1. **The Mittag-Leffler sum of the cotangent** (3 points). Prove the formula claimed in the lecture

\[
\pi \cot(\pi z) = \frac{1}{z} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z - k} + \frac{1}{k} \right)
\]

as follows:

- We know, looking at residues, that the difference

\[
d(z) := \pi \cot(\pi z) - \frac{1}{z} - \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z - k} + \frac{1}{k} \right)
\]

is an entire function (that is, it is holomorphic everywhere on \( \mathbb{C} \)). Compute the derivative of this difference (using \( \cot' = -\frac{1}{\sin^2} \)).

- Prove that \( d'(z + k) = d'(z) \) for all \( k \in \mathbb{Z} \) by reordering the sum.

- Prove that for \( |\text{Im}(z)| \to \infty, d'(z) \to 0 \). By the previous step you can assume w.l.o.g. that \( 0 < \text{Re}(z) \leq 1! \) (Consider the term involving \( \sin \) and the infinite sum separately.)

- Conclude, using Liouville’s theorem (why can it be applied?), that \( d'(z) \) is constant equal to 0. Hence \( d(z) \) is constant.

- Finally prove that \( d(-z) = -d(z) \) and conclude that \( d(z) = 0 \).

2. **Another formula of Euler.** Compute

\[
\sum_{k=1}^{\infty} \frac{1}{k^2}
\]

*Hint: Look at the coefficient \( a_1 \) in the Laurent series expansion at \( z = 0 \) of both sides of the identity of exercise 1. Here, it is convenient to write the Mittag-Leffler sum as \( \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2} \).*

3. **Wallis formula.** In the lecture, we derived from the identity in exercise 1 the following product expansion of the sine function:

\[
\frac{\sin(\pi z)}{\pi} = z \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{k^2} \right)
\]

Derive the Wallis formula

\[
\pi = 2 \frac{2 \cdot 2}{1 \cdot 3} \frac{4 \cdot 4}{3 \cdot 5} \frac{6 \cdot 6}{5 \cdot 7} \cdots
\]

from it.

*Please hand in your solutions on Friday, November 26, 2010 in the lecture hall*