Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011

Exercise sheet 2

1. **Rings of integers in quadratic fields.** Consider a square-free $N \in \mathbb{Z}$, $N \neq 0, 1$, and determine explicitly the "ring of integers" $\mathrm{Int}_{\mathbb{Z}}(\mathbb{Q}(\sqrt{N}))$ in $\mathbb{Q}(\sqrt{N})$.

   Hint: Use the fact, proven in the lecture, that an element $x \in \mathbb{Q}(\sqrt{N})$ is integral (over $\mathbb{Z}$) if and only if its (monic) minimal polynomial over $\mathbb{Q}$ has integral coefficients.

2. **Examples of non-normality.** (a) Let k be an algebraically closed field. Consider $R = k[x,y]/(x^2 - y^3)$ and consider $t = x/y$ in the quotient field K of R. Prove that the integral closure $\mathrm{Int}_R(K)$, i.e. the normalization of R, is $k[t]$, and hence $K = k(t)$. Give a geometric interpretation of this.

   (b) Also, by Ex. 1, the ring $\mathbb{Z}[5i]$ is not normal; its normalization is the ring $\mathbb{Z}[i]$. Give a geometric interpretation of this, i.e. describe the map $\mathrm{specm}(\mathbb{Z}[i]) \to \mathrm{specm}(\mathbb{Z}[5i])$.

   Hint for (b): Use localizations to reduce to a determination of the fibre above $(5, 5i)$.

3. **Unique factorization presupposes normality.** The rings $k[x,y](x^2 - y^3)$ and $\mathbb{Z}[5i]$ from Ex. 2 are not normal, hence cannot satisfy unique factorization. Give a counterexample (with justification) in each case.

4. **Finitely generated ideals.** We will see in the lecture, that $R = k[x,y]$ is noetherian for any field k, hence every ideal is finitely generated. Prove however, that for any N, there is an ideal of R which cannot be generated by $< N$ elements.

5. **The nilradical and Jacobson radical.** Let R be a commutative ring (with 1 as always).

   The *nilradical* $\mathrm{Nil}(R)$ of R is the set of all nilpotent elements. In Ex. 1 (h) on Ass. 1, you determined that it is equal to the intersection of all prime ideals.

   Similarly, the *Jacobson radical* $\mathrm{Jac}(R)$ of R is defined as the intersection of all maximal ideals.

   Obviously $\mathrm{Nil}(R) \subseteq \mathrm{Jac}(R)$ because maximal ideals are prime.

   (a) Show that $x \in \mathrm{Jac}(R)$ if and only if $1 - xy$ is a unit for all $y \in R$.

   (b) A ring is called *Jacobson*, if $\mathrm{Jac}(R') = \mathrm{Nil}(R')$ for every homomorphic image R' of R. Derive Hilbert's Nullstellensatz (strong form), as given in the lecture, from its following most general version:

   A finitely generated algebra S over a Jacobson ring R is Jacobson. The induced map $\mathrm{spec}(S) \to \mathrm{spec}(R)$ maps maximal ideals to maximal ideals and the corresponding extension of residue fields is finite.

   Hint: A field is clearly a Jacobson ring, hence $S = k[x_1, \dots, x_n]$ is Jacobson. The second statement immediately gives the weak form. The strong form can be obtained by exploiting the Jacobson property. Apply the fact that the intersection of all prime ideals gives the nilradical to S/I (where I is an ideal of S).

   (c) Prove that $\mathbb{Z}_{(p)}$ (the localization of $\mathbb{Z}$ at the prime (p)) is not Jacobson and that also the map $\mathrm{spec}(\mathbb{Z}_{(p)}[x]) \to \mathrm{spec}(\mathbb{Z}_{(p)})$ has *not* the property of mapping maximal ideals to maximal ideals.

Please hand in your solutions on Monday, January 24, 2011 in the lecture room