Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011

Exercise sheet 6

Choose 2 of the following exercises.

1. **Character tables.** Let Q be the quaternion group $\{1, -1, i, j, k, -i, -j, -k\}$ with the usual relations. Determine the complex character table of Q. Let D_8 be the dihedral group $\{1, z, z^2, z^3, s, sz, sz^2, sz^3\}$ with relations $z^4 = 1$, $s^2 = 1$ and $zs = sz^3$. Determine the complex character table of D_8.

2. **Representations of products.** Let G_1 and G_2 be finite groups. Prove, using complex characters and their orthogonality relations, that there is a bijection between (isomorphism classes of) irreducible complex representations V of $G_1 \times G_2$ and (isomorphisms classes of) pairs (V_1, V_2), where V_i is an irreducible complex representation of G_i, given by
$$(V_1, V_2) \mapsto V := V_1 \otimes V_2.$$
Here, on the right hand side, V_i is considered as a representation of $G_1 \times G_2$ by letting the other factor act as the identity.

3. **Induction and dual.** Let G be a finite group, $H \subseteq G$ a subgroup, and F a field. Let V be an F-representation of H and V^* be its dual. Prove $\mathrm{Ind}_H^G(V^*) \cong (\mathrm{Ind}_H^G V)^*$.

4. **A theorem of Frobenius.** Let G be a finite group and V be an irreducible $\mathbb{C}[G]$-module. Prove: $\dim_{\mathbb{C}} V | \#G$.

   Hint:

   (i) Prove that $\chi_V(\sigma)$ is integral over $\mathbb{Z}$ for all $\sigma \in G$.

   (ii) Recall the basis elements $c_i = \sum_{\sigma \in C_i} [\sigma]$ of $\mathrm{Cent}(\mathbb{C}[G])$, where the C_i are the conjugacy classes in G. Show that
$$c_i c_j = \sum_k m_{ijk} c_k, \tag{1}$$
   where $m_{ijk} \in \mathbb{Z}_{\geq 0}$.

   (iii) Prove that c_i acts on V by the scalar $\chi_V(c_i)/\dim_{\mathbb{C}} V = (\#C_i)\chi_V(\sigma)/\dim_{\mathbb{C}} V$ where $\sigma \in C_i$ is arbitrary.

   (iv) Lemma: Prove that $a \in \mathbb{C}$ is integral over $\mathbb{Z}$, if and only if there is a f.g. $\mathbb{Z}$-submodule $M \subset \mathbb{C}$, with $1 \in M$, such that $aM \subseteq M$.

   Use this and (1) to show that the numbers obtained in (iii) are integral over $\mathbb{Z}$.

   (v) Follow from (i), (iv), and the orthogonality of characters that $\dim_{\mathbb{C}} V | \#G$.

Please hand in your solutions on Monday, March 21, 2011 in the lecture room