Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 9

1. **The Mittag-Leffler sum of the cotangent** (3 points). Prove the formula claimed in the lecture
$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-k} + \frac{1}{k}\right)$$
as follows:

   - We know, looking at residues, that the difference
   $$d(z) := \pi\cot(\pi z) - \frac{1}{z} - \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-k} + \frac{1}{k}\right)$$
   is an entire function (that is, it is holomorphic everywhere on $\mathbb{C}$). Compute the derivative of this difference (using $\cot' = -\frac{1}{\sin^2}$).
   - Prove that $d'(z+k) = d'(z)$ for all $k \in \mathbb{Z}$ by reordering the sum.
   - Prove that for $|Im(z)| \to \infty$, $d'(z) \to 0$. By the previous step you can assume w.l.o.g. that $0 < Re(z) \leq 1$! (Consider the term involving sin and the infinite sum separately.)
   - Conclude, using Lioville's theorem (why can it be applied?), that $d'(z)$ is constant equal to 0. Hence $d(z)$ is constant.
   - Finally prove that $d(-z) = -d(z)$ and conclude that $d(z) = 0$.

2. **Another formula of Euler.** Compute
$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$
Hint: Look at the coefficient a_1 in the Laurent series expansion at $z = 0$ of both sides of the identity of exercise 1. Here, it is convenient to write the Mittag-Leffler sum as $\frac{1}{z} + 2z\sum_{k=1}^{\infty}\frac{1}{z^2-k^2}$.

3. **Wallis formula.** In the lecture, we derived from the identity in exercise 1 the following product expansion of the sine function:
$$\frac{\sin(\pi z)}{\pi} = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right).$$
Derive the Wallis formula
$$\pi = 2\,\frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\cdots$$
from it.

Please hand in your solutions on Friday, November 26, 2010 in the lecture hall