Choose 5 of the 7 exercises.

1. Let $R$ be a ring which is completely reducible as a left $R$-module. Prove that every left $R$-module is completely reducible.

   *Hint: Remember that for $M$ to be completely reducible, it suffices to show $M = \sum_{M' \subseteq \text{Irred}} M'$.*

2. **Lemma 2 of section 4.3.** Let $R$ be a ring. $M$ a completely reducible $R$-module, $R' := \text{End}_R(M)$, $R'' := \text{End}_{R'}(M)$. Show: $\text{End}_R(M^n) = \text{Mat}_{n \times n}(R')$ and for all $\alpha \in R''$, the map $(x_1, \ldots, x_n) \mapsto (\alpha x_1, \ldots, \alpha x_n)$ commutes with the action of $\text{End}_R(M^n)$.

3. **Density.** Let $R$ be a ring and $M$ a completely reducible $R$-module, $R' := \text{End}_R(M)$, $R'' := \text{End}_{R'}(M)$.

   Define a topology on $R''$ by letting a basis of the open sets to be cosets of the left ideals
   
   $$I(V) = \{ \alpha \in R'' \mid \alpha|_V = 0 \},$$

   where $V$ runs through the finitely-generated $R'$-submodules of $M$. Prove that this is defines the structure of a topological ring on $R''$ and that the image of $R$ is dense in $R''$.

4. **Ideals of $\text{End}_D(D^n)$.** Let $D$ be a division algebra (skew field). Prove that the association

   $$\{ V \subseteq D^n \text{ subspace} \} \rightarrow \{ J \subseteq \text{End}_D(D^n) \text{ left ideal} \}
   
   V \mapsto I(V) = \{ \alpha \in \text{End}_D(D^n) \mid \alpha|_V = 0 \}$$

   is an inclusion reversing bijection.

   *Hint: Define a map $Z$ going in the other direction. To show $I(Z(J)) = J$ for any left ideal $J$, start with the case of principal ideals. Then consider intersections of subspaces/sums of ideals.*

5. Determine explicitly a direct sum decomposition of $R = \text{End}_D(D^n)$ as left module over itself into irreducible left $R$-modules. (We know that they have to be all isomorphic to $D^n$ as $R$-modules).

6. **Frobenius’ Theorem on real division algebras.** If $F = \mathbb{R}$, prove that $\mathbb{R}$, $\mathbb{C}$, and the Hamiltonian quaternions $\mathbb{H}$ are the only skew fields (up to isomorphism) $D$ which are finite-dimensional $F$-algebras.

   *Hint: Let $D$ be a f.d. $\mathbb{R}$-algebra which is division. If $D \neq \mathbb{R}$, every element $i \in D \setminus \mathbb{R}$ generates a field extension of $\mathbb{R}$ so $\mathbb{R}[i] \cong \mathbb{C}$ and w.l.o.g. $i^2 = -1$. This renders $D$ into an $\mathbb{R}[i]$-vector space by left multiplication. Show that right multiplication by $i$ can have eigenvalues $i$ and $-i$ only and that $D = D^{+i} \oplus D^{-i}$ for the corresponding eigenspaces. Prove that $D^{+i} = \mathbb{R}[i]$. If $D^{-i} = 0$ then $D \cong \mathbb{C}$. If there is an $x \in D^{-i}$ prove that right multiplication with it exchanges $D^{+i}$ and $D^{-i}$. Conclude that $D \cong \mathbb{H}$."

7. **Representations of the symmetric group in 3 elements.** Determine explicitly the algebra isomorphisms $\mathbb{C}[S_3] \cong \mathbb{C} \oplus \mathbb{C} \oplus \text{Mat}_{2 \times 2}(\mathbb{C})$ and $\mathbb{R}[S_3] \cong \mathbb{R} \oplus \mathbb{R} \oplus \text{Mat}_{2 \times 2}(\mathbb{R})$.

Please hand in your solutions on Monday, March 14, 2011 in the lecture room.