1. **Yoneda’s lemma.** Let \( C \) be any category. Prove Yoneda’s lemma: The functor
\[
C^\text{op} \to \text{Funct}(C, \text{Sets})
\]
\[ O \mapsto \text{Hom}(O, \cdot) \]
is fully faithful.

2. **A simple universal object.** Consider the category \( C \) whose objects are pairs \((S, e)\) where \( S \) is a set and \( e \) is a map \( S \to S \). Morphisms \((S, e) \to (S', e')\) are maps \( \alpha : S \to S' \) such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & S' \\
e & \downarrow & \downarrow e' \\
S & \xrightarrow{\alpha} & S'
\end{array}
\]
is commutative. We have a forgetful functor \( V : C \to \text{Sets} \) given by \((S, e) \mapsto S\).

(a) Prove by explicit construction that \( V \) has a left adjoint \( F \).

(b) Prove that \( F(\{\cdot\}) \) (together with its canonical element \( \xi \in V(F(\{\cdot\})) \)) represents \( V \) (here \( \{\cdot\} \) denotes a set with 1 element). *Note that this is also equivalent to \((F(\{\cdot\}), \xi)\) being an initial object in the category \( CV \) constructed in the lecture.*

(c) Explain (in words, not too scrupulously) that if we denote \((\mathbb{N}, \nu) := F(\{\cdot\})\) and \( 1 := \xi \) then the representability amounts to the Peano axioms for \( \mathbb{N} \) as set of natural numbers, \( \nu \) as its “successor function” and \( 1 \in \mathbb{N} \) as the “1” element.

3. **Adjoints.** Let \( V : \text{TopSp} \to \text{Sets} \) be the forgetful functor. Show that it has a right and a left adjoint.

4. **Examples of a non-noetherian rings.** Let \( R \) be the ring of continuous functions on the interval \([0, 1]\). Prove that \( R \) is not noetherian.

Let \( R \) be the subring of \( k[X,Y] \) generated by \( Y, XY^2, X^2Y^3, X^3Y^4, \ldots, X^iY^{i+1}, \ldots \). Prove that \( R \) is not noetherian. *Note that this is a subring of a noetherian ring!*

5. **Artinian rings.** Let \( R \) be a noetherian ring. Prove that the following are equivalent:

(a) \( R \) is artinian,

(b) the topology on \( \text{spec}(R) \) is discrete.

*Hint: \((a) \Rightarrow (b)\) was shown in the lecture (even without assuming \( R \) noetherian a priori). For the converse, use the noetherian hypothesis to see that \( \text{spec}(R) \) is a finite set. Use the decomposition theorem 1.3.4. to decompose \( R \) into a product of rings. For each constituent (which is then local) you have to show that \( m^n = (0) \) for the maximal ideal \( m \) and some \( n \).*