Fritz Hörmann — MATH 316: Complex Analysis — Fall 2010

Mid-term exam, October 22, 2010

Name: ______________________________________

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1. Which of the following complex numbers are 8th roots of 16:

   ● $-1-i$ ● $\sqrt{2}\exp(\frac{\pi i}{4})$ ○ $2\exp(\frac{\pi i}{4})$

   ○ $\sqrt{2}\exp(\frac{\pi i}{8})$ ● $-\sqrt{2}$ ○ $2i$

2. Which of the following functions are holomorphic in a neighborhood of 0:

   ○ $f(z) = |z|$,

   ○ $f(z) = \overline{z}$,

   ● $f(z) = x^2 - y^2 + 2xyi$, where $z = x + iy$.

3. Let $U \subset \mathbb{C}$ be a open **star-like** subset and $f : U \to \mathbb{C}$ be a holomorphic function. Which of the following statements are always true:

   ● The path-integral of f over any *closed* path $\varphi : [a, b] \to U$ is zero.

   ● f has a primitive, defined on the whole set U.

4. Let now $U \subset \mathbb{C}$ be an **arbitrary** open subset and $f : U \to \mathbb{C}$ be a holomorphic function. Which of the following statements are (still) always true:

   ○ The path-integral of f over any *closed* path $\varphi : [a, b] \to U$ is zero.

   ○ f has a primitive, defined on the whole set U.

5. Let again $U \subset \mathbb{C}$ be an **arbitrary** open subset and $f : U \to \mathbb{C}$ be a holomorphic function. Which of the following statements are always true:

   ○ f can be expanded into a power series, which converges on the whole set U.

   ● For any closed disc $\overline{B_r(z_0)} \subset U$,
   f can be expanded in a power series converging in the interior of that disc.

— Please turn over —

6. Which of the following subsets $U \subset \mathbb{C}$ is an elementary domain, that is, a connected open subset, such that **every** holomorphic $f : U \to \mathbb{C}$ has a primitive on U:

   - ○ $U = B_1(0) \setminus \{0\}$.
   - ● $U = \{z \in \mathbb{C}^* \mid \arg(z) \neq 0\}$.
   - ● $U = \{z \in \mathbb{C} \mid -1 < Re(z) < 1,\ -1 < Im(z) < 1\}$.
   - ○ $U = \mathbb{C} \setminus \mathbb{R}$.

7. Which of the following statements are true:

   - ● Every holomorphic function $f : \mathbb{C} \to B_1(0)$ is constant.
   - ● Every non-constant polynomial $f(z) = a_n z^n + \cdots + a_0$, with complex coefficients $a_k \in \mathbb{C}, k = 0 \ldots n$, has a zero in $\mathbb{C}$.
   - ○ The coefficients $a_k, k = 0 \ldots \infty$ of a power series expansion of a holomorphic function (defined on a small disc) satisfy $|a_k| < 1$ for all sufficiently large k.
   - ● Every *bijective* holomorphic function $f : B_1(0) \to B_1(0)$ is a Möbius transformation.
   - ● Let $U \subset \mathbb{C}$ be open and connected. The image of a non-constant holomorphic function $f : U \to \mathbb{C}$ is open.

8. The exponential function maps the open rectangle

$$U := \{z \in \mathbb{C} \mid -4 < Re(z) < 4;\ -4 < Im(z) < 4\}$$

   to the following set:

   - ○ It is not uniquely defined.
   - ○ $\{z \in \mathbb{C} \mid e^{-4} < Re(z) < e^4;\ -4 < Im(z) < 4\}$.
   - ● $\{z \in \mathbb{C} \mid e^{-4} < |z| < e^4\}$.
   - ○ $\{z \in \mathbb{C}^* \mid e^{-4} < Re(z) < e^4;\ -4 < \arg(z) < 4\}$.

9. The function $f(z) := z^2$ maps the upper half plane

$$U := \{z \in \mathbb{C} \mid Im(z) > 0\}$$

   to the following set:

   - ● $\{z \in \mathbb{C}^* \mid \arg(z) \neq 0\}$.
   - ○ $\{z \in \mathbb{C} \mid Im(z) \neq 0\}$.
   - ○ $\{z \in \mathbb{C} \mid Re(z) < 0\}$.

10. What is $\int_\varphi \frac{1}{z} dz$, for the path $\varphi(t) = \exp(2\pi i t), t \in [0, 1]$?

    ○ -2π ○ $-2\pi i$ ○ 0 ○ 2π ● $2\pi i$

11. What is $\int_\varphi \frac{1}{z} dz$, for the path $\varphi(t) = 2 + \exp(2\pi i t), t \in [0, 1]$?

    ○ $2 + 2\pi$ ○ $2 + 2\pi i$ ● 0 ○ 2π ○ $2\pi i$