Fritz Hörmann — MATH 571: Higher Algebra II — Winter 2011

Exercise sheet 7

1. **The snake lemma.** Let R be a ring. Given a commutative diagram of (left) R modules

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

with exact rows, construct a sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$$

and prove its exactness.

2. **The five lemma.** Let R be a ring. Given a commutative diagram of (left) R modules

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where $\alpha, \beta, \delta, \epsilon$ are isomorphisms, prove that γ is an isomorphism, too.

You may prove this directly or use the snake lemma.

3. $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$**.** Using the obvious free resolution of $\mathbb{Z}/2\mathbb{Z}$,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

calculate $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ by the elementary definition given on Monday. Carry out explicitly the construction of the 2 possible extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$ starting from the two elements in $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.

Please hand in your solutions on Friday, April 1, 2011 in the lecture room