Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Exercise sheet 9

1. **The Mittag-Leffler sum of the cotangent** (3 points). Prove the formula claimed in the lecture
$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-k} + \frac{1}{k}\right)$$
as follows:

   - We know, looking at residues, that the difference
   $$d(z) := \pi\cot(\pi z) - \frac{1}{z} - \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-k} + \frac{1}{k}\right)$$
   is an entire function (that is, it is holomorphic everywhere on $\mathbb{C}$). Compute the derivative of this difference (using $\cot' = -\frac{1}{\sin^2}$).
   - Prove that $d'(z+k) = d'(z)$ for all $k \in \mathbb{Z}$ by reordering the sum.
   - Prove that for $|Im(z)| \to \infty$, $d'(z) \to 0$. By the previous step you can assume w.l.o.g. that $0 < Re(z) \leq 1$! (Consider the term involving sin and the infinite sum separately.)
   - Conclude, using Lioville's theorem (why can it be applied?), that $d'(z)$ is constant equal to 0. Hence $d(z)$ is constant.
   - Finally prove that $d(-z) = -d(z)$ and conclude that $d(z) = 0$.

STEP 1: We proceed as given in the hints and first calculate the derivative $d'(z)$ of the difference $d(z)$:
$$d'(z) = -\frac{\pi^2}{\sin^2(\pi z)} + \sum_{k\in\mathbb{Z}} \frac{1}{(z-k)^2}.$$
The commutation of summation and differentiation is justified by Weierstrass' theorem because the series is compactly convergent on $\mathbb{C} \setminus \mathbb{Z}$, by construction.
STEP 2: Next we prove that $d'(z+1) = d'(z)$ (this is in fact already true for $d(z)$):
$$d'(z+1) = -\frac{\pi^2}{\sin^2(\pi z + \pi)} + \sum_{k\in\mathbb{Z}} \frac{1}{(z+1-k)^2}.$$
Since $\sin^2$ is periodic with period π, the first term stays the same. In the second term we substitute k with $k+1$ and obtain
$$d'(z+1) = -\frac{\pi^2}{\sin^2(\pi z)} + \sum_{k\in\mathbb{Z}} \frac{1}{(z-k)^2} = d'(z).$$
STEP 3: Let $\{z_1, z_2, z_3, \ldots\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} |Im(z_n)| = \infty$. We want to show that $\lim_{n\to\infty} d'(z_n) = 0$. By the previous step, we may assume w.l.o.g. that $0 < Re(z_n) \leq 1$. Write $\sin(z) = \frac{\exp(iz)-\exp(-iz)}{2i}$ and observe that $|\exp(iz_n)| = \exp(-Im(z_n))$ and $|\exp(-iz_n)| = \exp(Im(z_n))$. Hence $\lim_{n\to\infty} |\sin(z_n)| = \infty$ and so the term $\frac{\pi^2}{\sin^2(\pi z_n)}$ converges to 0.

Now we want to show the same for the infinite sum. This is the only a bit more involved step. We use the fact that $\sum_{k=1}^{\infty} k^2$ converges, which is not difficult to prove. We have now for $k \geq 0$

$$|z_n - k|^2 = Im(z_n)^2 + (Re(z_n) + k)^2 \geq Im(z_n)^2 + k^2.$$

We may write

$$|\sum_{k=0}^{\infty} \frac{1}{(z_n - k)^2}| \leq \sum_{k=0}^{N} \frac{1}{|Im(z_n)|^2} + \sum_{k=N+1}^{\infty} \frac{1}{k^2}.$$

For any $\epsilon > 0$, we may find an N such that $\sum_{k=N+1}^{\infty} \frac{1}{k^2} < \frac{\epsilon}{2}$ (convergence of the sum). In addition, we may now find an M such that $\frac{N+1}{|Im(z_n)|^2} < \frac{\epsilon}{2}$ for all $n > M$ (assumption on the sequence). Since ϵ was arbitrary this shows, that the value of the infinite sum

$$\sum_{k=0}^{\infty} \frac{1}{(z_n - k)^2}$$

converges to 0 if $n \to \infty$. For the sum going from $-\infty$ to -1, the same argument works.
STEP 4: The previous step shows that d' is bounded on the strips $\{z \in \mathbb{C} \mid 0 \leq Re(z) \leq 1,\ M \leq \pm Im(z)\}$ for some big M. (If it would be unbounded, we could find a sequence $\{z_1, z_2, \dots\}$ with imaginary part going to infinity and unbounded $|d'(z_n)|$ contradicting the previous step.) On the rectangle $\{z \in \mathbb{C} \mid 0 \leq Re(z) \leq 1,\ -M \leq Im(z) \leq +M\}$, d' is bounded because all singularities are removable by construction and so d' is continuous. The second step thus shows that d' is bounded everywhere. Hence by Lioville's theorem d' is constant. The third step however shows that $\lim_{n\to\infty} d'(z_n) = 0$ for any sequence with unbounded imaginary part. Therefore this constant has to be 0. Therefore the original d is constant.
STEP 5: Since cot is an odd function, $\frac{1}{z}$ is an odd function and the infinite sum is symmetric under substitution $k \mapsto -k$, d is an odd function, that is $d(-z) = -d(z)$. Since it is also constant by the previous step, we finally get $d(z) = 0$ for all $z \in \mathbb{C}$.

2. **Another formula of Euler.** Compute

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hint: Look at the coefficient a_1 in the Laurent series expansion at $z = 0$ of both sides of the identity of exercise 1. Here, it is convenient to write the Mittag-Leffler sum as $\frac{1}{z} + 2z\sum_{k=1}^{\infty} \frac{1}{z^2-k^2}$.

We first show that the infinite sum can be rewritten in the shown way:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-k} + \frac{1}{k}\right) = \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{k} + \frac{1}{z+k} + \frac{1}{-k}\right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k}\right) = \sum_{k=1}^{\infty} \left(\frac{z+k}{z^2-k^2} + \frac{z-k}{z^2-k^2}\right) = \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}.$$

Now we get the first terms of the Laurent expansion of the right hand side of the identity from exercise 1:

$$\frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2} = \frac{1}{z} + 0 - 2\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) z + a_2 z^2 + a_3 z^3 + \dots$$

The coefficient a_1 is therefore equal to $-2\sum_{k=1}^{\infty}\frac{1}{k^2}$.
On the other hand, we get the coefficient a_1 of the left hand side as

$$a_1 = \frac{(z\pi\cot(\pi z))''(0)}{2!}$$

because the expression has a simple pole at $z = 0$.
Now calculate:

$$(z\pi\cot(\pi z))' = \pi\cot(\pi z) - z\frac{\pi^2}{\sin^2(\pi z)}$$

and

$$(z\pi\cot(\pi z))'' = -\frac{2\pi^2}{\sin^2(\pi z)} + z\frac{\pi^3 2\cos(\pi z)}{\sin^3(\pi z)} = 2\pi^2\frac{-\sin(\pi z) + z\pi\cos(\pi z)}{\sin^3(\pi z)}.$$

Now $\sin(\pi z) = \pi z - \pi^3\frac{z^3}{6} + \ldots$ and $z\pi\cos(\pi z) = \pi z - \frac{\pi^3 z^3}{2} + \cdots$ and $\sin^3(\pi z) = \pi^3 z^3(1 + \cdots)$.
Putting these things together, we get for the value at 0:

$$2\pi^2\frac{\left(\frac{\pi^3}{6} - \frac{\pi^3}{2}\right)}{\pi^3} = -\frac{2\pi^2}{3}.$$

Hence $a_1 = -\frac{\pi^2}{3}$ and therefore

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

3. **Wallis formula.** In the lecture, we derived from the identity in exercise 1 the following product expansion of the sine function:

$$\frac{\sin(\pi z)}{\pi} = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right).$$

   Derive the Wallis formula

$$\pi = 2\,\frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\cdots$$

   from it.

Letting $z = \frac{1}{2}$ we get:

$$\frac{1}{\pi} = \frac{1}{2}\prod_{k=1}^{\infty}\left(1 - \frac{2^{-2}}{k^2}\right).$$

Taking the reciprocal, we get:

$$\pi = 2\prod_{k=1}^{\infty}\left(1 - \frac{1}{(2k)^2}\right)^{-1}.$$

Brining things to a common denominator, this may be rewritten as

$$\pi = 2\prod_{k=1}^{\infty}\left(\frac{(2k)^2 - 1}{(2k)^2}\right)^{-1}.$$

And finally:

$$\pi = 2\prod_{k=1}^{\infty}\frac{(2k)^2}{(2k-1)(2k+1)}.$$

This is Wallis' formula.
Remark: The formula, as nice as it is, is not very suitable for a computation of π. *A product over the first 1000 terms is approximately equal to* 3.1408 *(*$\pi \simeq 3.1416$*).*