These are notes for a talk I gave at the summer school on derivators at the University of Freiburg in August 2014. They are a report on ongoing research. Publications of the indicated results will follow soon.

The reader is assumed to have read some introductory text on usual (not fibered) derivators as, for instance, the author’s notes [14] for this summer school or Moritz Groth’s paper [11].

I would like to thank all participants of the summer school for the nice week, and for their very useful questions, comments and remarks.
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# Introduction to the six functors

In mathematics we often are given a collection of (derived) categories, one for each “base space” $S$ in another category $\mathcal{S}$, which could, for instance, be the category of schemes, topological spaces, analytic manifolds, etc. The fiber $\mathcal{C}_S$ above such a base space $S$ is a derived category of “sheaves” over this space, for example, coherent sheaves, $l$-adic sheaves, abelian sheaves, $D$-modules, motives, etc. These frequently form a bifibered category, that is, they are equipped with push-forward and pull-back functors along morphisms in $\mathcal{S}$. Often, one has also a monoidal structure (tensor product) in the fibers. Frequently there is a certain duality such that, in the end, we have six types of functors:

\[
\begin{align*}
&f^* \quad f_* \quad \text{for each } f \in \text{Mor}(\mathcal{S}) \\
&f_! \quad f^! \quad \text{for each } f \in \text{Mor}(\mathcal{S}) \\
& \otimes \quad \mathcal{HOM} \quad \text{in each fiber over an } S \in \mathcal{S}
\end{align*}
\]

The functors on the left hand side are left adjoints of the functors on the right hand side. $f_!$ is “the dual of $f_*$” and is called mostly push-forward with proper support, because in the topological setting (abelian sheaves over topological spaces) this is what it is derived from. Its right adjoint $f^!$ is called the exceptional pull-back. These functors come along with a bunch of compatibilities between them. Roughly all left-adjoints (resp. right-adjoints) commute with each other.

## 1.1.

More precisely, part of the datum of the six functors are the following natural isomorphisms (in the “left adjoints” column):

<table>
<thead>
<tr>
<th>left adjoints</th>
<th>right adjoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\ast, \ast)$</td>
<td>$(fg)^* \sim \rightarrow g^* f^*$</td>
</tr>
<tr>
<td>$(!, !)$</td>
<td>$(fg)<em>! \sim \rightarrow f</em>! g_!$</td>
</tr>
<tr>
<td>$(!, \ast)$</td>
<td>$g^* f_! \sim \rightarrow F_! G^*$</td>
</tr>
<tr>
<td>$(\otimes, \ast)$</td>
<td>$f^* (- \otimes -) \sim \rightarrow f^* - \otimes f^*$</td>
</tr>
<tr>
<td>$(\otimes, !)$</td>
<td>$f_! (- \otimes f^*) \sim \rightarrow (f_! -) \otimes -$</td>
</tr>
<tr>
<td>$(\otimes, \otimes)$</td>
<td>$(- \otimes -) \otimes - \sim \rightarrow - \otimes (- \otimes -)$</td>
</tr>
<tr>
<td>&amp; $\mathcal{HOM}(- \otimes -, -) \sim \rightarrow \mathcal{HOM}(-, \mathcal{HOM}(-, -))$</td>
<td></td>
</tr>
</tbody>
</table>

Here $f, g, F, G$ are morphisms in $\mathcal{S}$, which in the $(!, \ast)$-row, are related by a cartesian diagram:

\[
\begin{array}{ccc}
G & \sim \rightarrow & \mathcal{HOM}(f^* -, -) \\
\downarrow & & \downarrow \\
\mathcal{HOM}(-, f_! -) & \sim \rightarrow & \mathcal{HOM}(f^* -, f^! -) \\
\end{array}
\]
In the right hand side column, for convenience, we wrote down the corresponding adjoint natural transformations. In each case the left hand side natural isomorphism determines the right hand side one and vice versa. (In the ($\otimes$, $!$)-case there are 2 versions of the commutation between the right adjoints; in this case any of the three isomorphisms determines the other two). The ($!, *$)-isomorphism (between left adjoints) is called base change, the ($\otimes$, $!$)-isomorphism is called the projection formula, and the ($*, \otimes$)-isomorphism is usually part of the definition of a monoidal functor. The ($\otimes, \otimes$)-isomorphism is the associativity of the tensor product and part of the definition of a monoidal category. The ($*, *$)-isomorphism, resp. ($!, !$)-isomorphism, express that the corresponding functors arrange as a pseudo-functor with values in categories.

The story does not end here because, of course, there have to be compatibilities among those natural isomorphisms. To scare the reader, some of them are listed in figures 1–4. Instead of trying to give a complete list of them (or even only a generating list from which all of them would follow) we proceed in a more abstract way (like in the ideas of fibered category or multicategory) and get a precise definition of a six functor context without having to specify any of these compatibilities explicitly. The natural isomorphisms of 1.1 will be derived from a composition law in a fibered multicategory and all compatibilities will be just a consequence of associativity of this composition law.

The six functors are the right framework to study duality theorems like Serre duality, Poincaré duality, various (Tate) dualities for the (co)homology of groups, etc.

**Example 1.2** (Serre duality). Let $k$ be a field and consider a $k$-scheme $\pi : S \to \text{spec}(k)$ which is proper and smooth of dimension $n$. Consider a locally free sheaf $E$ on $S$ and consider the following isomorphism (one of the 2 adjoints of the projection formula):

$$
\pi_! \mathcal{HOM}(E, \pi^! k) \sim \mathcal{HOM}(\pi_* E, k)
$$

In this case, we have $\pi_! E = \pi_* E$ because $\pi$ is proper, and $\pi^! k = \Omega^n_S[n]$. Taking $i$-th homology of complexes we arrive at

$$
H^{i+n}(S, E^\vee \otimes \Omega^n_S) \cong H^{-i}(S, E)^*.
$$

This is the classical formula of Serre duality.

**Example 1.3** (Poincaré duality). Let $k$ be a field and consider an $n$-dimensional topological manifold $X$. Let $E$ be a local system of $k$-vector spaces on $X$. Consider the isomorphism (again one of the 2 adjoints of the projection formula):

$$
\pi_! \mathcal{HOM}(E, \pi^! k) \sim \mathcal{HOM}(\pi_* E, k)
$$

We have $\pi^! k = \mathcal{L}_{or}[n]$, the orientation sheaf over $k$ of $X$. Taking $i$-th homology of complexes we arrive at

$$
H^{i+n}(X, E^\vee \otimes \mathcal{L}_{or}) \cong H^{-i}_c(X, E)^*.
$$

This is the classical formula of Poincaré duality.
Example 1.4 (Group (co)homology). Let $G$ be a group and consider the classifying stack $\llbracket \cdot / G \rrbracket$ and the projection $\llbracket \cdot / G \rrbracket \to \cdot$. The six functor formalism, say for abelian sheaves (note: abelian sheaves on $\llbracket \cdot / G \rrbracket = G$-representations in abelian groups), encodes duality theorems between those, like Tate duality. In this case $\pi_*$ is group cohomology and $\pi_!$ is group homology. If $G$ is finite, we also have a natural morphism $\pi_! \to \pi_*$ whose cone (homotopy cokernel) is Tate cohomology.
Figure 1: pentagon axiom (compatibility of $(\otimes, \otimes)$ iso’s).

Figure 2: definition of monoidal functor (compatibility between $(\ast, \otimes)$ and $(\otimes, \otimes)$ iso’s).

Figure 3: example of compatibility between $(\ast, \otimes)$ and $(\otimes, \otimes)$ iso’s.
Figure 4: example of compatibility between $(!,\ast)$ and $(\ast,\otimes)$ and $(!,\otimes)$ and $(\ast,\ast)$ iso’s.
## 2 A very incomplete history

Recently there has been increasing interest in the six functors in various contexts. To indicate some of these developments, we mention some related works without any aim whatsoever towards completeness:

<table>
<thead>
<tr>
<th>Year</th>
<th>Author(s)</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960’s</td>
<td>Grothendieck, [1, 2, 3, 21]</td>
<td>schemes, top. spaces, coherent sheaves, sheaves</td>
</tr>
<tr>
<td></td>
<td>Verdier, Deligne</td>
<td>schemes, etale sheaves</td>
</tr>
<tr>
<td>1980’s</td>
<td>Bernstein</td>
<td>char 0 varieties, $D$-modules</td>
</tr>
<tr>
<td>:</td>
<td></td>
<td></td>
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<tr>
<td>2001</td>
<td>Voevodski</td>
<td>abstract theory</td>
</tr>
<tr>
<td>2006</td>
<td>Ayoub [4, 5]</td>
<td>schemes, motives</td>
</tr>
<tr>
<td>2008</td>
<td>Lazlo, Olsson [16, 17]</td>
<td>(classical) stacks, etale and $\ell$-adic sheaves</td>
</tr>
<tr>
<td>2009</td>
<td>Cisinski, Deglise [8]</td>
<td>schemes, motives</td>
</tr>
<tr>
<td>2009</td>
<td>Lipman, Hashimoto [18]</td>
<td>schemes, diag. of schemes, coherent sheaves</td>
</tr>
<tr>
<td>2012</td>
<td>Zheng, Liu [19]</td>
<td>(higher) stacks, etale sheaves ($\infty$-categorical methods)</td>
</tr>
<tr>
<td>2013</td>
<td>Zheng [22]</td>
<td>DM stacks, constructible sheaves</td>
</tr>
</tbody>
</table>
3  (Co)homological descent

3.1. Assume that the category $S$ carries a Grothendieck (pre-)topology. The question we want to adress now is: “How can the functors $f_!$ and $f^*$ be ‘glued’ with respect to this topology?”

More precisely, consider a space $S \in S$ and a hypercover $S_\bullet$ of $S$, i.e. a simplicial object $S_\bullet \in \text{Fun}(\Delta^{op}, S)$

$$S_\bullet := \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0$$

with a map $p : S_\bullet \rightarrow S$ which is, in a certain sense, locally trivial. For example $S_\bullet$ could be associated with a Čech cover $U \rightarrow S$, that is, the simplicial object $S_i := U \times_S \cdots \times_S U$

$$i + 1\text{-times}$$

with the obvious maps. Let $\pi$ be the projection to the final object of $S$ (assumed here to exist).

3.2. (COHOMOLOGICAL DESCENT.) Given an object $E$ over $S$, we say that our situation satisfies cohomological descent, whenever we have an isomorphism of the form

$$\pi_* E \cong \text{Tot}^{\prod} (\cdots \leftarrow \pi_{2,*} p_2^* E \leftarrow \pi_{1,*} p_1^* E \leftarrow \pi_{0,*} p_0^* E)$$

(1)

The complex appearing in brackets is the complex associated with the cosimplicial object $\Delta_i \mapsto \pi_{i,*} p_i^* E$ where the morphisms are given by the various units. Note that this question is not really well defined because, when working with derived categories, $\text{Tot}^{\prod}$ does not make sense at all, and if we work with complexes (on-the-nose, not up to quasi-isomorphism) instead, a coherent simplicial diagram of complexes can not be constructed because the $\pi_{i,*}$, resp. $\pi_*$ are only derived functors. Usually the reader is used to the situation where the $S_i$ have trivial cohomology and accordingly the $\pi_{i,*}$ come from an exact functor between abelian categories. So working with complexes on-the-nose and taking $\text{Tot}^{\prod}$ of those, the right hand side becomes well-defined. Then the question is made precise if we understand on the left hand side the derived $\pi_*$. The task in general is therefore

1. to give a meaning to the RHS of (1) in the derived setting (when the $\pi_{i,*}$ are just derived functors),

2. to find criteria under which (1) is an isomorphism (preferably for every hypercover at least).

In this summer school, we learned that the total complex appearing in equation (1) above is just the homotopy limit over the cosimplicial object $\Delta_i \mapsto \pi_{i,*} p_i^* E$. Hence pursuing the idea of derivators we have to find means of constructing this diagram in a coherent way (i.e. as an object in the value of the associated derivator at $\Delta$) and not only as a diagram in the derived category.
3.3. (Homological descent) Dually, given an object \( E \) over \( S \) we might ask, whether we have a identity of the form

\[
\pi_! E \cong \text{Tot}^\oplus \left( \cdots \longrightarrow \pi_2 \cdot p_!^2 E \longrightarrow \pi_1 \cdot p_!^1 E \longrightarrow \pi_0 \cdot p_!^0 E \right)
\]  

(2)

The complex is associated with the simplicial object \( \Delta_i \mapsto \pi_2 \cdot p_!^2 E \) where the morphisms are given by the various counits. Now the question makes even less sense because \( p_!^2 \) is, in most cases, only constructed as a morphism in the derived category. Again, the question of homological descent amounts to 1. give meaning to the RHS of (2) in the derived setting (when the \( \pi_i \cdot ! \) and \( p_i^! \) are classically just functors between derived categories) and 2. find criteria under which (2) is an isomorphism (preferably for every hypercover at least).

3.4. As an easy topological example for homological descent (in which we can give meaning to the question) consider an (in general non-compact) real \( C^\infty \)-manifold \( X \) of dimension \( n \) and the constant sheaf \( \mathbb{R} \) on it. Let \( \{ U_i \}_i \) be a finite Cech cover. We can compute \( \pi_! \mathbb{R} \) as the the total complex of

\[
E_{p,q} := \bigoplus_{i_1, \ldots, i_p} H^c_c(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{E}^q)
\]

where \( \mathcal{E}^q \) is the sheaf of \( C^\infty \)-differential forms of degree \( q \). Here a section with compact support on a smaller open is mapped to a section with compact support on a larger open. If the \( U_{i_1} \cap \cdots \cap U_{i_p} \) is sufficiently nice, \( H^c_{p,q} \) is only different from zero if \( q = n \), hence \( \pi_! \mathbb{R} \) is represented by the complex

\[
\cdots \longrightarrow \bigoplus_{i_1, i_2} H^c_{n}(U_{i_1} \cap U_{i_2}, \mathbb{R}) \longrightarrow \bigoplus_i H^c_{n}(U_i, \mathbb{R}).
\]

This example works because we have representations of the \( \pi_i \cdot ! \mathbb{R} \) for which a coherent complex can be constructed.

3.5. More generally, if \( S_\bullet \) is just any simplicial object in \( S \), for example the presentation of a stack (or even a higher stack), can we define a functor \( \pi_! \) by

\[
\text{Tot}^\oplus \left( \cdots \longrightarrow \pi_2 \cdot ! \mathcal{E}_2 \longrightarrow \pi_1 \cdot ! \mathcal{E}_1 \longrightarrow \pi_0 \cdot ! \mathcal{E}_0 \right)
\]

where the \( \mathcal{E}_i \) are objects over \( S_i \) and we are given (quasi-)isomorphisms \( S(\delta)^* \mathcal{E}_i \rightarrow \mathcal{E}_j \) for all \( \delta : \Delta_j \rightarrow \Delta_i \) in a compatible way? We call the collection \( \{ \mathcal{E}_i \}_i \) together with these quasi-isomorphisms a cocartesian object over \( S_\bullet^{op} \).

And can we define a functor \( \pi_! \) by

\[
\text{Tot}^\oplus \left( \cdots \longrightarrow \pi_2 \cdot ! \mathcal{E}_2 \longrightarrow \pi_1 \cdot ! \mathcal{E}_1 \longrightarrow \pi_0 \cdot ! \mathcal{E}_0 \right)
\]

where the \( \mathcal{E}_i \) are objects over \( S_i \) and we are given (quasi-)isomorphisms \( \mathcal{E}_i \rightarrow S(\delta)^! \mathcal{E}_j \) for all \( \delta : \Delta_j \rightarrow \Delta_i \) in a compatible way? We call the collection \( \{ \mathcal{E}_i \}_i \) together with these quasi-isomorphisms a cartesian object over \( S_\bullet \).
Two main questions, which will be addressed in section 8, then become: “Given a morphism between simplicial objects \( S \to T \), when are the categories of cartesian (resp. cocartesian) objects equivalent?” “When are the categories of cartesian objects on \( S \) and cocartesian objects over \( S^{op} \) equivalent?”.
4 Coherence of the six functors

4.1. Before proceeding, we come back to the question of making precise what a “six functor context” is. Whenever we are given a compatible bunch of functors like, for example, a tensor product, or a collection of pull-back morphisms, there is a general procedure: There is always a “better structure” in which the compatibility between functors becomes encoded in a composition law and its associativity. The first example is the one of a collection of pull-back functors (i.e. a pseudo-functor with values in categories):

<table>
<thead>
<tr>
<th>collection of functors and compatibilities</th>
<th>better structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^*$</td>
<td></td>
</tr>
<tr>
<td>$(gf)^* \sim g^<em>f^</em>$</td>
<td>opfbered category</td>
</tr>
<tr>
<td>$(fgh)^* \sim (gh)^<em>f^</em>$</td>
<td>$C \to S^{op}$</td>
</tr>
<tr>
<td>$h^<em>(fg)^</em> \sim h^*g^<em>f^</em>$</td>
<td></td>
</tr>
<tr>
<td>Hom$(f^*A, B) = \text{Hom}_f(A, B)$</td>
<td></td>
</tr>
</tbody>
</table>

Here we use the notation Hom$_f(A, B)$ to denote the preimage of $f$ under the opfibration $C \to S^{op}$. The second example is a tensor product (i.e. a monoidal category):

<table>
<thead>
<tr>
<th>collection of functors and compatibilities</th>
<th>better structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\otimes$</td>
<td></td>
</tr>
<tr>
<td>$(- \otimes -) \otimes - \sim - \otimes (- \otimes -)$</td>
<td>multicategory $C$</td>
</tr>
<tr>
<td>$(- \otimes -) \otimes (- \otimes -)$</td>
<td></td>
</tr>
<tr>
<td>$((- \otimes -) \otimes -)$</td>
<td></td>
</tr>
<tr>
<td>$(- \otimes (- \otimes -))$</td>
<td></td>
</tr>
<tr>
<td>$(- \otimes (- \otimes -)) \otimes -$</td>
<td></td>
</tr>
<tr>
<td>Hom$((A_1 \otimes (A_2 \otimes \cdots)), B) = \text{Hom}(A_1, \ldots, A_n; B)$</td>
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</tbody>
</table>

The third is a combination of the first two (i.e. a monoidal categegory together with a monoidal pseudo-functor [18 (3.6.7) b]):
Here $S^{\text{op}}$ is turned into a multicategory in the following way: A morphism $f \in \text{Hom}(S_1, \ldots, S_n; T)$ is a collection of morphims $f_i : T \to S_i$ for all $i$. This multicategory is representable (or opfibered over ·, see below), i.e. is a monoidal category with the tensor product being given by $\times$.

This multicategory is representable (i.e. opfibered over ·), closed (i.e. fibered over ·) and self-dual with tensor product and internal hom both given by $\times$ and having as unit the final object. It may seem strange that we did not take the structure of 2-morphisms into account. When considering the fibered derivator picture this will be corrected. Here we would have to define a notion of fibered 2-multicategory which probably can be done but is cumbersome. However, considering isomorphism classes of correspondences, in addition, encodes a certain compatibility of $!$ and $*$ for isomorphisms.
4.3. We now proceed to give the precise definition of a (op)fibered multicategory. Details can be found, for instance, in [12, 13]. The reader should keep in mind that a multicategory abstracts the properties of multi-linear maps, and indeed every monoidal category gives rise to a multicategory setting

$$\text{Hom}(A_1, \ldots, A_n; B) := \text{Hom}((A_1 \otimes (A_2 \otimes (\ldots))), B).$$ (3)

**Definition 4.4.** A multicategory $C$ consists of

- a class of objects $\text{Ob}(C)$,
- for every $n \in \mathbb{Z}_{\geq 0}$, and objects $X_1, \ldots, X_n, Y$ a class $\text{Hom}(X_1, \ldots, X_n; Y)$,
- an associative composition for objects $X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z$ and for each integer $1 \leq i \leq m$:
  $$\text{Hom}(X_1, \ldots, X_n; Y_i) \times \text{Hom}(Y_1, \ldots, Y_m; Z) \to \text{Hom}(Y_1, \ldots, Y_{i-1}, X_1, \ldots, X_n, Y_{i+1}, \ldots, Y_m; Z),$$
- for each $X$ object an identity: $\text{id}_X \in \text{Hom}(X; X)$,

satisfying the usual properties. A symmetric (braided) multicategory is given by an action of the symmetric (braid) groups, i.e. isomorphisms

$$\alpha : \text{Hom}(X_1, \ldots, X_n; Y) \to \text{Hom}(X_{\alpha(1)}, \ldots, X_{\alpha(n)}; Y)$$

for $\alpha \in S_n$, resp. $\alpha \in B_n$, forming an action, and compatible with composition in the obvious way (substitution of strings in the braid group).

In some references the composition is defined in a seemingly more general way; in the presence of identities these descriptions are, however, equivalent.

4.5. We leave the obvious definition of a functor between multicategories to the reader. Similarly there is a definition of a opmulticategory, in which we have classes

$$\text{Hom}(X; Y_1, \ldots, Y_n)$$

and similar data. If $C$ is a multicategory, there is a natural opmulticategory $C^{\text{op}}$. The trivial category is considered a multicategory setting all $\text{Hom}(\{\}; \ldots, \{\}; \{\})$ to the 1 element set. It is the final object in the category of multicategories.

To clarify the precise relation between multicategories and monoidal categories we have to define cartesian and cocartesian morphisms. It turns out that we can actually give a definition which is a common generalization of cocartesian morphisms in opfibered categories and those morphisms expressing the existence of a tensor product:
Definition 4.6. Consider a functor of multicategories \( p : C \to S \). We call a morphism
\[
\xi \in \text{Hom}(X_1, \ldots, X_n; Y)
\]
in \( C \) **cocartesian** w.r.t. \( p \), if for all \( Y_1, \ldots, Y_m, Z \) with \( Y_i = Y \), and for all \( f \in \text{Hom}(p(Y_1), \ldots, p(Y_m); p(Z)) \) the map
\[
\alpha \mapsto \alpha \circ \xi
\]
is bijective.
We call a morphism
\[
\xi \in \text{Hom}(X_1, \ldots, X_n; Y)
\]
in \( C \) **cartesian** w.r.t. \( p \) at the \( i \)-th slot, if for all \( Y_1, \ldots, Y_m, f \in \text{Hom}(p(Y_1), \ldots, p(Y_m); p(X_i)) \)
\[
\text{Hom}(Y_1, \ldots, Y_m; X_i) \to \text{Hom}_{p(\xi)}(Y_1, \ldots, Y_{i-1}, X_1, \ldots, X_n, Y_{i+1}, \ldots, Y_m; Z).
\]
\[\alpha \mapsto \xi \circ \alpha\]
is bijective.

\( p : C \to S \) is called a **opfibered multicategory** if every for every \( g \in \text{Hom}(S_1, \ldots, S_n; T) \) in \( S \) and objects \( X_i \) with \( p(X_i) = S_i \) there is some \( Y \) over \( T \) and some cocartesian morphism \( \xi \in \text{Hom}(X_1, \ldots, X_n; Y) \) with \( p(\xi) = g \).

\( p : C \to S \) is called a **fibered multicategory** if for every \( 1 \leq j \leq n \) and \( g \in \text{Hom}(S_1, \ldots, S_n; T) \) in \( S \) and objects \( X_i \) for \( i \neq j \) with \( p(X_i) = S_i \), and \( Y \) over \( T \) there is some \( X_j \) and some cartesian morphism w.r.t. the \( j \)-th slot \( \xi \in \text{Hom}(X_1, \ldots, X_n; Y) \) with \( p(\xi) = g \).

\( p : C \to S \) is called a **bifibered multicategories** if it is both fibered and opfibered.

A **morphism of (op)fibered multi-categories** is a commutative diagram of functors
\[
\begin{array}{ccc}
C_1 & \xrightarrow{F} & C_2 \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{G} & S_2
\end{array}
\]
such that \( F \) maps (co-)cartesian morphisms to (co-)cartesian morphisms.

It turns out that the composition of cartesian morphisms is cartesian (and similarly for cocartesian morphisms).

\[\overset{1}{\text{As with fibered categories there are weaker notions of cartesian which still uniquely determine a cartesian morphism (up to isomorphism) from given objects over a given multimorphism, however, do not imply that they are stable under composition. Similarly for cocartesian morphisms.}}\]
Lemma 4.7. 1. An opfibered multicategory \( p : \mathcal{C} \to \{\cdot\} \) is a monoidal category in a natural way. Conversely any monoidal category gives rise to an opfibered multicategory \( p : \mathcal{C} \to \{\cdot\} \) (via equation 3). A multicategory \( \mathcal{C} \) is a closed category if and only if it is fibered over \( \{\cdot\} \). In particular, the fibers of an (op)fibered multicategory \( p : \mathcal{C} \to \mathcal{S} \) are always closed/monoidal in the following sense: Given any functor of multicategories \( 2 \times x : \{\cdot\} \to \mathcal{S} \), the category \( \mathcal{C}_x \) of objects over \( x \) is closed/monoidal.

2. Given (op)fibered multicategories \( p : \mathcal{C} \to \mathcal{D} \) and \( q : \mathcal{D} \to \mathcal{E} \) also the composition \( q \circ p \) is an (op)fibered multicategory. In particular, if we have an opfibered multicategory \( p : \mathcal{C} \to \mathcal{S} \) and if \( \mathcal{S} \to \{\cdot\} \) is opfibered (i.e. \( \mathcal{S} \) is monoidal) then also \( \mathcal{C} \to \{\cdot\} \) is opfibered (i.e. \( \mathcal{C} \) is monoidal). The same holds dually. A morphism \( \alpha \) is (co)cartesian for \( q \circ p \) if and only if \( \alpha \) is (co)cartesian for \( q \) and \( q(\alpha) \) is (co)cartesian for \( p \).

In the case of an opfibration \( p : \mathcal{C} \to \{\cdot\} \), the tensor product \( A \otimes B \) is reobtained as the target of a cocartesian morphism in \( \text{Hom}(A,B; A \otimes B) \) which exists for any \( A, B \) by definition. Similarly, the unit is just the target of a cocartesian morphism in \( \text{Hom}(\{\cdot\}; 1) \) which exists by definition (the existence is also required for the empty set of objects).

The second part of the Lemma encapsulates the distinction between internal and external tensor product in a four (or six) functor context, see 4.11.

Example 4.8. Let \( \mathcal{S} \) be a usual category. If \( \mathcal{S} \) has coproducts, then \( \mathcal{S} \) may be turned into a symmetric multicategory setting

\[
\text{Hom}(X_1, \ldots, X_n; Y) := \text{Hom}(X_1; Y) \times \cdots \times \text{Hom}(X_n; Y).
\]

Let \( p : \mathcal{C} \to \mathcal{S} \) be an opfibered (usual) category. Any object \( X \) induces a canonical functor of multicategories \( x : \{\cdot\} \to \mathcal{S} \) with image \( X \), hence the fibers of an opfibered multicategory \( p : \mathcal{C} \to \mathcal{S} \) are monoidal and the datum \( p \) is equivalent to giving a pseudo-functor such that push-forwards \( f^* \) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a covariant monoidal pseudo-functor in e.g. [18, (3.6.7)].

Example 4.9. Similarly, if \( \mathcal{S} \) has products, \( S^{\text{op}} \) may be turned into a symmetric multicategory (or \( \mathcal{S} \) into a symmetric opmulticategory) setting

\[
\text{Hom}(X_1, \ldots, X_n; Y) := \text{Hom}(Y; X_1) \times \cdots \times \text{Hom}(Y; X_n).
\]

Let \( p : \mathcal{C} \to S^{\text{op}} \) be a opfibered (usual) category. Then an opfibered multicategory structure on \( p \) is equivalent to giving a monoidal structure on the fibers such that pull-backs \( f^* \) (along morphisms in \( \mathcal{S} \)) are monoidal functors and such that the compatibility morphisms between them are morphisms of monoidal functors. This is called a contravariant monoidal pseudo-functor in e.g. [18, (3.6.7)].

\footnote{This specifies also morphisms in \( \text{Hom}(X, \ldots, X; X) \), for all \( n \), compatible with composition.}
The point is that the notion of opfibered multicategory is *not restricted to* the situation of Examples 4.8 and 4.9. In particular, with the definition of $S^\text{cor}$ as in 4.2 we arrive at the sought-for precise definition of the six functors:

**Definition 4.10.** Let $S$ be a category. A **Grothendieck six functor context** on $S$ is a bifibered symmetric multicategory

\[ C \to S^\text{cor}. \]

4.11. We have a morphism of opfibered (over $\{\cdot\}$) multicategories $S^\text{op} \to S^\text{cor}$ where $S^\text{op}$ is equipped with the multicategory structure as in 4.9. However there is no reasonable morphism of opfibered multicategories $S \to S^\text{cor}$ (There is no compatibility involving only $\otimes$ and $!$). From a Grothendieck six functor context, we get the operations $g_+$, $g^*$ as pullback and push-forward along the correspondence

\[
g: X \to Y; X
\]

We get $f^!$ and $f_!$ as pullback and pushforward along the correspondence

\[
f: X \to Y; X
\]

We get $A \otimes B$ for objects $A,B$ above $X$ as the target of “the” cartesian morphism

\[ \otimes \in \text{Hom}_{\xi_X}(A,B; A \otimes B) \]

over the correspondence

\[ \xi_X = \left( \begin{array}{ccc} X & X & X \\ X & X & X \end{array} \right) \]

Alternatively, we have

\[ A \otimes B := \Delta^*(A \boxtimes B) \]

Here $\Delta^*$ is the pushforward along the correspondence

\[
\left( \begin{array}{ccc} X & X \\ \Delta & f \\ X \times X & X \end{array} \right)
\]

induced by the canonical $\xi_X \in \text{Hom}(X,X; X)$, and $\boxtimes$ is the absolute monoidal product which exists due to the fact that by Lemma 4.7 the composition $C \to \cdot$ is opfibered, too, i.e. $C$ is monoidal.
Exercise 4.1. Derive from the definition of bifibered multicategory over $\mathcal{S}^{\text{cor}}$ that the absolute monoidal product $A \boxtimes B$ can be reconstructed from the fibre-wise product as $\text{pr}_1^* A \boxtimes \text{pr}_2^* B$ on $X \times Y$, whereas the absolute $\text{HOM}(A, B)$ is given by $\mathcal{HOM}(\text{pr}_1^* A, \text{pr}_2^* B)$ on $X \times Y$. In particular $\mathcal{D} A := \text{HOM}(A, 1)$ is given by $\mathcal{HOM}(A, \pi^!)$ for $\pi : X \to \cdot$ the final morphism.

Lemma 4.12. Given a Grothendieck six functor context on $\mathcal{S}$ for the six operations as extracted in 4.11 there exist naturally all compatibility isomorphisms listed in 1.1.

Proof. They are now all consequences of the fact that compositions of cocartesian morphisms are cocartesian. For example, the projection formula $(\otimes, !)$ is derived from the following composition in $\mathcal{S}^{\text{cor}}$:

$$
\begin{array}{c}
\ (X \quad Y) \quad (X \quad Y) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
5 Fibered derivators

5.1. As we have seen, the questions of homological and cohomological descent cannot be treated
in a satisfactory way by considering a “classical” Grothendieck six functor context \( C \to S^{\text{cor}} \) whose
fibers are derived categories (of sheaves, \( D \)-modules, motives, etc.). This does not allow to consider
operations between coherent diagrams in \( C \) over arbitrary diagrams in \( S \) or even over diagrams of
correspondences i.e. diagrams in \( S^{\text{cor}} \). Enhancing only the fibers as derivators, is also obviously
not sufficient.

There are two approaches to correct this:

1. To enlarge the domain of a derivator to “diagrams in \( S \)” (or here even to “diagrams in \( S^{\text{cor}} \)”).
   This road has been taken for example in the work of Ayoub, Cisinski and Deglise \[4, 5, 8\]
   under the name algebraic derivator.

2. To consider fibered (multi-)derivators, which are morphisms of pre-(multi-)derivators, satisfying
   some axioms.

The two approaches are morally equivalent. In view of the philosophy that it should be preferred
to consider a compact structure like fibered multicategories instead of a bunch of functors and
compatibilities, we prefer the second approach. In particular with the first approach, taking the
monoidal aspect into account, i.e. to study a six functor context, we would run into the same
coherence problems as before.

Let \( \text{Dia} \) be a category of diagrams (full subcategory of the category of small categories satisfying
some closure properties).

**Definition 5.2.** A pre-derivator of domain \( \text{Dia} \) is a contravariant (strict) 2-functor
\[
D : \text{Dia}^{1-\text{op}} \to \text{CAT}
\]
into the “category”\(^3\) of categories.

A pre-multi-derivator of domain \( \text{Dia} \) is a contravariant (strict) 2-functor
\[
D : \text{Dia}^{1-\text{op}} \to \text{MCAT}
\]
into the “category” of multicategories. A morphism of pre-derivators is a natural transformation.
For a morphism \( \alpha : I \to J \) in \( \text{Dia} \) the corresponding functor \( D(\alpha) \)
\[
D(J) \to D(I)
\]
will be denoted by \( \alpha^* \).

We call a pre-multi-derivator symmetric (resp. braided), if the images are symmetric (resp. braided),
and the morphisms \( \alpha^* \) are compatible with the actions of the symmetric (resp. braid
groups).

\(^3\)where “category” has classes replaced with 2-classes (or, if you prefer, is constructed w.r.t. a larger universe).
Definition 5.3. We consider the following axioms on a pre-(multi-)derivator $\mathbb{D}$:

(Der1) For $I, J$ in $\text{Dia}$, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence. Moreover $\mathbb{D}(\emptyset)$ is not empty.

(Der2) For $I$ in $\text{Dia}$ the ‘underlying diagram’ functor

$$\text{dia} : \mathbb{D}(I) \to \text{Hom}(I, \mathbb{D}())$$

is conservative.

In addition, we consider the following axioms for a morphism of pre-(multi-)derivators $p : \mathbb{D} \to \mathbb{S}$ (here we write down the left versions of the axioms; they all have corresponding dual right versions):

(FDer0 left) For each $I$ in $\text{Dia}$ the morphism $p$ specializes to an opfibered (multi-)category and the morphisms $\alpha : I \to J$ induce a diagram

$$\begin{array}{ccc}
\mathbb{D}(J) & \xrightarrow{\alpha^*} & \mathbb{D}(I) \\
\downarrow & & \downarrow \\
\mathbb{S}(J) & \xrightarrow{\alpha^*} & \mathbb{S}(I)
\end{array}$$

of opfibered (multi-)categories, i.e. the top horizontal functor maps cocartesian arrows to cocartesian arrows.

(FDer3 left) For each morphism $\alpha : I \to J$ in $\text{Dia}$ and $S \in \mathbb{S}(J)$ the functor $\alpha^* : \mathbb{D}(S) \to \mathbb{D}(I)\alpha^*S$ has a left-adjoint $\alpha^*_S$ (we often omit the base $S$ from the notation, it will always be clear from the context).

(FDer4 left) For each morphism $\alpha : I \to J$ in and object $j \in J$ and the 2-cell

$$\begin{array}{cc}
I \times_J j & \xrightarrow{i} & I \\
\downarrow & \alpha^* & \downarrow \\
\{j\} & \xrightarrow{j} & J
\end{array}$$

we get that the induced natural transformation of functors $\alpha_j^*(\mathbb{S}(\mu)) \mu^* \to j^* \alpha_1$ is an isomorphism.

---

4 The diagram $I \times_J j$ is the 2-pullback of the diagram $\begin{pmatrix}
I \\
\downarrow \\
\{j\} & \xrightarrow{j} & J
\end{pmatrix}$ and is also called slice or comma category.

5 This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.
(FDer5 left) (only for the multi-deriver case needed). For any $I \in \text{Dia}$ and the morphism $\pi : I \to \cdot$, and a morphism $\xi \in \text{Hom}(S_1, \ldots, S_n; T)$ in $S(\cdot)$ for some $n \geq 1$, the natural transformations of functors

$$\pi(\alpha^*\xi)_\bullet(\pi^*\cdot, \pi^*\cdot, \ldots, \pi^*\cdot) \cong \xi(\cdot, \ldots, \pi_\cdot, \cdot, \ldots, \cdot)$$

are isomorphisms.

In axiom (FDer4 left), which makes only sense in the presence of (FDer0 left) and (FDer3 left), $(S(\mu))_\bullet$ is an arbitrary choice of push-forward along $S(\mu)$. Similarly in (FDer5 left), $\xi_\bullet$ is a functor defined by choosing a cocartesian arrow, which makes only sense in the presence of (FDer0 left).

**Question 5.4.** It seems natural to allow also multi-categories, in particular operads, as domain for a fibered multi-deriver. The author however did not succeed in writing down a neat generalization of (FDer3–4) which would encompass (FDer5).

**Definition 5.5.** A morphism of pre-(multi-)derivators $p : D \to S$ with domain $\text{Dia}$ is called a left fibered (multi-)deriverator with domain $\text{Dia}$, if axioms (Der1–2) hold for $D$ and $S$ and (FDer0–5 left) hold for $p$. Similarly it is called a right fibered (multi-)deriverator with domain $\text{Dia}$, if instead the corresponding dual axioms (FDer0–5 right) hold. It is called fibered if it is both left and right fibered.

**5.6.** Let $S \in S(\cdot)$ be an object. In particular, given a (left, resp. right) fibered (multi-)deriverator $p : D \to S$, the association

$$I \mapsto D(I)p^*S,$$

where $p : I \to \cdot$ is the projection, defines a (left, resp. right) derivator in the usual sense which we call the fibre of $p$ over $S$. In the multi-setting, the fibre is monoidal in the sense of [10, Definition 2.4], if $S$ defines a section $\{\cdot\} \to S$ of pre-multi-derivators. The following two last axioms involve only these fibers.

More generally, if $S \in S(J)$ we may consider the association

$$I \mapsto D(I \times J)p^*S,$$

where $p : I \times J \to J$ is the projection. This defines again a (left, resp. right) derivator in the usual sense which we call the fibre of $p$ over $S$.

**Definition 5.7.** Let $p : D \to S$ be a (left and right) fibered derivator. We call $D$ pointed (relative to $p$) if the following axiom holds:

(FDer6) For any $S \in S(\cdot)$, the category $D(\cdot)S$ has a zero object.

**Definition 5.8.** Let $p : D \to S$ be a (left and right) fibered derivator. We call $D$ stable (relative to $p$) if the fibers of $p$ are strong

$^6$See [11, Definition 1.13] for this condition
(FDer7) For any $S \in \mathcal{S}(\cdot)$, in the category $\mathbb{D}(\square)_{/S}$ an object is homotopy cartesian if and only if it is homotopy cocartesian.

5.9. Recall from [11] that axiom (FDer7) implies that the fibers of a stable fibered derivator are triangulated categories in a natural way. Since push-forward, resp. the (relative and absolute) tensor product commute with homotopy colimits (FDer5 left, cf. also 5.10 below) they induce, in particular, triangulated functors between the fibers.

5.10. Let $\mathbb{D} \to \mathcal{S}$ be a left fibered derivator with some domain $\text{Dia}$. Let $\alpha : I \to J$ be a morphism in $\text{Dia}$ and let $f : S \to T$ be a morphism in $\mathcal{S}(J)$. Axiom (FDer0 left) implies that we have a canonical isomorphism

$$\left(\alpha^*(f)\right)_* \alpha^* \to \alpha^* f_*$$
determined by the choice of push-forward functors. We get an associated exchange morphism

$$\alpha^! \left(\alpha^*(f)\right)_* \to f_* \alpha^! \tag{4}$$

One can show that axiom (FDer4 left) implies that this is an isomorphism. In other words $f_*$ commutes with homotopy left Kan extensions (in particular with homotopy colimits). This also follows from (FDer0 left) and (FDer0 right) because, in that case, $f_*$ is a left adjoint. In particular (FDer5 left) follows from (FDer4 left) if we are considering plain fibered derivators (not multi-derivators). Also in the multi-case, (FDer5 left) follows from (FDer0 left) and (FDer0 right) together.

Analogously to Lemma 4.7, we have the following:

**Proposition 5.11.** Let

$$\begin{array}{ccc}
E & \xrightarrow{p_1} & \mathbb{D} & \xrightarrow{p_2} & \mathcal{S}
\end{array}$$

be two left (resp. right) fibered (multi-)derivators. Then also the composition $p_2 \circ p_1 : E \to \mathcal{S}$ is a left (resp. right) fibered (multi-)derivator.

The canonical source for fibered multi-derivators, at least if the base $\mathcal{S}$ is associated with a usual multi-category $\mathcal{S}$, is the following:

**Theorem 5.12.** Suppose we are given a bifibered multi-category $\mathcal{D} \to \mathcal{S}$ and a collection of model category structures

$$(\mathcal{D}_S, \text{Fib}_S, \text{Cof}_S, \mathcal{W}_S) \tag{5}$$
on each $\mathcal{D}_S$ such that for every morphism $f \in \text{Hom}(S_1, \ldots, S_n; T)$ the “push-forward”

$$X_1, \ldots, X_n \mapsto f_*(X_1, \ldots, X_n)$$
is a left Quillen functor in $n$-variables.

---

7Considering $n = 1, 2$ is sufficient!
Consider the disjoint union $\mathcal{W}$ of the $\mathcal{W}_S$ and for each diagram $I$ the class $\mathcal{W}_I$ of morphisms in $\text{Fun}(I, \mathcal{D})$ which are point-wise in $\mathcal{W}$.

Then the association

$$\mathcal{D} : I \mapsto \text{Fun}(I, \mathcal{D})[\mathcal{W}_I^{-1}]$$

is a fibered multi-derivator over $\mathcal{S}$ whose fibers are just the associated derivators (monoidal if a section $\{\} \to \mathcal{S}$ of multi-categories exists) of the model categories \([5]\). It is pointed (resp. stable) if the model categories \([5]\) are pointed (resp. stable).

The theorem is shown analogously to Cisinski \([6, \text{Théorème 6.11.}]\) by restricting first to the class of directed, resp. inverse diagrams.

The fact that it is reasonable to just invert the union of the $\mathcal{W}_S$ to get the fibered category associated with the derived push-forwards and pull-backs the author learned from Deligne \([3, \text{Exposé XVII, §9}]\).

5.13. In view of what was said in the introduction, every left fibered derivator $\mathcal{D} \to \mathcal{S}$ does indeed give rise to a 2-functor

$$\mathcal{D} : \text{Dia}(\mathcal{S})^{1-\text{op}} \to \text{CAT}$$

$$D = (I, F) \mapsto \mathcal{D}(D) := \mathcal{D}(I)_F$$

where $\text{Dia}(\mathcal{S})$ is the 2-category of pairs $(I, F)$ where $I \in \text{Dia}$ and $F \in \mathcal{S}(I)$ and a morphism $(I, F) \to (J, G)$ is given by a functor $\alpha : I \to J$ and a morphism $f : F \to \alpha^*G$. The pullback $\mathcal{D}(\alpha, f)$, denoted also by $(\alpha, f)^*$, is given by $f^*\alpha^*$. The 2-morphisms are defined in the obvious way.

Note: there is a dual version $\text{Dia}^{op}(\mathcal{S}) = \text{Dia}(\mathcal{S}^{op})^{2-\text{op}}$ where $f : F \to \alpha^*G$ is replaced by $f : \alpha^*G \to F$.

All of the following has a corresponding dual version, which we won’t state explicitly.

5.14. For the rest of this section, we assume that $\mathcal{S}$ is a category and $\mathcal{S}$ is its associated pre-derivator. We also assume that $\mathcal{S}$ has fibered products. Then there is a notion of comma object $D_1 \times_{D_2} D_3$ in $\text{Dia}(\mathcal{S})$ which satisfies a similar universal property in the 2-category $\text{Dia}(\mathcal{S})$ as the usual slice category satisfies in $\text{Dia}$.

Definition 5.15. Let $p : \mathcal{D} \to \mathcal{S}$ be a left fibered derivator (which also satisfies $\text{FDer0 right}$), where $\mathcal{S}$ is the pre-derivator associated with a category $\mathcal{S}$. We call a morphism $f : U \to X$ in $\mathcal{S}$ $\mathcal{D}$-local if the following hold:

(Dloc1) The morphism $f$ satisfies base change, i.e. for a pullback-diagram

$$
\begin{array}{ccc}
U \times_X Y & \xrightarrow{F} & Y \\
\downarrow{G} & & \downarrow{g} \\
U & \xrightarrow{f} & X
\end{array}
$$

\[8\]The localization $\mathcal{D}[W^{-1}]$ is defined for multi-categories in a similar manner as for categories (only 1-ary morphisms become inverted!).

\[9\]which starts with the words “Le rédacteur insiste pour que le lecteur s'abstienne de lire ce §.”
where \( g \) is any morphism in \( \mathcal{S} \), and if \( \tilde{F} \) and \( \tilde{f} \) are cartesian, and \( \tilde{g} \) is cocartesian in a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{G} & & \downarrow{\tilde{g}} \\
C & \xrightarrow{\tilde{f}} & D
\end{array}
\]

in \( \mathcal{D}({\cdot}) \) above it, then also \( \tilde{G} \) is cocartesian.

\( (Dloc2) \) The functor

\[
f^* : \mathcal{D}({\cdot})_X \to \mathcal{D}({\cdot})_U
\]

commutes with homotopy colimits, too.

Similarly a morphism \( f \) in \( \mathcal{S}(I) \) is called local if it is object-wise local.

The associated pseudo-functor satisfies the following:

**Theorem 5.16.** Let \( \mathcal{D} \to \mathcal{S} \) be a left fibered derivator with existing pull-backs (i.e. \( FDer0 \) right) holds true, too) such that \( \mathcal{S} \) is the pre-derivator associated with a category \( \mathcal{S} \) (with existing fibered products). Then the associated pseudo-functor satisfies the following properties:

1. For a morphism of diagrams \( (\alpha, f) : D_1 \to D_2 \) the corresponding pullback

\[
(\alpha, f)^* : \mathcal{D}(D_2) \to \mathcal{D}(D_1)
\]

has a left-adjoint \( (\alpha, f)_! \).

2. For a slice diagram:

\[
\begin{array}{ccc}
D_1 \times_{D_3} D_2 & \xrightarrow{p_1} & D_1 \\
\downarrow{p_2} & \phi^{-\alpha} & \downarrow{\beta_1} \\
D_2 & \xrightarrow{\beta_2} & D_3
\end{array}
\]

the corresponding exchange morphism

\[
p_2^! p_1^* \to \beta_2^* \beta_1!
\]

is an isomorphism in \( \mathcal{D}(D_2) \) provided \( \beta_2 \) is \( \mathcal{D} \)-local (in particular, if the underlying morphism \( f_2 \) of \( \beta_2 \) is an isomorphism — in which case we say that \( \beta_2 \) is of pure diagram type).

1. If \( D_1 = (I_1, F_1) \) and \( D_2 = (I_2, F_2) \) then for \( \alpha : I_1 \to I_2 \) and \( f : F_1 \to \alpha^* F_2 \) the left adjoint \( (\alpha, f)_! \) is obviously just given by \( \alpha_! f \).
6  (Co)homological descent revisited

6.1. We will explain in this section how the formalism of fibered derivators gives a neat solution to the problems of homological and cohomological descent. For this we assume that $\mathcal{S}$ is a category equipped with a Grothendieck pre-topology and $\mathcal{S}$ is its associated pre-deriverator.

**Definition 6.2.** A fibered derivator $p: \mathcal{D} \to \mathcal{S}$ is called **local** w.r.t. the pre-topology on $\mathcal{S}$, if

1. Every morphism $U_i \to S$ which is part of a covering family is $\mathcal{D}$-local (see 5.15).

2. For a covering $\{f_i : U_i \to S\}$ the family

$$f^*_i : \mathcal{D}(S) \to \mathcal{D}(U_i)$$

is jointly conservative.

**Definition 6.3.** A set $W$ of morphisms in $\text{Dia}(\mathcal{S})$ is called a **localizer**, if it satisfies the following properties:

(L1) $W$ is weakly saturated.

(L2) If $D = (I,F) \in \text{Dia}(\mathcal{S})$, and if $I$ has a final object $e$, then the projection $D \to (\{e\}, F(e))$ is in $W$.

(L3) If there is a commutative diagram in $\text{Dia}$

$$
\begin{array}{ccc}
I_1 & \xrightarrow{\alpha} & I_2 \\
\downarrow & & \downarrow \\
I_3 & \xleftarrow{\beta} & I_3
\end{array}
$$

and $w: D_1 := (I_1,F_1) \to D_2 := (I_2,F_2)$ is an extension of the horizontal morphism and

$$w_i : D_1 \times_{I_3} \{i\} \to D_2 \times_{I_3} \{i\}$$

is in $W$ for all $i$, then $w \in W$.

(L4) If there is a commutative diagram of diagrams

$$
\begin{array}{ccc}
D_1 & \xrightarrow{w} & D_2 \\
\downarrow & & \downarrow \\
(\{\cdot\}, S) & \xleftarrow{(\cdot)} & (\{\cdot\}, S)
\end{array}
$$

and $\{U_j \to S\}$ is a cover and the corresponding morphism

$$w_j : D_1 \times_S U_j \to D_2 \times_S U_j$$

is in $W$ for all $j$, then $w \in W$. 

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If $S$ is the trivial category, this is precisely the definition of localizer of Grothendieck.

The class of localizers is obviously closed under intersection, hence there is a smallest localizer $W_{\text{min}}$. In the case $S = \{\cdot\}$ Cisinski [7] showed that $W_{\text{min}} = W_{\infty}$, the class of functors $\alpha$ such that $N(\alpha)$ is a weak equivalence. There should be a connection between this more general notion and the homotopy theory of simplicial presheaves on $S$. In particular

**Lemma 6.4.** Let $\mathcal{W}$ be a localizer in $\text{Dia}(S)$. If $F : (\Delta^{\text{op}}, S) \to (\Delta^{\text{op}}, T)$ is a morphism of simplicial diagrams in $S$ which is a finite hypercover, then $F$ is in $\mathcal{W}$.

or more simply:

**Example 6.5 (Mayer-Vietoris).** The easiest example of an interesting morphism in $\mathcal{W}$ arises from a cover $\{U_1 \to S, U_2 \to S\}$ consisting of two morphisms. Then the projection

$$p : \begin{pmatrix} U_1 \cap U_2 & \rightarrow & U_1 \\ \downarrow & & \downarrow \\ U_2 & \rightarrow & S \end{pmatrix}$$

is in $\mathcal{W}$, as is easily derived from the axioms (L1–4).

**Definition 6.6.** Let $I$ be a diagram and $\alpha : I \to E$ be a functor. We say that an object $A \in D(I)$ is $E$-(co)cartesian, if for any $\mu : i \to j$ in $I$ mapping to an identity in $E$, the corresponding morphism $D(\mu)(A) \in \text{Mor}(D(\cdot))$ is (co)cartesian. This defines a full subcategory $D(I)^{E\text{-cart}}$ (resp. $D(I)^{E\text{-cocart}}$) of $D(I)$, and $D(I)_F^{E\text{-cart}}$ (resp. $D(I)_F^{E\text{-cocart}}$) of $D(I)_F$ for any $F \in S(I)$. If $E$ is the trivial category, we omit it and talk about (co)cartesian objects.

**Main theorem 6.7.** Let $D \to S$ be a stable fibered derivator with domain $\text{Dia}$ which is well-generated. The set $W_D$ of those $f : D_1 \to D_2$ in $\text{Dia}(S)$ such that $f^* : D(D_2) \to D(D_1)$ induces an equivalence

$$f^* : D(D_2)^{E\text{-cart}} \to D(D_1)^{E\text{-cart}}$$

form a localizer.

Note that, in particular, $f^*$ induces an equivalence between categories of cartesian objects, if $f$ is a finite hypercover $(\Delta^{\text{op}}, S) \to (\Delta^{\text{op}}, T)$. There is a variant including arbitrary hypercovers which, however, requires more axioms.

6.8. To compare this to the question raised in 3.2. Let $D_1 = (I, F)$ and let $D_2$ be of the form $(\cdot, S)$ and let $f$ be given by $\varphi : F \to p^*S$, where $p : I \to \cdot$ is the projection, then obviously $D(D_2)^{E\text{-cart}} = D(D_2) = D(S)$ and the inverse functor to $f^*$ is just $f_*$, which is the push-forward $\varphi_\bullet$ along $\varphi$ followed by a homotopy colimit (see Theorem 5.16). In general, the inverse is given by $\varphi_\bullet$ followed by a cartesian projection.

---

[10] This means that the fibers of the derivators, which are triangulated categories, are well-generated in the sense of Neeman [15 20]. Actually, it suffices to consider the fibers over objects in $S(\cdot)$.
6.9. Resuming Example 6.5 If $D$ is local, and $A \in D(S)$ this yields

$$A \cong pp^* A,$$

i.e. the homotopy colimit of

$$i_{1,2} * i_{1,2}^* A \longrightarrow i_{1,1} * i_{1,1}^* A$$

$$\downarrow$$

$$i_{2,2} * i_{2,2}^* A$$

is isomorphic to $A$ which, since $D \rightarrow S$ is stable, translates to the usual distinguished triangle

$$i_{1,2} * i_{1,2}^* A \longrightarrow i_{1,1} * i_{1,1}^* A \oplus i_{2,2} * i_{2,2}^* A \longrightarrow A \longrightarrow i_{1,2} * i_{1,2}^* A[1]$$

in the language of triangulated categories.

6.10. An intrinsic example of a fibered derivator, where the base is not associated with a usual category: If $D$ is a stable derivator (not fibered) which is well-generated then we get an induced fibered derivator:

$$D^{\text{cart}} \rightarrow H$$

where $D^{\text{cart}}$ is the pre-derivator

$$I \mapsto (F \in \mathbb{H}(I), X \in D(\int F^{I-\text{cart}}))$$

and where $\mathbb{H}$ is the derivator of homotopy types, i.e. weak equivalence classes of diagrams (equivalently: of topological spaces). Note that in the non-fibered setting “cart” and “cocart” are synonymous. Here $F$ is considered to be a functor in diagrams $I \rightarrow \text{Dia}$ and $\int F$ is the corresponding Grothendieck construction. In particular for $I = \cdot$, we have a corresponding pseudo-functor

$$\mathcal{HOT} = \mathbb{H}(\cdot) \rightarrow \mathcal{CAT}$$

$$J \mapsto D(J)^{\text{cart}}$$

6.11. Of course, if $S$ is a category with a Grothendieck topology, $S$ is its associated pre-derivator, and $D \rightarrow S$ is a local stable fibered derivator\footnote{well-generated — to be able to apply the theory.}, then we would similarly like to get a fibered derivator

$$D^{\text{cart}} \rightarrow S^{ss}$$

where $S^{ss}$ is the derivator of simplicial (pre-)sheaves on $S$. The results in this section go in this direction.
7 The goal

7.1. Let $S$ be a category with fibered products. $S^{\text{cor}}$ is in a natural way a bicategory in which all 2-morphisms are invertible. There exists an associated symmetric pre-multi-derivator $S^{\text{cor}}$. It can be defined precisely as follows:

**Definition 7.2.** We define the simplicial class $S^{\text{cor}}$, which is, in fact an $\infty$-category and even the nerve of a bi-category. Let $S_0^{\text{cor}}$ be the class of objects in $S$, let $S_1^{\text{cor}}$ be the class of diagrams of the form

$$
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{}
\end{array}
$$

$Y$

Let $S_2^{\text{cor}}$ be the class of diagrams of the form

$$
\begin{array}{ccc}
C & \xrightarrow{f_1} & A \\
\downarrow{g_1} & & \downarrow{f_2}
\end{array}
$$

$$
\begin{array}{ccc}
A & \xleftarrow{g_2} & B \\
\downarrow{f_3} & & \downarrow{g_3}
\end{array}
$$

where the square is cartesian.

$S_i^{\text{cor}}$ for $i \geq 3$ are defined by forcing the unique lifting property w.r.t. $\partial \Delta_i \hookrightarrow \Delta_i$.

We define $S^{\text{cor}}$ to be the associated pre-derivator. One can extend this construction to see that it is a symmetric pre-multi-derivator in a natural way.

**Definition 7.3.** We define a Grothendieck six functor context (derivator version) over $S$ to be a stable (left and right) fibered symmetric multi-derivator

$$D \to S^{\text{cor}}.$$  

\[\text{12}\] The pre-derivator associated with a simplicial class (or $\infty$-category): Let $S$ be a simplicial class, i.e. a functor

$$\Delta \to \text{CLASS}$$

into the “category” of classes. We associate with it the pre-derivator

$$S : I \mapsto \text{Ho}(\text{Hom}(N(I), S)).$$

This means that the category $S(I)$ has as objects the morphisms $\alpha : N(I) \to S$. The class of morphisms is freely generated by morphisms of the following form: $\mu : N(I \times \Delta_1) \to S$ considered to be a morphism from its restriction to $N(I \times \{0\})$ to its restriction to $N(I \times \{1\})$ modulo the relations given by morphisms $\nu : N(I \times \Delta_2) \to S$, i.e. if $\nu_1, \nu_2$ and $\nu_3$ are the restrictions of $\nu$ to the 3 faces of $\Delta_2$, then we have $\nu_3 = \nu_2 \circ \nu_1$. 28
The goal is to construct such contexts in interesting situations (e.g. etale constructible sheaves over schemes, or (pro-)quasi-coherent sheaves over schemes). Imitating the classical constructions of $f_!$ using compactifications and $f^!$ as its right adjoint using Brown representability, we are able to achieve this. The degree of generality (in particular which category Dia can be taken) is not yet completely clear. The input is, in any case, a bifibration of monoidal model categories like in 5.12 over $S^{op}$ satisfying a bunch of additional axioms, including the existence of compactifications.
8 The six functors for stacks

Consider a stable fibered derivator $D \rightarrow S$ as in the last section which is well-generated. We neglect the multi- (=monoidal) aspect in this section.

8.1. Consider again the case that $S$ is equipped with a Grothendieck pretopology. Let $X \circ, Y \circ$ be simplicial objects, i.e. objects of $\text{Fun}(\Delta^{\text{op}}, S)$. Let $\alpha: X \circ \rightarrow Y \circ$ be a morphism. As mentioned in 3.5, we would like to define 4 functors $\alpha_!, \alpha^!, \alpha_*, \alpha^*$ between the two categories

$$D((X \circ)^{\text{op}})^{\text{cocart}} \quad D((Y \circ)^{\text{op}})^{\text{cart}}$$

where $(X \circ)^{\text{op}}$ is the opposite diagram in $\text{Fun}(\Delta, S^{\text{op}})$. If $X \circ$ represents a 1-stack, which we will assume in a second then, by cohomological descent, these categories do only depend on this stack up to equivalence (this assumes that the restriction $D \rightarrow S^{\text{op}}$ is colocal, cf. 6.2).

We already have the adjoint pair of functors $\alpha^*$ and $\alpha_*$. Note (cf. also 6.8) that $\alpha_*$ is the usual pull-back $\alpha^*$ (pull-back in the fibered derivator $D \rightarrow S^{\text{op}}$) followed by a right cocartesian projection. In the same way, we have already well defined adjoint functors $\alpha_!$ and $\alpha^!$ between the two categories

$$D(X \circ)^{\text{cart}} \quad D(Y \circ)^{\text{cart}}$$

which, by homological descent, do only depend on the stack up to equivalence (provided that the restriction $D \rightarrow S$ is local, cf. 6.2).

We now have

**Proposition 8.2.** If $X \circ$ presents a 1-stack and the fibered derivator has certain additional locality properties w.r.t. the Grothendieck pretopology, then we have an equivalence

$$D(X \circ)^{\text{cart}} \cong D((X \circ)^{\text{op}})^{\text{cocart}}$$

Therefore we dispose of the 4 functors in either version. These satisfy the base-change formula w.r.t. the fiber product of 1-stacks.

**Idea of proof.** Here the utility of the general derivator version of a four (or six) functor context is revealed: We construct a diagram of correspondences $X^{\circ} \in \text{Fun}(\Delta \times \Delta^{\text{op}}, S^{\text{cor}})$ (for this it is crucial that $X \circ$ presents a 1-stack) with maps of diagrams in $S^{\text{cor}}$

$$\xymatrix{ & X^{\circ} \ar[dl] \ar[dr] & \\ X \circ & & X^{\circ}^{\text{op}} }$$

and show that those induce equivalences

$$D(X \circ)^{\text{cart}} \cong D(X^{\circ})^{\Delta^{\text{op}}-\text{cart}, \Delta-\text{cocart}} \cong D((X \circ)^{\text{op}})^{\text{cocart}}$$

The base change formula can be proven using the symmetric diagrams $X^{\circ}$. The proof is completely formal.
Unfortunately, so far, the author has not been able to make this construction coherent, that is, to construct a fibered derivator over correspondences of stacks. This would probably be necessary to extend the result to higher stacks.
References


