sin^2(\pi - x)$, we get

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{a+\sin^2(\theta)} = \frac{1}{4}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\sin^2(\theta)}$$

A function R is given by

$$R(z_1, z_2) = \frac{1}{a + z_1^2}$$

and we get

$$f(z) = z^{-1}\frac{1}{a + (\frac{z-z^{-1}}{2i})^2} = \frac{-4z}{z^4 - (4a+2)z^2 + 1} = \frac{-4z}{(z^2 - 2a - 1 + 2\sqrt{a^2+a})(z^2 - 2a - 1 - 2\sqrt{a^2+a})}.$$

Again, only $2a+1-2\sqrt{a^2+a}$ lies in $B_1(0)$, hence $z_1 = \sqrt{2a+1-2\sqrt{a^2+a}}$ and $z_2 = -\sqrt{2a+1-2\sqrt{a^2+a}}$ are the singularities of our function in $B_1(0)$. They are **simple** poles. Therefore, we get, letting $p(z) = z^4 - (4a+2)z^2 + 1$.

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \sin^2(\theta)} &= 2\pi \mathrm{Res}_{z_1}(\frac{-4z}{p(z)}) + 2\pi \mathrm{Res}_{z_2}(\frac{-4z}{p(z)}) \\ &= 2\pi\frac{-4z_1}{p'(z_1)} + 2\pi\frac{-4z_2}{p'(z_2)}.\end{aligned}$$

We have $\frac{-4z}{p'(z)} = \frac{-1}{z^2-2a-1}$. Therefore this expression is equal to:

$$2\pi\frac{1}{2\sqrt{a^2+a}} + 2\pi\frac{1}{2\sqrt{a^2+a}}.$$

Putting everything together, we get

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{a + \sin^2(\theta)} = \frac{\pi}{2\sqrt{a^2+a}}.$$

(c) In this case, we could find a function R as before because $\cos(3\theta)$ may be expressed as a polynomial in $\sin(\theta)$ and $\cos(\theta)$. This is a little bit cumbersome, however. Its better to mimic the transformation given in the lecture.

$$\int_0^{2\pi} \frac{\cos(3\theta)\mathrm{d}\theta}{5 - 4\cos(\theta)} = \int_0^{2\pi} \frac{\frac{e^{3i\theta}+e^{-3i\theta}}{2}}{5 - 4\frac{e^{i\theta}+e^{-i\theta}}{2}}\mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{e^{3i\theta} + e^{-3i\theta}}{10 - 4e^{i\theta} - 4e^{-i\theta}}\mathrm{d}\theta = \frac{1}{i}\int_0^{2\pi} \frac{(e^{6i\theta}+1)}{10e^{4i\theta} - 4e^{5i\theta} - 4e^{3i\theta}} ie^{i\theta}\mathrm{d}\theta.$$

And introducing the path $\varphi(t) = e^{it}$, where $t \in [0, 2\pi]$, we have $\varphi'(t) = i\varphi(t)$, and the integral may be written as the path-integral:

$$= \frac{1}{i}\int_\varphi \frac{z^6+1}{10z^4 - 4z^5 + 4z^3}\mathrm{d}z = -\frac{1}{2i}\int_\varphi \frac{z^6+1}{z^3(2z-1)(z-2)}\mathrm{d}z.$$

Only the singularitities 0 and $\frac{1}{2}$ do lie in the unit disc, hence we get:

$$\begin{aligned}&= -\pi \mathrm{Res}_0(\frac{z^6+1}{z^3(2z-1)(z-2)}) - \pi \mathrm{Res}_{\frac{1}{2}}(\frac{z^6+1}{z^3(2z-1)(z-2)}) \\ &= -\pi\frac{(\frac{z^6+1}{(2z-1)(z-2)})''|_{z=0}}{2!} - \pi\frac{(\frac{1}{2})^6+1}{2(\frac{1}{2})^3(\frac{1}{2}-2)} \\ &= -\pi\frac{21}{8} + \pi\frac{65}{24} = \frac{\pi}{12}.\end{aligned}$$

2. **Real integrals II:** Calculate with the aid of the residue theorem:

   (a) $\int_0^\infty \frac{x^2}{x^4+5x^2+6}\mathrm{d}x$,

   (b) $\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\mathrm{d}x$, $a \in \mathbb{R}, a > 0$,

   (c) $\int_0^\infty \frac{\cos(x)}{x^2+a^2}\mathrm{d}x$, $a \in \mathbb{R}, a > 0$.

   Hint: Bring the integrals first into a form involving the range $(-\infty, \infty)$ using parity of the function involved. Then use the theorems provided in the lecture. Be careful with (c). First write $\cos(x) = \frac{\exp(ix)+\exp(-ix)}{2}$. Then make a change of variables $x \mapsto -x$ for the summand involving $\exp(-ix)$ to bring it in the form of theorem 6.6.2.

(a) Since the integrand is an even function of x, we get

$$\int_0^\infty \frac{x^2}{x^4+5x^2+6}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+5x^2+6}\mathrm{d}x.$$

Since the degree of x^4+5x^2+6 exceeds the degree of x^2 by two, we may apply the residue theorem to calculate this integral. The singularities of the integrand lie at the zeros of $p(z) = z^4+5z^2+6 = (z^2+2)(z^2+3)$. Only the ones in the upper half plane are counted, namely $\sqrt{2}i$ and $\sqrt{3}i$. These are simple poles, hence we get:

$$= 2\pi i \mathrm{Res}_{\sqrt{2}i}(\frac{z^2}{p(z)}) + 2\pi i \mathrm{Res}_{\sqrt{3}i}(\frac{z^2}{p(z)}) = 2\pi i(\frac{2i}{p'(\sqrt{2}i)} + \frac{3i}{p'(\sqrt{3}i)}) = 2\pi(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}).$$

using $p'(z) = 4z^3 + 10z$. Putting everything together, we get:

$$\int_0^\infty \frac{x^2}{x^4+5x^2+6}\mathrm{d}x = \pi(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}).$$

(b) Since the integrand is an even function of x, we get

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\mathrm{d}x.$$

Since the degree of $(x^2+a^2)^3$ exceeds the degree of x^2 by more than two, we may apply the residue theorem to calculate this integral. The singularities of the integrand are at $\pm ai$, only ia in the upper half plane is counted. Hence

$$\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\mathrm{d}x = 2\pi i \mathrm{Res}_{ai}(\frac{x^2}{(x-ia)^3(x+ai)^3}) = 2\pi i \frac{(\frac{x^2}{(x+ai)^3})''|_{z=ai}}{2!} = \frac{\pi}{8a^3}.$$

Putting everything together, we get:

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\mathrm{d}x = \frac{\pi}{16a^3}.$$

(c) Since the integrand is an even function of x, we get

$$\int_0^\infty \frac{\cos(x)}{x^2+a^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(x)}{x^2+a^2}\mathrm{d}x = \frac{1}{4}\int_{-\infty}^\infty \frac{e^{ix}+e^{-ix}}{x^2+a^2}\mathrm{d}x = \frac{1}{4}\left(\int_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2}\mathrm{d}x + \int_{-\infty}^\infty \frac{e^{-ix}}{x^2+a^2}\mathrm{d}x\right)$$

We transform the second integral by $x \mapsto -x$ and get:

$$= \frac{1}{4}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\mathrm{d}x + \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\mathrm{d}x\right) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\mathrm{d}x$$

This integral is of the form considered in the lecture because the degree of $x^2 + a^2$ is bigger than 1. Hence we can apply the residue theorem. The singularities are at $z = \pm ai$, but only ia in the upper half plane is to be considered. Hence we get

$$= \pi i \mathrm{Res}_{ai}\left(\frac{e^{ix}}{(x-ai)(x+ai)}\right) = \pi i \frac{e^{-a}}{2ai} = \frac{\pi}{2ae^a}.$$

3. **Using Rouché's theorem I:** How many zeros (counted with multiplicity) has

   $$g(z) = z^7 - 2z^5 + 6z^3 - z + 1$$

   in $B_1(0)$?

   Hint: Choose a suitable among the monomials z^7, $-2z^5$, $6z^3$, $-z$, resp. 1 as the function f in Rouché's theorem.

We choose $f(z) = 6z^3$ and get the following estimate for $z \in \mathbb{C}$ with $|z| = 1$:

$$|g(z) - f(z)| = |z^7 - 2z^5 - z + 1| \leq |z^7| + |-2z^5| + |z| + 1 = 1 + 2 + 1 + 1 = 5 < |6z^2| = |f(z)|.$$

Hence $g(z)$ is not zero for z with $|z| = 1$ and Rouché's theorem tells us, that f and g have the same number of zeros (counted with multiplicity) in $B_1(0)$. f, however, has one zero at $z = 0$ with multiplicity 3. Therefore also g has 3 zeros (counted with multiplicity) in $B_1(0)$.

4. **A fixed point:** Let h be a holomorphic function on $B_{1+\varepsilon}(0)$ and assume $|h(z)| < |z|$ for all z with $|z| = 1$. Show that there is exactly one $z \in B_1(0)$ with $h(z) = z$.

A fixed point $h(z) = z$ is a zero of the function $g(z) := h(z) - z$. Taking $f(z) = -z$ we get the estimate

$$|g(z) - f(z)| = |h(z)| < |z| = |f(z)|.$$

Hence $g(z)$ is not zero for z with $|z| = 1$ and has precisely 1 zero in $B_1(0)$ because f obviously has one. This is the required fixed point of h.

5. **Using Rouché's theorem II:** How many zeros (counted with multiplicity) has

   $$g(z) = z^4 - 6z + 3$$

   on the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$?

   Hint: First show, that g has no zeros on the circles of radius 1 and 2 respectively. Then apply Rouché's theorem twice.

First let $f(z) := z^4$. For $z \in \mathbb{C}$ with $|z| = 2$, we get the estimate

$$|g(z) - f(z)| = |-6z + 3| \leq |6z| + |3| = 15 < 16 = 2^4 = |f(z)|$$

Therefore $g(z)$ has no zero in the circle with radius 2 around 0 and precisely 4 zeros (counted with multiplicity) in $B_2(0)$.

Now let $f(z) := -6z$. For $z \in \mathbb{C}$ with $|z| = 1$, we get the estimate

$$|g(z) - f(z)| = |z^4 + 3| \leq |z^4| + |3| = 4 < 6 = |-6z| = |f(z)|$$

Therefore $g(z)$ has no zero in the circle with radius 1 around 0 either and precisely 1 simple zero in $B_1(0)$.
The other 3 zeros (counted with multiplicity), therefore, must lies in the open annulus

$$\{z \in \mathbb{C} \mid 1 < |z| < 2\}.$$