Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011

Exercise sheet 5

Choose 5 of the 7 exercises.

1. Let R be a ring which is completely reducible as a left R-module. Prove that every left R-module is completely reducible.

   Hint: Remember that for M to be completely reducible, it suffices to show $M = \sum_{M' \subseteq M \, irred.} M'$.

2. **Lemma 2 of section 4.3.** Let R be a ring. M a completely reducible R-module, $R' := \mathrm{End}_R(M)$, $R'' := \mathrm{End}_{R'}(M)$. Show: $\mathrm{End}_R(M^n) = \mathrm{Mat}_{n\times n}(R')$ and for all $\alpha \in R''$, the map $(x_1, \dots, x_n) \mapsto (\alpha x_1, \dots, \alpha x_n)$ commutes with the action of $\mathrm{End}_R(M^n)$.

3. **Density.** Let R be a ring and M a completely reducible R-module, $R' := \mathrm{End}_R(M)$, $R'' := \mathrm{End}_{R'}(M)$.

   Define a topology on R'' by letting a basis of the open sets to be cosets of the left ideals
   $$I(V) = \{\alpha \in R'' \mid \alpha|_V = 0\},$$
   where V runs through the *finitely-generated* R'-submodules of M. Prove that this is defines the structure of a topological ring on R'' and that the image of R is dense in R''.

4. **Ideals of** $\mathrm{End}_D(D^n)$**.** Let D be a division algebra (skew field). Prove that the association
   $$\begin{array}{rcl} \{V \subseteq D^n \ \text{ subspace }\} & \to & \{J \subseteq \mathrm{End}_D(D^n) \ \text{ left ideal}\} \\ V & \mapsto & I(V) = \{\alpha \in \mathrm{End}_D(D^n) \mid \alpha|_V = 0\} \end{array}$$
   is an inclusion reversing bijection.

   Hint: Define a map Z going in the other direction. To show $I(Z(J)) = J$ for any left ideal J, start with the case of principal ideals. Then consider intersections of subspaces/sums of ideals.

5. Determine explicitly a direct sum decomposition of $R = \mathrm{End}_D(D^n)$ as left module over itself into irreducible left R-modules. (We know that they have to be all isomorphic to D^n as R-modules).

6. **Frobenius' Theorem on real division algebras.** If $F = \mathbb{R}$, prove that $\mathbb{R}$, $\mathbb{C}$, and the Hamiltonian quaternions $\mathbb{H}$ are the only skew fields (up to isomorphism) D which are finite-dimensional F-algebras.

   Hint: Let D be a f.d. $\mathbb{R}$-algebra which is division. If $D \neq \mathbb{R}$, every element $i \in D \backslash \mathbb{R}$ generates a field extension of $\mathbb{R}$ so $\mathbb{R}[i] \cong \mathbb{C}$ and w.l.o.g. $i^2 = -1$. This renders D into an $\mathbb{R}[i]$-vector space by left multiplication. Show that right multiplication by i can have eigenvalues i and $-i$ only and that $D = D^{+i} \oplus D^{-i}$ for the corresponding eigenspaces. Prove that $D^{+i} = \mathbb{R}[i]$. If $D^{-i} = 0$ then $D \cong \mathbb{C}$. If there is an $x \in D^{-i}$ prove that right multiplication with it exchanges D^{+i} and D^{-i}. Conclude that $D \cong \mathbb{H}$.

7. **Representations of the symmetric group in 3 elements.** Determine explicitly the algebra isomorphisms $\mathbb{C}[S_3] \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{Mat}_{2x2}(\mathbb{C})$ and $\mathbb{R}[S_3] \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathrm{Mat}_{2x2}(\mathbb{R})$.

Please hand in your solutions on Monday, March 14, 2011 in the lecture room