Generalized automorphic sheaves and the proportionality principle of Hirzebruch-Mumford

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Abstract

We axiomatize the algebraic structure of toroidal compactifications of Shimura varieties and their automorphic vector bundles. We propose a notion of generalized automorphic sheaf which includes the sheaves of sections of automorphic vector bundles with all kinds of prescribed vanishing orders along strata in the compactification, their quotients, as well as e.g. Jacobi forms, and almost holomorphic modular forms. Using this machinery we give a short and purely algebraic proof of the proportionality theorem of Hirzebruch and Mumford. The main motivation was however to create a theory which can be applied to other compactified moduli spaces to be able to investigate "modular forms" on them and their "Fourier-Jacobi expansions" purely algebraically.

Notation

We write $[n]$ for the unordered set $\{1,\dots,n\}$ and Δ_n for the poset $\{1 \le 2 \cdots \le n\}$ also regarded as a category. For a scheme, formal scheme, or stack X we write $[\ X\text{-}\mathbf{Coh}\ ]$ (or sometimes $[\ \mathcal{O}_X\text{-}\mathbf{Coh}\ ]$) for the category of coherent sheaves on X and $[\ X\text{-}\mathbf{Qcoh}\ ]$ for the category of quasi-coherent sheaves. For an algebraic group G we denote by $\mathfrak{g}$ its Lie algebra.

Contents

1 Introduction

This article was motivated by the search for an axiomatization of the algebraic structure of toroidal compactifications of Shimura varieties and their automorphic vector bundles, which might also be applied to other moduli spaces (which, for instance, carry certain families of Calabi-Yau threefolds) to study "modular forms" on them and their "Fourier expansions" and "Fourier-Jacobi expansions" purely algebraically. While this is the content of work in progress, this article focuses on the axiomatization, and explains that the axioms fit the situation for toroidal compactifications of Shimura varieties. To show that the language is sufficiently powerful we distill a few simple axioms that imply the famous proportionality theorem of of Hirzebruch [6] and Mumford [12], thus providing a purely algebraic proof thereof.

We now describe the axiomatization more in detail. All varieties and formal schemes are understood over a field k of characteristic zero. We define a *formally toroidal scheme* (Definition 2.1.3) to be a formal scheme together with an action of $\mathbb{M}_m^n$, where $\mathbb{M}_m$ is the multiplicative monoid on the affine line, which looks like the completion of a (partially) compactified $\mathbb{G}_m^n$-torsor on a variety along a boundary stratum. An *abstract toroidal compactification* (Definition 2.3.2) is defined as a smooth variety $\overline{M}$ with a divisor of strict normal crossings D such that the completions along all strata (of the stratification defined by D) carry the structure of a formally toroidal scheme in a compatible way w.r.t. the partial ordering of the strata. In Section 2.4 we explain that toroidal compactifications of mixed Shimura varieties in the sense of Pink [13] indeed give rise to such objects.

Moreover, we introduce the notion of *automorphic data* (Definition 3.1.1) on an abstract toroidal compactification. If $D = \varnothing$ this is just the datum of a "compact dual" $M^\vee$ and a "period torsor" B equipped with morphisms

$$M \xleftarrow{\pi} B \xrightarrow{p} M^\vee$$

where $\pi : B \to M$ is a right-torsor under a reductive group P_M and $M^\vee$ a component of the moduli space of parabolics of P_M (a flag variety). The morphism p is P_M-equivariant.

This situation is well-known in the theory of Shimura varieties. In this case B is called the *standard principal bundle* and is (philosophically) the bundle of trivializations of the de Rham realization of the universal motive (associated with a representation ρ of the defining group P_M) together with its natural P_M-structure. The morphism p in this case is induced by the Hodge filtration. The diagram can also be seen as a morphism $\Xi : M \to [M^\vee/P_M]$ to the quotient stack. If $M^\vee$ contains a k-rational point then the quotient stack is isomorphic to the classifying stack $[\cdot/Q_M]$ of a parabolic $Q_M \subset P_M$. Therefore the datum is essentially the same as a Q_M-torsor over M. This

allows to define a vector bundle $\Xi^*\mathcal{E}$ on M associated with any representation $\mathcal{E}$ of Q_M (or with a P_M-equivariant vector bundle on $M^\vee$). These are called *automorphic vector bundles*.
This situation generalizes to the case in which D is non-trivial. In this case automorphic data consist of the following: for any stratum Y a diagram

$$C_{\overline{Y}}(\overline{M}) \xleftarrow{\pi} B_Y \xrightarrow{p} M_Y^\vee$$

where $C_{\overline{Y}}$ means formal completion along $\overline{Y}$, and $\pi : B_Y \to C_{\overline{Y}}(\overline{M})$ is again a right-torsor under a — now arbitrary — linear algebraic group P_M and $M^\vee$ is a component of the moduli space of quasi-parabolics of P_M. The morphism p is again P_M-equivariant. Furthermore the action of $\mathbb{M}_m^{n_Y}$ lifts to B_Y (the lifted action is part of the datum) such that p becomes invariant. For any two strata $Z \le Y$ we suppose given an open embedding $M_Z^\vee \hookrightarrow M_Y^\vee$ and a morphism $B_Z \to B_Y$ both equivariant for a given homomorphism $P_Z \to P_Y$.
Such a datum is present on toroidal compactifications of Shimura varieties. This is probably less well-known, and was first described in this form in [7] (cf. [8, 2.5]). It exists (philosophically) because the P_M-structure of the de Rham realization of the universal motive becomes a P_Y-structure near the boundary stratum $\overline{Y}$ (in the formal sense) because of a natural weight filtration on the realization there, leading to a family of mixed Hodge structures.
The more general situation of an (abstract) toroidal compactification equipped with automorphic data allows one to define *generalized automorphic sheaves* (Definition 3.4.3) on $\overline{M}$. In the situation of toroidal compactifications of (mixed) Shimura varieties these include for instance:

- sheaves of sections of automorphic vector bundles with certain vanishing conditions along the boundary (e.g. bundles of cusp forms, subcanonical extensions, etc.),
- the structure sheaf $\mathcal{O}_D$ of the boundary or the structure sheaf $\mathcal{O}_{\overline{Y}}$ of a closed stratum thereof,
- bundles of Jacobi-forms,
- bundles $\Omega^i(\overline{M})$ and jet bundles of automorphic vector bundles,
- bundles of “almost holomorphic” modular forms.

To define generalized automorphic sheaves, the category of P_M-equivariant vector bundles on $M^\vee$ is not sufficient as input category. For this purpose, we define an *Abelian* category, the *Fourier-Jacobi category* (Definition 3.4.1). The objects are specified by a collection of functors

$$F_Y : \mathbb{Z}^{n_Y} \to [\ [M_Y^\vee/P_Y]\text{-}\mathbf{Coh}\ ]$$

for each stratum Y, where $n_Y = \operatorname{codim}(\overline{Y})$. These are supposed to fulfill a finiteness condition, namely they should be left Kan extensions of functors defined on some bounded subregion of $\mathbb{Z}^{n_Y}$. In particular, the sheaves $F_Y(v + \lambda e_i)$ become constant for sufficiently large λ and we require that they are isomorphic to $F_W(\mathrm{pr}(v))$ restricted to $M_Y^\vee$ where W is a larger stratum. It is explained in 3.4.3 that such a datum $\{F_Y\}$ defines a sheaf $\Xi^*(\{F_Y\})$ on $\overline{M}$. The essential tool to define those sheaves is the theory of descent on formal/open coverings developed in [9]. This theory enables to glue $\Xi^*(\{F_Y\})$ from sheaves on the various completions. The latter are, by definition, formally toroidal, and the functor F_Y describes the parts of $C_{\overline{Y}}(\Xi^*(\{F_Y\}))$ of varying weight under $\mathbb{G}_m^{n_Y}$.

Example 1. Let $\overline{M}$ be the compactification of a (fine) moduli space of elliptic curves. There are only two types of strata: $Y = M$ is the open stratum or Y is a point (a cusp). In the first case

$P_M = \mathrm{GL}_2$ and $M^\vee = \mathbb{P}^1 = P_M/Q_M$ while in the second case $P_Y = \begin{pmatrix} * & * \\ & 1 \end{pmatrix}$ and $M^\vee = \mathbb{A}^1 = P_Y/\mathbb{G}_m$. The bundle of modular forms of weight k (with "vanishing" order $\nu_Y \in \mathbb{Z}$ at the cusp Y) is given by the following input datum:

$$F_M := \mathcal{L}^{\otimes k}$$

for the open stratum, where $\mathcal{L}$ is the standard one-dimensional representation of weight k of Q_M, and

$$F_Y : v \mapsto \begin{cases} \mathcal{L}^{\otimes k}|_{\mathbb{A}^1} & \text{if } v \geq \nu_Y, \\ 0 & \text{otherwise,} \end{cases}$$

for the cusps.

Example 2. Let M' be the universal elliptic curve over a (fine) moduli space of elliptic curves. Let $\overline{M}$ the Poincaré line bundle over M' associated with the standard polarization. It is the partial compactification of a $\mathbb{G}_m$-torsor M over M'. The variety M is a mixed Shimura variety associated with the group $P_M = \mathrm{GL}_2 \ltimes W$, where W is a Heissenberg group, a central extension of $\mathbb{G}_a^2$:

$$0 \longrightarrow U \cong \mathbb{G}_a \longrightarrow W \longrightarrow V \cong \mathbb{G}_a^2 \longrightarrow 0.$$

(Here GL_2 acts on V via the natural 2-dimensional representation and on U via the determinant.) In this case there is only one boundary stratum $Y \cong M'$ apart from M. Consider the following input datum:

$$F_M := 0$$

and

$$F_Y : v \mapsto \begin{cases} \mathcal{L}^{\otimes k} & \text{if } v = i, \\ 0 & \text{otherwise.} \end{cases}$$

for $\mathcal{L}$ as before, extended (considered as a representation) to the present Q_M in the only possible way. The associated generalized automorphic sheaf is then the bundle of Jacobi forms of weight k and index i (it has support on $Y \cong M'$). Here we completely ignored the behaviour along the boundary of M' for simplicity, which can be achieved by considering a full compactification of M instead.

We finally consider the notion of (logarithmic) connection on automorphic data, and certain (purely algebraic) axioms regarding it:

- (F) flatness of the logarithmic connection (3.1.2),
- (T) infinitesimal Torelli (3.1.3),
- (M) unipotent monodromy condition (3.1.4),
- (B) boundary vanishing condition (3.1.6).

(These axioms are of course not all expected to hold in this form for generalizations of the theory to other moduli spaces.) For example (F) and (T) imply that — on the open stratum — the formation of automorphic vector bundles commutes with the formation of sheaves of differential forms and jet bundles (Section 3.3). If (M) holds, even the sheaves of differential forms and the jet bundles — on the compactification — can be defined as generalized automorphic sheaves (Section 3.5), as opposed to their logarithmic variants which are always usual automorphic vector bundles. Finally,

if in addition (B) holds, Hirzebruch-Mumford proportionality holds for the compactification (Section 4.2). In the compact case (M) and (B) are vacuous, and this becomes much easier. The validity of the axioms for automorphic data on toroidal compactifications of (mixed) Shimura varieties is explained in section 3.6.
We prove the proportionality theorem of Hirzebruch and Mumford in Section 4 in the following form:

Theorem 4.2.1. Let $\overline{M}$ be a toroidal compactification of dimension n equipped with automorphic data with logarithmic connection which satisfies the axioms (F, T, M, B) and such that P_M is reductive. There is $c \in \mathbb{Q}$ such that for all polynomials p of degree n in the graded polynomial ring $\mathbb{Q}[c_1, c_2, \dots, c_n]$ and all P_M-equivariant vector bundles $\mathcal{E}$ in $[\ [M^\vee/P_M]$-**Coh** $]$ the proportionality

$$p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) = c \cdot p(c_1(\Xi^*\mathcal{E}), \dots, c_n(\Xi^*\mathcal{E}))$$

holds true.

The idea of the proof is as follows. Following Atiyah [2] the polynomials in the Chern classes of vector bundles can be computed as an element in $H^n(\overline{M}, \omega) \cong k$, resp. $H^n(M^\vee, \omega) \cong k$ by a purely homological algebra construction starting from the extensions

$$0 \longrightarrow \Omega^1 \otimes \mathcal{E} \longrightarrow J^1\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0. \tag{1}$$

for $\mathcal{E}$ and similar extensions for $\Xi^*\mathcal{E}$. This construction works in every Abelian tensor category. It suffices therefore to find an Abelian tensor category $\mathcal{A}$ which maps via an exact tensor functor to

$$[\ \overline{M}\text{-}\mathbf{Coh}\ ], \text{ and } [\ M^\vee\text{-}\mathbf{Coh}\ ],$$

respectively, such that an extension like (1) exists in $\mathcal{A}$ and maps to the extensions $J^1\mathcal{E}$, and $J^1(\Xi^*\mathcal{E})$, respectively. Furthermore, this Abelian tensor category has to satisfy the property that $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{O}, \omega')$ is one-dimensional where ω' is the pre-image of both $\omega_{\overline{M}}$ and $\omega_{M^\vee}$.
In the compact case the category $[\ [M^\vee/P_M]$-**Coh** $]$ of P_M-equivariant vector bundles on $M^\vee$ can be taken as $\mathcal{A}$. This does not work in general because $\Xi^*\omega_{M^\vee} = \omega_{\overline{M}}(\log)$ and mostly $H^n(\overline{M}, \omega(\log)) = 0$.
In the non-compact case, the Fourier-Jacobi categories can be taken as $\mathcal{A}$. Here the boundary vanishing condition comes into play which, by an easy homological algebra argument, implies that $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{O}, \omega')$ is indeed one-dimensional. (Strictly speaking we only construct the tensor product on a subcategory of "torsion-free" objects in the Fourier-Jacobi-categories and show that Ξ^* respects it. For the reasoning above this is however sufficient.)
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2 Toroidal compactifications

2.1 Formally toroidal schemes

2.1.1. Let k be a field of characteristic 0, fixed for the whole article. Let $\mathbb{M}_m$ be $\mathbb{A}^1$ with its unital multiplicative monoid structure over k and, as usual, $\mathbb{G}_m \hookrightarrow \mathbb{M}_m$ the open subscheme of the multiplicative group. Denote by ε the unit of $\mathbb{M}_m$ or $\mathbb{G}_m$ and by μ the multiplication.

Let n be a positive integer and let X be a formal scheme over k with an action of $\mathbb{M}_m^n$, i.e. with a given morphism

$$\mathbb{M}_m^n \times X \xrightarrow{\rho} X$$

such that

$$\begin{array}{ccc} \mathbb{M}_m^n \times \mathbb{M}_m^n \times X & \xrightarrow{\mathrm{id} \times \rho} & \mathbb{M}_m^n \times X \\ \downarrow{\scriptstyle \mu \times \mathrm{id}} & & \downarrow{\scriptstyle \rho} \\ \mathbb{M}_m^n \times X & \xrightarrow{\rho} & X \end{array}$$

is commutative and such that the composition

$$X \xrightarrow{\varepsilon \times \mathrm{id}} \mathbb{M}_m^n \times X \xrightarrow{\rho} X$$

is the identity.

Lemma 2.1.2. *Let $X = \operatorname{spf} R$ be an affine formal scheme over k. It is equivalent to give an action of $\mathbb{M}_m^n$ on X or a (topological) $\mathbb{Z}_{\geq 0}^n$-**grading** on R, i.e. collection of subrings $R_v \subseteq R$ for each $v \in \mathbb{Z}_{\geq 0}^n$ such that*

1. *For all $v, w \in \mathbb{Z}_{\geq 0}^n$, we have*
$$R_v \cdot R_w \subseteq R_{v+w}.$$
2. *Each $x \in R$ has a* unique *expression as an infinite converging sum*
$$x = \sum_{v \in \mathbb{Z}_{\geq 0}^n} x_v$$
with $x_v \in R_v$.

We denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n$.

Definition 2.1.3. *A formal scheme X with an action of $\mathbb{M}_m^n$ is called* **formally toroidal** *if there is an affine covering by $\operatorname{spf} R$'s such that the action restricts to $\mathbb{M}_n^m \times \operatorname{spf} R \to \operatorname{spf} R$ and such that*

1. *All R_v have the discrete topology.*
2. *The induced map*
$$R_0[R_{e_1}, \ldots, R_{e_n}] \to R$$
has dense image and induces an isomorphism between the completion of $R_0[R_{e_1}, \ldots, R_{e_n}]$ at the ideal $(R_{e_1}, \ldots, R_{e_n})$ and R.
3. *The R_{e_i} (and hence by 2. all R_v) are locally free R_0-modules of rank 1.*

It follows that, up to restricting to a smaller open cover, we have

$$R \cong R_0[\![x_1, \ldots, x_n]\!]$$

with its natural topological $\mathbb{Z}_{\geq 0}^n$-grading. The x_i however are only determined up to $R_0^\times$.

2.1.4. On a formally toroidal scheme X we also have a ring-sheaf $\mathcal{O}_{X_0}$ which locally gives the R_0's and the $\mathcal{O}_{X,v}$ which are coherent $\mathcal{O}_{X_0}$-submodules of $\mathcal{O}_X$. The topological space X together with $\mathcal{O}_{X,0}$ is a scheme and it is isomorphic to the categorical quotient (in the category of formal schemes) of X w.r.t. the action of $\mathbb{M}_m^n$. It is denoted by X_0. Furthermore there is an obvious section (a closed embedding) $X_0 \hookrightarrow X$.

Example 2.1.5. *The standard example starts from a $\mathbb{G}_m^n$-bundle on a variety X which gets partially compactified by glueing in the partial compactification $\mathbb{G}_m^n \hookrightarrow \mathbb{M}_m^n$ and then completed at the section given by the origin of $\mathbb{M}_m^n$.*

2.2 Modules and differentials

In the following we consider the integers $\mathbb{Z}$ as a category via the natural inclusion of posets into categories. In other words, there is a morphism (and a unique one) $n \to n'$ if and only if $n \leq n'$.

Proposition 2.2.1. *Let X with action of $\mathbb{M}_m^n$ be a noetherian formally toroidal scheme. It is equivalent to give*

1. *a coherent $\mathcal{O}_X$-Module M with an extension of the $\mathbb{G}_m^n$-action (not necessarily the $\mathbb{M}_m^n$-action);*

2. *a collection of coherent $\mathcal{O}_{X_0}$-modules M_w for $w \in \mathbb{Z}^n$ together with an associative multiplication morphism for $v \in \mathbb{Z}^n_{\geq 0}$:*
$$\mathcal{O}_{X,v} \otimes_{\mathcal{O}_{X_0}} M_w \to M_{v+w}$$
giving for $v = 0$ just the module-structure, such that there are $N', N \in \mathbb{Z}$ with the property that for all w such that $w_i \geq N$, all i, and $v = e_i$ the morphism is an isomorphism and for all w such that some $w_i < N'$ the module M_w is zero;

3. *a functor with values in $\mathcal{O}_{X_0}$-coherent sheaves*
$$\begin{aligned} M : \mathbb{Z}^n &\to [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ] \\ v &\mapsto M(v) \end{aligned}$$
such that there are $N, N' \in \mathbb{Z}$ with the property that for all i and for all w with $w_i \geq N$ the morphism $M(w \to w + e_i)$ is an isomorphism and for all w such that $w_i < N'$ for some i the module $M(w)$ is zero. In other words the functor is isomorphic to the left Kan extension of a functor $\Delta^n_{N-N'} \to [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ]$ where $\Delta_{N-N'}$ is considered as an interval $[N', N] \subset \mathbb{Z}$.

Proof (sketch). $1 \leftrightarrow 2$: Given a module M the associated M_v is just the $\mathcal{O}_{X_0}$-submodule of elements transforming with weight v under $\mathbb{G}_m^n$. Conversely, the module M is given as the *product* of the modules M_v.
$2 \leftrightarrow 3$: A collection M_v is associated with the functor $v \mapsto M(v) := M_v \otimes \mathcal{O}_{X,-v}$. Here $\mathcal{O}_{X,v}$ for arbitrary $v \in \mathbb{Z}^n$ is defined by $\bigotimes_i \mathcal{O}_{X,e_i}^{\otimes \alpha_i}$ for $v = \sum \alpha_i e_i$.
And a morphism $v \to w$ in $\mathbb{Z}^n$ is mapped to the morphism
$$M_v \otimes_{\mathcal{O}_{X,0}} \mathcal{O}_{X,-v} \to M_w \otimes_{\mathcal{O}_{X,0}} \mathcal{O}_{X,-w}$$
induced by
$$\mathcal{O}_{X,w-v} \otimes_{\mathcal{O}_{X,0}} M_v \to M_w.$$
The functoriality of the functor M is equivalent to the associativity of the multiplication on the module M. □

Definition 2.2.2. *Let X with an action of $\mathbb{M}_m^n$ be a noetherian formally toroidal scheme. Coherent $\mathcal{O}_X$-modules with compatible $\mathbb{G}_m^n$-action as in Proposition 2.2.1 form an Abelian category which we denote by $[\ \mathcal{O}_X\textbf{-TCoh}\ ]$.*

Lemma 2.2.3. *Under the correspondence above, we have $M(v)$ are torsion-free $\mathcal{O}_{X,0}$-modules and $M(v \to w)$ are monomorphisms for all $v \leq w$, if and only if M is torsion-free.*

Remark 2.2.4. *We define the full subcategory $\mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])^{f.g.}$ of $\mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])$ as those functors M which are bounded below and have the property stated in Proposition 2.2.1, 3. Hence we have an equivalence*

$$[\ \mathcal{O}_X\textbf{-TCoh}\ ] \cong \mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])^{f.g.}.$$

2.2.5. Let M be a coherent sheaf on X with a compatible action of $\mathbb{G}_m^n$. We have its associated functor $M : \mathbb{Z}^n \to [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ]$. As said, there is an N such that $M(\sum \alpha_i e_i)$ is (essentially) constant in α_i if $\alpha_i > N$. We denote this sheaf by $\lim_{\alpha \to \infty} M(v + \alpha e_i)$. Note that also expressions like $\lim_{\alpha_1 \to \infty, \dots, \alpha_j \to \infty} M(v + \alpha_1 e_{i_1} + \dots + \alpha_j e_{i_j})$ do make sense (up to isomorphism). Given an injection $\beta : [j] \hookrightarrow [n]$ we will regard this construction w.r.t to the *missing* indices in the image of β as a functor

$$\lim_{\beta} : \mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])^{f.g.} \to \mathrm{Fun}(\mathbb{Z}^j, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])^{f.g.}.$$

We just write "lim" for this construction w.r.t. all indices.

2.2.6. For coherent, *torsion-free* sheaves M and N we can describe the tensor product $M \otimes N$ with its natural $\mathbb{M}_m^n$ action by the functor

$$M \otimes N(v) = \sum_{v_1 + v_2 = v} M(v_1) \otimes N(v_2)$$

where the sum is formed in $(\lim M) \otimes (\lim N)$.

2.2.7. For any injection $\beta : [j] \hookrightarrow [n]$ define a sheaf $\mathcal{O}_X[\beta^{-1}]$ as the sheafification of the pre-sheaf, defined (for small enough U) by

$$U \mapsto \mathcal{O}_X(U)[x_{\beta(1)}^{-1}, \dots, x_{\beta(j)}^{-1}]$$

where the x_i are generators of $\mathcal{O}_{X,e_i}$. To a coherent (in the sense of modules on ringed spaces) $\mathcal{O}_X[\beta^{-1}]$-module with $\mathbb{G}_m^n$-action we may still associate (in the same way as in Proposition 2.2.1) a functor in $\mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])$. This yields a *fully-faithful* functor

$$[\ \mathcal{O}_X[\beta^{-1}]\textbf{-TCoh}\ ] \to \mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ]$$

which has the property that the functors in the image are constant in the direction of the $e_{\beta(i)}$. Let $\beta^\perp : [n-j] \hookrightarrow [n]$ be a (to β) complementary injection. The diagram

$$\begin{array}{ccccc} [\ \mathcal{O}_X\textbf{-TCoh}\ ] & \longrightarrow & & & [\ \mathcal{O}_{X_0}[\beta^{-1}]\textbf{-TCoh}\ ] \\ \downarrow & & & & \downarrow \\ \mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])^{f.g.} & \xrightarrow{\lim_{\beta^\perp}} & \mathrm{Fun}(\mathbb{Z}^{n-j}, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])^{f.g.} & \xrightarrow{p_{\beta^\perp}^*} & \mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ]) \end{array}$$

is commutative. Here $p_{\beta^\perp}^*$ is the pullback induced by the projection $p_{\beta^\perp} : \mathbb{Z}^n \to \mathbb{Z}^{n-j}$ induced by $\beta^\perp$. The sheaf $\mathcal{O}_X[\beta^{-1}]$ can be completed afterwards w.r.t. any of the ideals generated by $\mathcal{O}_{X,e_i}$ for

$i \notin \mathrm{im}(\beta)$. This process might be repeated. Any sheaf R of $\mathcal{O}_X$-algebras so obtained (which carries still an action of $\mathbb{G}_m^n$) still yields a *fully-faithful* functor

$$[\ R\textbf{-TCoh}\ ] \to \mathrm{Fun}(\mathbb{Z}^n, [\ \mathcal{O}_{X_0}\textbf{-Coh}\ ])$$

whose image in contained in those functors which are constant in the direction of the $e_{\beta(i)}$ for those i such that (locally) a generator X_i has been inverted. An inverse functor on the essential image might be quite complicated to describe. It is given as a subset of the infinite product that was considered in Proposition 2.2.1 but the sequences might be e.g. bounded below in some direction, point-wise w.r.t. another direction. Since we will not need it we will not elaborate on this.
A $\mathbb{G}_m^n$-equivariant coherent $\mathcal{O}_X[\mathrm{id}_{[n]}^{-1}]$-Module $\widetilde{M}$ is equivalent to just an $\mathcal{O}_{X_0}$-module via $\widetilde{M} \mapsto \widetilde{M}(0)$. Each $\mathcal{O}_{X_0}$-module M_0 in turn has a **canonical extension** to an $\mathcal{O}_X$-Module with $\mathbb{M}_m^n$-action, given by means of the functor

$$M_0(v) = \begin{cases} M_0 & \text{if } v \in \mathbb{Z}_{\geq 0}^n, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently by $M := M_0 \otimes_{\mathcal{O}_{X,0}} \mathcal{O}_X$ with its natural $\mathbb{M}_m^n$-action. We denote the full subcategory of $[\ \mathcal{O}_X\textbf{-TCoh}\ ]$ consisting of canonical extensions by $[\ \mathcal{O}_X\textbf{-TCoh-can}\ ]$.
We have a morphism 'constant term' of functors:

$$\text{c.t.} : M \otimes_{\mathcal{O}_X} \mathcal{O}_X[\beta^{-1}] \to \lim_{\beta} M.$$

2.2.8. There is the following exact sequence (equivariant w.r.t. the action of $\mathbb{M}_m^n$) of coherent sheaves on X:

$$0 \longrightarrow \Omega_{X_0} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow \textstyle\sum_i \mathcal{O}_{X,e_i} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_X \longrightarrow 0$$

where $\sum_i \mathcal{O}_{X,e_i} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_X$ is isomorphic to the bundle $\Omega_{X|X_0}$. Ω_X is not a canonical extension. There is the larger bundle $\Omega_X(\log)$ which is locally generated by Ω_X and by the rational differentials $\frac{\mathrm{d}x_i}{x_i}$. The latter are invariant under the action of $\mathbb{M}_m^n$. We proceed to describe the associated functors of the $\mathbb{M}_m^n$-equivariant vector bundles Ω_X and $\Omega_X(\log)$.
Consider the Atiyah extensions on X_0 associated with the line bundles $\mathcal{O}_{X,e_i}$

$$0 \longrightarrow \Omega_{X_0} \longrightarrow E_i \xrightarrow{p_i} \mathcal{O}_{X_0} \longrightarrow 0$$

and their amalgamed sum

$$0 \longrightarrow \Omega_{X_0} \longrightarrow E \xrightarrow{\oplus p_i} \textstyle\bigoplus_i \mathcal{O}_{X_0} \longrightarrow 0 \tag{2}$$

Then $\Omega_X(\log)$ is just the canonical extension of E, i.e. it is given by the functor

$$\Omega_X(\log)(v) = \begin{cases} E & \text{if } v \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2.9. *The functor associated with Ω_X is given by*

$$\Omega_X(v) = \begin{cases} \{e \in E \mid p_i(e) = 0\ \forall i : \alpha_i = 0\} & \text{if } v = \sum \alpha_i e_i \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

as a subfunctor of $\Omega_X(\log)$.

2.3 Abstract toroidal compactifications

2.3.1. Let M be a smooth k-variety. Consider an open embedding $M \hookrightarrow \overline{M}$ into a smooth k-variety (which will mostly assumed to be proper), such that $D := \overline{M} \smallsetminus M$ is a divisor with strict normal crossings. Consider the coarsest statification $\overline{M} = \bigcup_{Y \in \mathcal{S}} Y$ into locally closed subsets such that all components of D are closures of a stratum in the finite set $\mathcal{S}$. M itself will be the unique open stratum. Let n_Y be the codimension of $\overline{Y}$. Consider furthermore a formally toroidal action ρ_Y of $\mathbb{M}_m^{n_Y}$ on the formal completion $X = C_{\overline{Y}}(\overline{M})$ of $\overline{M}$ along $\overline{Y}$ which establishes $\overline{Y}$ as the invariant subscheme X_0. For a pair of strata Y, Z we write $Z \le Y$ if $Z \subset \overline{Y}$.

Definition 2.3.2. *The embedding $M \hookrightarrow \overline{M}$ together with the collection $\{\rho_Y\}_Y$ is called a (***partial***, if $\overline{M}$ is not proper)* **toroidal compactification** *if for each pair $Z \le Y$ of strata we have an* injective *map $\beta_{ZY} : [n_Y] \hookrightarrow [n_Z]$ such that the natural morphism of formal schemes*

$$C_{\overline{Z}}(\overline{M}) \longrightarrow C_{\overline{Y}}(\overline{M})$$

is equivariant w.r.t. the action of $\mathbb{M}_m^{n_Y}$, where $\mathbb{M}_m^{n_Y}$ acts via β_{ZY} and ρ_Z on $C_{\overline{Z}}(\overline{M})$.

Remark 2.3.3. *The map β_{ZY} is uniquely determined by the condition in the definition and hence for strata $W \le Z \le Y$ we have $\beta_{WZ}\beta_{ZY} = \beta_{WY}$.*

We will regard objects on $\overline{M}$ such as coherent sheaves etc. always with a compatible action of the $\mathbb{G}_m^{n_Y}$ (not necessarily $\mathbb{M}_m^{n_Y}$) on their completion on $C_{\overline{Y}}(\overline{M})$ for all strata Y in a compatible way.

Definition 2.3.4. *In particular, we have a category* [$\mathcal{O}_{\overline{M}}$**-TCoh**] *of coherent sheaves with compatible $\mathbb{G}_m^{n_Y}$-actions on the various completions. It has a full subcategory* [$\mathcal{O}_{\overline{M}}$**-TCoh-can**] *of those sheaves with compatible $\mathbb{G}_m^{n_Y}$-action whose completions are all canonical extensions (2.2.7).*

For example $\Omega^i(\overline{M})$, $T(\overline{M})$ and $\mathcal{O}_{\overline{M}}$ and are sheaves in [$\mathcal{O}_{\overline{M}}$**-TCoh**]. The former two are no canonical extensions, however.

2.3.5. Each closed stratum $\overline{Y}$ is itself a (partial) toroidal compactification. The completion $C_{\overline{Z}}(\overline{Y})$ is the following formal subscheme of $C_{\overline{Z}}(\overline{M})$. It is in local coordinates given by $R_0[\![R_{e_i}, \dots, R_{e_{n_Z}}]\!]$ modulo the ideal generated by $R_{e_{\beta(1)}}, \dots, R_{e_{\beta(n_Y)}}$ (where $\beta = \beta_{ZY}$). It carries an action of $\mathbb{G}_m^{n_Z - n_Y}$. Here the missing indices not in the image of β can be numbered in any way. We denote the corresponding injective map by $\beta_{ZY}^\perp : [n_Z - n_Y] \hookrightarrow [n_Z]$. With the restriction $\beta'_{WZ} : [n_Z - n_Y] \hookrightarrow [n_W - n_Y]$ of the transition maps β_{WZ} for $W \le Z \le Y$ the scheme $\overline{Y}$ becomes a toroidal compactification. The following commutative diagram shows the compatibility of the chosen numberings:

$$\begin{array}{ccc} [n_Z - n_Y] & \xhookrightarrow{\beta'_{WZ}} & [n_W - n_Y] \\ \big\downarrow{\scriptstyle \beta_{ZY}^\perp} & & \big\downarrow{\scriptstyle \beta_{WY}^\perp} \\ [n_Z] & \xhookrightarrow{\beta_{WZ}} & [n_W] \end{array}$$

Lemma 2.3.6. *Let E be a coherent sheaf on $\overline{M}$ with compatible $\mathbb{G}_m^{n_Y}$-action on the respective completion E_Y on $C_{\overline{Y}}(\overline{M})$. Then for any stratum $Z \le Y$ and $v \in \mathbb{Z}^{n_Y}$ we have that*

$$E_Y(v)$$

is the coherent sheaf on $\overline{Y}$ which (w.r.t. to the restricted structure of toroidal compactification of 2.3.5) corresponds to the functor w.r.t. Z:

$$z \mapsto E_Z(\beta_{ZY}(v) + \beta_{ZY}^\perp(z)).$$

Lemma 2.3.7 (Glueing lemma). *Consider for each stratum Y be a functor*

$$F_Y : \mathbb{Z}^{n_Y} \to [\ \overline{Y}\textbf{-TCoh-can}\ ]$$

which satisfies the conditions of Proposition 2.2.1, 3., where $[\ \overline{Y}\textbf{-TCoh-can}\ ]$ is the category of toroidal coherent sheaves on $\overline{Y}$ which are canonical extensions (see 2.3.4). Consider for all $Z \leq Y$ an isomorphism of functors

$$\kappa_{ZY} : \iota_{ZY}^* F_Y \xrightarrow{\sim} \lim_{\beta_{ZY}} F_Z \tag{3}$$

which are compatible w.r.t. $Y \leq Z \leq W$ in the obvious way. Here $\iota_{ZY} : C_{\overline{Z}}(\overline{M}) \to C_{\overline{Y}}(\overline{M})$ is the natural morphism of formal schemes.
Then there exists a coherent sheaf E on $\overline{M}$ with compatible actions of $\mathbb{G}_m^{n_Y}$ on $C_{\overline{Y}}(E)$ for all Y, such that there are isomorphisms of functors

$$\lambda_Y : C_{\overline{Y}}(E)(v)|_Y \cong F_Y(v)|_Y$$

which are compatible with the functors κ_{ZY} in the sense that for all $v \in Z^{n_Y}$ the diagram

$$\begin{array}{ccc} C_{\overline{Y}}(E)|_Y(v) & \xleftarrow[\lambda_Y]{} & F_Y(v) \\ \downarrow & & \downarrow{\scriptstyle \kappa_{ZY}} \\ C_{\overline{Z}}(E)[\beta_{ZY}^{-1}](v) & \xleftarrow[\lambda_Z]{} & (\lim_{\beta_{ZY}} F_Z)(v) \end{array} \tag{4}$$

is commutative. In particular E is isomorphic to F_M on the open stratum M. E is uniquely determined (up to unique isomorphism) by this property and the isomorphisms κ.

Proof. We apply [9, Main theorem 7.6]. The sheaves of $\mathcal{O}_{\overline{M}}$-algebras R_Y of [loc. cit.] are isomorphic to the restriction of the sheaf $C_{\overline{Y}}(\mathcal{O}_X)$ to any open subset $U \subset \overline{M}$ such that $U \cap \overline{Y} = Y$. We write as usual $C_{\overline{Y}}(\mathcal{O}_X)|_Y$ for this sheaf. All ring sheaves are considered on the topological space underlying $\overline{M}$. Note that they are *not* quasi-coherent as $\mathcal{O}_X$-modules, except for the open stratum M itself.
For any pair of strata $Z \leq Y$ the sheaf of $\mathcal{O}_{\overline{M}}$-algebras $R_{Y,Z}$ of [loc. cit.] is, by definition, equal to $C_{\overline{Y}}(R_Z \otimes_{\mathcal{O}_{\overline{M}}} \mathcal{O}_U)$ where U is any open subset such that $U \cap \overline{Y} = Y$ and where the tensor product is formed in the category of ring sheaves. The sheaf of $\mathcal{O}_{\overline{M}}$-algebras $C_{\overline{Y}}(R_Z \otimes_{\mathcal{O}_{\overline{M}}} \mathcal{O}_U)$ is also isomorphic to a completion of the localization $C_{\overline{Z}}(\mathcal{O}_X)[\beta_{YZ}^{-1}]$ since $\overline{Y}$ is given in formal local coordinates in $C_{\overline{Z}}(\mathcal{O}_X)$ by the zero locus of $x_{\beta(1)}, \ldots, x_{\beta(n_Y)}$ for $\beta = \beta_{ZY}$.
By the nature of toroidal compactification of $\overline{M}$ we have an action of $\mathbb{G}_m^{n_Y}$ on R_Y and an action of $\mathbb{G}_m^{n_Z}$ on $R_{Y,Z}$ which are compatible (via β_{ZY}) with the inclusion

$$R_Y \hookrightarrow R_{Y,Z}.$$

The category of R_Y-coherent sheaves with $\mathbb{G}_m^{n_Y}$-action is equivalent to the category

$$\mathrm{Fun}(\mathbb{Z}^{n_Y}, [\ \mathcal{O}_Y\textbf{-TCoh}\ ])^{f.g.}.$$

Hence the given collection of functors $\{F_Y\}_Y$ gives such objects by restricting F_Y to Y.
From the category of $R_{Y,Z}$-coherent sheaves with $\mathbb{G}_m^{n_Z}$-action we have still a fully-faithful embedding into the sub-category of

$$\mathrm{Fun}(\mathbb{Z}^{n_Z}, [\ \mathcal{O}_Z\textbf{-TCoh}\ ])$$

consisting of functors which are constant in the directions e_i for $i \notin \mathrm{im}(\beta_{ZY})$. For each $Z \leq Y$ we get such an object taking $\lim_{\beta_{ZY}} F_Z$. The glueing datum required by [loc. cit.] can therefore be given by diagram (4). Hence, by the main theorem of [loc. cit.], we get the requested sheaf of $\mathcal{O}_{\overline{M}}$-modules which is by construction an object in $[\ \mathcal{O}_{\overline{M}}\textbf{-TCoh}\ ]$. $\square$

2.4 Toroidal compactifications of (mixed) Shimura varieties

2.4.1. The standard examples of abstract toroidal compactifications in the sense of Definition 2.3.2 are toroidal compactifications of Shimura varieties [1]. Since we are interested only in the situation over a field, we can use the theory of canonical models of toroidal compactifications of mixed Shimura varieties due to Pink [13, 2.1] which has been extended in [7] (cf. also [8, 2.5]) to the integral (good reduction) case. For the automorphic data referred to in the next section we rely on [8, 2.5] also for the rational case. In that case the ideas for the proofs of the theorems in [8, 2.5.] (which are given in [7]) are essentially due to Harris [3–5].

2.4.2. For each pure (or mixed) rational Shimura datum $\mathbf{X} = (P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$ in the sense of [8, 2.2.3][1] or [13, 2.1], and for each sufficiently small compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ there is an associated Shimura variety $\mathrm{M}({}^K\mathbf{X})$ which is a smooth quasi-projective variety defined over the reflex field $E(\mathbf{X})$.
Furthermore for each smooth K-admissible rational polyhedral cone decomposition Δ for $\mathbf{X}$ (cf. [8, 2.2.23]) there is a (partial) toroidal compactification $\mathrm{M}({}^K_\Delta\mathbf{X})$ which contains $\mathrm{M}({}^K\mathbf{X})$ as an open subvariety whose complement is a divisor with strict normal crossings. This and the following is a summary of [8, Main Theorem 2.5.9]). If Δ is chosen (and this is always possible) to be projective and complete then $\mathrm{M}({}^K_\Delta\mathbf{X})$ is a smooth projective variety defined over the reflex field $E(\mathbf{X})$. This situation thus gives rise to a stratification of $\mathrm{M}({}^K_\Delta\mathbf{X})$ as considered in 2.3.1. Each stratum corresponds furthermore to a rational polyhedral cone in Δ. For each stratum Y in this stratification there is a mixed Shimura datum $\mathbf{B}_Y = (P_{\mathbf{B}_Y}, \mathbb{D}_{\mathbf{B}_Y}, h_{\mathbf{B}_Y})$ such that $P_{\mathbf{B}_Y}$ is a subgroup of $P_{\mathbf{X}}$ (actually this is a certain normal subgroup of the $\mathbb{Q}$-parabolic of $P_{\mathbf{X}}$ describing the corresponding boundary component in the Baily-Borel compactification). $\mathbf{B}_Y$ is determined only up to conjugation. Furthermore, Δ restricts to a rational polyhedral cone decomposition Δ_Y for $\mathbf{B}_Y$. The partial toroidal compactification of the mixed Shimura variety $\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y)$ has a matching stratum $\widetilde{Y}$ and there is an isomorphism of formal schemes (assuming that K is small enough)

$$C_{\overline{Y}}\mathrm{M}({}^K_\Delta\mathbf{X}) \cong C_{\overline{\widetilde{Y}}}\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y).$$

Furthermore the mixed Shimura variety $\mathrm{M}({}^{K_Y}\mathbf{B}_Y)$ is a torus torsor over another mixed Shimura variety $\mathrm{M}({}^{K_Y'}\mathbf{B}_Y/U)$ where U is a subgroup of $U_{\mathbf{B}_Y}$ (the center of the unipotent radical of $P_{\mathbf{B}_Y}$) and the action of the torus extends to $\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y)$ (cf. [8, 2.5.8]). The acting torus gets identified with $\mathbb{G}_m^{n_Y}$ by means of the basis of the n_Y-dimensional rational polyhedral cone describing Y. By construction of the toroidal compactification this action extends to $\mathbb{M}_m^{n_Y}$ in such a way that $C_{\overline{\widetilde{Y}}}\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y)$ becomes a formally toroidal scheme in the sense of 2.1.3. The functoriality of the theory implies that the actions of the tori match for pairs of strata $Z \le Y$. Thus $\overline{M} := \mathrm{M}({}^K_\Delta\mathbf{X})$ is an abstract toroidal compactification in the sense of Definition 2.3.2.

3 Automorphic data

3.1 Automorphic data on an abstract toroidal compactification

Let $\overline{M}$ be an abstract toroidal compactification (Definition 2.3.2).

Definition 3.1.1. Automorphic data *on the toroidal compactification $\overline{M}$ consists of a collection $\{P_Y, M_Y^\vee, B_Y, \dots\}_Y$ indexed by the strata Y of $\overline{M}$ where*

[1]where the integrality property has to be ignored.

1. *P_Y is a linear algebraic group (not necessarily reductive).*

2. *$M_Y^\vee$ is an open and closed subscheme of the moduli space of quasi-parabolic subschemes of P_Y. We will call these spaces* **generalized flag varieties**. *If P_Y is reductive then they are projective. We consider the action of P_Y on $M_Y^\vee$ as a right-action.*

3. *We are given a diagram*
$$C_{\overline{Y}}(\overline{M}) \xleftarrow{\pi} B_Y \xrightarrow{p} M_Y^\vee$$
in which π is a right P_Y-torsor and p is a P_Y-equivariant map.

4. *We are given a lift of the $\mathbb{M}_m^{n_Y}$-action to B_Y in a P_Y-equivariant way, and p is $\mathbb{M}_m^{n_Y}$-invariant. We assume that B_Y is a canonical extension, i.e. isomorphic to $\Pi^{-1}B_Y$ for some bundle on $\overline{Y}$ with its induced $\mathbb{M}_m^{n_Y}$-action, where $\Pi : C_{\overline{Y}}\overline{M} \to \overline{Y}$ is the projection. (Such a datum is basically equivalent to a Q_Y-principal bundle on $\overline{Y}$ where Q_Y corresponds to a k-rational point of $M^\vee$ if it exists.)*

For strata $Z \le Y$ we suppose given closed embeddings of algebraic groups $\alpha_{ZY} : P_Z \hookrightarrow P_Y$ which induce open embeddings $M_Z^\vee \hookrightarrow M_Y^\vee$ *and given P_Z- and $\mathbb{M}_m^{n_Y}$-equivariant morphisms $B_Z \to B_Y$ such that the diagram of formal schemes*

$$\begin{array}{ccccc} C_{\overline{Z}}(\overline{M}) & \xleftarrow{\pi} & B_Z & \xrightarrow{p} & M_Z^\vee \\ \downarrow & & \downarrow & & \downarrow \\ C_{\overline{Y}}(\overline{M}) & \xleftarrow{\pi} & B_Y & \xrightarrow{p} & M_Y^\vee \end{array}$$

commutes. The morphisms have to be functorial w.r.t. three strata $W \le Z \le Y$.

In other words, if $M^\vee$ contains a k-rational point Q_M, automorphic data is roughly given by a Q_M-principal bundle B_M on $\overline{M}$ such that the structure group restricts to Q_Y on the formal completion along $\overline{Y}$ in an $\mathbb{M}_m^{n_Y}$-equivariant way. Here Q_Y is any parabolic in $M_Y^\vee(k)$.

3.1.2. Consider the following sequence of vector bundles on B_Y (which are all $\mathbb{M}_n^{n_Y}$-equivariant and canonical extensions). We assume given a logarithmic Ehresmann connection on B_Y, i.e. a section s_Y which is P_Y-invariant and $\mathbb{M}_n^{n_Y}$-equivariant:

$$0 \longrightarrow \mathcal{O}_{B_Y} \otimes \mathfrak{g}_Y = T_{B_Y}^{\pi-\mathrm{vert}} \longrightarrow T_B(\log) \underset{\xleftarrow{s_Y}}{\longrightarrow} \pi^* T_{C_{\overline{Y}}(\overline{M})}(\log) \longrightarrow 0.$$

Since everything is $\mathbb{M}_n^{n_Y}$-equivariant and a canonical extension, this is equivalent to give a section of the sequence

$$0 \longrightarrow \mathcal{O}_{B_Y|_{\overline{Y}}} \otimes \mathfrak{g}_Y = T_{B_Y|_{\overline{Y}}}^{\pi-\mathrm{vert}} \longrightarrow T_B(\log)|_{\overline{Y}} \underset{\xleftarrow{s'_Y}}{\longrightarrow} \pi^* T_{C_{\overline{Y}}(\overline{M})}(\log)|_{\overline{Y}} \longrightarrow 0.$$

Furthermore these sections are supposed to be compatible w.r.t. the relation $Z \le Y$ on strata. Such a datum will be called **automorphic data with logarithmic connection** on the toroidal compactification $\overline{M}$.
We define the P_Y-sub vector bundle $T_{B_Y}^{\mathrm{horz}}$ as the image of s_Y, and get a P_Y-equivariant decomposition:

$$T_{B_Y}(\log) = T_{B_Y}^{\pi-\mathrm{vert}} \oplus T_{B_Y}^{\mathrm{horz}}.$$

We say that the connection is **flat**, if

(F) $T_{B_Y}^{\mathrm{horz}}$ is closed under the Lie bracket.

We call P_π^{vert} and P_π^{horz} the corresponding projection operators. If s_Y is flat, it induces a homomorphism of ring-sheaves

$$\nu : \pi^{-1}\mathcal{D}_{M_Y}(\log) \to \mathcal{D}_{B_Y}(\log). \tag{5}$$

3.1.3. We say that the automorphic data satisfies **Torelli**[2], if we have in addition

(T) a direct sum decomposition

$$T_{B_Y}(\log) = T_{B_Y}^{p-\mathrm{vert}}(\log) \oplus T_{B_Y}^{\mathrm{horz}}$$

where $T_{B_Y}^{p-\mathrm{vert}}(\log)$ is the intersection of $T_{B_Y}^{p-\mathrm{vert}}$ with $T_{B_Y}(\log)$ in T_{B_Y}.

We then denote by P_p^{vert} and P_p^{horz} the corresponding projection operators.
Torelli (T) induces an isomorphism

$$p^*T_{M^\vee} \cong \pi^*T_M(\log).$$

In the same way, if s_Y is flat and Torelli holds, we also get a homomorphism of ring-sheaves

$$\mu : p^{-1}\mathcal{D}_{M_Y^\vee} \to \mathcal{D}_{B_Y}(\log). \tag{6}$$

3.1.4. Note that, by the structure of toroidal compactification, we have a sequence dual to sequence (2)

$$0 \longrightarrow \oplus_{i=1}^{n_Y} \mathcal{O}_{C_{\overline{Y}}(\overline{M})} \cdot \mathrm{can}_{i,\overline{M}} \longrightarrow T_{C_{\overline{Y}}(\overline{M})}(\log) \longrightarrow \Pi^* T_{\overline{Y}} \longrightarrow 0$$

where $\mathrm{can}_{i,\overline{M}}$ are the fundamental vector fields for the $\mathbb{G}_m^{n_Y}$-action on $\overline{M}$, and Π is the projection to $\overline{Y}$.
We also consider the following compatibility axiom (called the **unipotent monodromy condition**)

(M) We have $P_\pi^{\mathrm{vert}}(\mathrm{can}_{i,B_Y}) \in \mathfrak{u}_Y^{(i)} \otimes \mathcal{O}_{B_Y}$, where $\mathfrak{u}_Y^{(i)}$ is a sub-Lie-algebra of $\mathfrak{p}_Y$ given by a 1-dimensional *normal unipotent* subgroup $\mathbb{G}_a \cong U^{(i)} \subset P_Y$.

Axiom (M) has the following immediate consequence:

Lemma 3.1.5. *We have $p(P_\pi^{\mathrm{vert}}(\mathrm{can}_{i,B_Y})) \in p^*T_{M^\vee}^{(i)}$ (or equivalently $p(P_\pi^{\mathrm{horz}}(\mathrm{can}_{i,B_Y})) \in p^*T_{M^\vee}^{(i)}$), where $T_{M^\vee}^{(i)}$ is the subbundle of $T_{M^\vee}$ induced by a subalgebra $\mathfrak{u}_Y^{(i)}$ of $\mathfrak{p}_Y$ given by a 1-dimensional* normal unipotent *subgroup $\mathbb{G}_a \cong U^{(i)} \subset P_Y$.*

3.1.6. We also consider the following axiom (called the **boundary vanishing condition**):

(B) For all strata $Y \neq M$ we have: $H^i([M_Y^\vee/P_Y], \omega_{M_Y^\vee}) = 0$ for $i \geq \dim(Y)$.

(cf. Section 3.2 for the notation). Here $\omega_{M_Y^\vee} = \Omega_{M_Y^\vee}^n$ is the highest power of the P_Y-equivariant sheaf of differential forms on $M_Y^\vee$.

[2] this rather corresponds to classical *infinitesimal* Torelli theorems

3.2 Generalized flag varieties and representations of quasi-parabolic subgroups

3.2.1. For a linear algebraic group P and a quasi-parabolic subgroup Q we have several functors between Q-representations, P-representations and (equivariant) coherent sheaves on the quasi-projective variety $M^\vee = Q\backslash P$ (generalized flag variety)[3]. These functors are best understood in the language of Artin stacks. We will not use this theory explicitly but mention it as a guiding principle because it so much clarifies the relations. All representations are, of course, understood to be *algebraic*. We have the following diagram of morphisms of Artin stacks where all stacks are quotient stacks (even schemes in the right-most column):

$$\begin{array}{ccccc} [\cdot/Q] & \xrightarrow[\sim]{a} & [M^\vee/P] & \xleftarrow{c} & M^\vee \\ & & \downarrow b & & \downarrow d \\ & & [\cdot/P] & \underset{e}{\overset{f}{\rightleftarrows}} & \mathrm{spec}(k) \end{array} \tag{7}$$

We denote the categories of (quasi-)coherent sheaves on a stack X by [X-**(Q)Coh**] or sometimes by [$\mathcal{O}_X$-**(Q)Coh**]. For the particular stacks above, we get

[$[\cdot/Q]$-**Coh**]	category of finite-dimensional algebraic Q-representations in k-vector spaces;
[$[\cdot/P]$-**Coh**]	category of finite-dimensional algebraic P-representations in k-vector spaces;
[$[M^\vee/P]$-**Coh**]	category P-equivariant finite dimensional vector bundles on $M^\vee$;
[$M^\vee$-**Coh**]	category of coherent sheaves on $M^\vee$;
[spec(k)-**Coh**]	category of finite-dimensional k-vector spaces,

and similarly for the categories of quasi-coherent sheaves.
The corresponding pull-back and (derived) push-forward functors between the categories of (quasi-)coherent sheaves are given as follows.

- a_* associates with a Q-representation V a locally free P-equivariant sheaf on $M^\vee$. The total space can be described as $(V \times P)/Q$ where Q acts on V and P. It defines an equivalence of the category of finite-dimensional Q-representations and coherent P-equivariant sheaves on $M^\vee$.
- a^* is the inverse of a_*, evaluation at the choosen base point of $M^\vee$.
- b_* global sections on $M^\vee$, remembering the induced P-action. The right derived functors are cohomology on $M^\vee$ equipped with the induced P-action.
- b^* associates with a P-representation V the coherent sheaf $V \otimes \mathcal{O}_{M^\vee}$ with the natural P-action.
- c^* forgets the P-action.
- d_* global sections on $M^\vee$. The right derived functors are the cohomology on $M^\vee$.
- d^* associates with a vector space V the coherent sheaf $V \otimes \mathcal{O}_{M^\vee}$.
- e_* induction $\mathrm{Ind}_e^P(-)$, associates with a vector space V the P-representation $V \otimes \mathcal{O}(P)$.
- e^* forgets the P-action.

[3] Hence, in contrast to the last section, we explicitly assume for simplicity that $M^\vee$ has a k-rational point with corresponding quasi-parabolic Q.

f_* associates with a P-representation the vector space of P-invariants. This functor is exact if P is reductive. Otherwise the right derived functors are the (Hochschild) group cohomology of P with values in the respective representation.

f^* equips a vector space V with the trivial P-representation.

The composed functor a^*b^* is the forgetful functor considering a P-representation as a Q-representation. Its right adjoint, the composed functor b_*a_*, is therefore also called $\mathrm{Ind}_Q^P(-)$ but it is not exact in general.

For a stack X, we denote by $H^i(X,\mathcal{E})$ the higher derived functors of π_* evaluated at $\mathcal{E}$, where π is the structural morphism. For example $H^i([\cdot/P],\mathcal{E})$ denotes the (Hochschild) cohomology of P with values in the representation $\mathcal{E}$.
We will use the following Lemma and its obvious consequences when one of the functors is exact without further mentioning.

Lemma 3.2.2. *For all compositions of push-forward functors along morphisms of Artin stacks we have corresponding Grothendieck spectral sequences of composed functors.*

Proof. See e.g. [10] for elementary statements regarding the stacks appearing in this section. □

3.3 Jet bundles on generalized flag varieties

3.3.1. We start with a general discussion of jet bundles and differential operators. Let X be a smooth k-variety and $X^{(n)}$ the n-th diagonal, i.e.

$$X^{(n)} := \mathcal{O}_{X\times X}/\mathcal{J}^n$$

where $\mathcal{J}$ is the ideal sheaf of the diagonal. Let $\mathcal{E}$ be a vector bundle on X.
We have the two projections:

$$\begin{array}{ccc} & X^{(n)} & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ X & & X \end{array}$$

One defines the n-th jet bundle $J^n\mathcal{E}$ by

$$J^n\mathcal{E} = \mathrm{pr}_{1,*}\,\mathrm{pr}_2^*\,\mathcal{E}$$

which is always equipped with a surjective map

$$J^n\mathcal{E} \to \mathcal{E},$$

induced by the unit $\mathcal{E} \to \Delta_*\Delta^*\mathcal{E}$. Since $\mathcal{O}_{X^{(n)}} = \mathrm{pr}_1^*\,\mathcal{O}_X = \mathrm{pr}_2^*\,\mathcal{O}_X$ there is also a splitting of this map in the case $\mathcal{E} = \mathcal{O}$:

$$\mathcal{O} \to J^n\mathcal{O}.$$

3.3.2. For two vector bundles $\mathcal{E}$ and $\mathcal{F}$ the sheaf of differential operators (of degree $\le n$) is defined as

$$\mathcal{D}^{\le n}(\mathcal{E},\mathcal{F}) := \mathcal{HOM}_{\mathcal{O}_X}(J^n\mathcal{E},\mathcal{F}).$$

The bundle $J^n\mathcal{E}$ has also a different $\mathcal{O}_X$-module structure coming from pr_2, which we denote as an action on the right. We have

$$J^n\mathcal{O}_X \otimes \mathcal{E} \cong J^n\mathcal{E}$$

where the tensor-product is formed w.r.t. this second $\mathcal{O}_X$-module structure.

3.3.3. There is an inclusion

$$\mathcal{D}^{\leq n}(\mathcal{E},\mathcal{F}) \hookrightarrow \mathcal{HOM}_k(\mathcal{E},\mathcal{F})$$

into the sheaf of k-linear morphisms of sheaves (but not $\mathcal{O}_X$-linear). For an open subset $U \subset X$, a section $s \in H^0(U,\mathcal{E})$ here is considered to be a morphism

$$\mathcal{O}_U \to \mathcal{E}_U$$

and the composition

$$\mathcal{O}_U \to \mathrm{pr}_{1,*}\,\mathrm{pr}_1^*\,\mathcal{O}_U = \mathrm{pr}_{1,*}\,\mathrm{pr}_2^*\,\mathcal{O}_U \to \mathrm{pr}_{1,*}\,\mathrm{pr}_2^*\,\mathcal{E} = J^n\mathcal{E}$$

yields a section in $H^0(U,J^n\mathcal{E})$ and then, via application of an element of $H^0(U,\mathcal{HOM}(J^n\mathcal{E},\mathcal{F}))$ a section in $H^0(U,\mathcal{F})$. The second $\mathcal{O}_X$-module structure on $J^n\mathcal{E}$ here dualizes to pre-composition with a section of $\mathcal{O}_X$. We write $\mathcal{D}_X^{\leq n} := \mathcal{D}^{\leq n}(\mathcal{O}_X,\mathcal{O}_X)$. The ring sheaf $\mathcal{D}_X := \mathrm{colim}_n\,\mathcal{D}_X^{\leq n}$ is generated by $\mathcal{O}_X$ and $\mathcal{T}_X$ with the only relations coming from the Lie bracket of vector fields and differentiation of functions.
Similarly to the case of jet bundles, we have

$$\mathcal{D}^{\leq n}(\mathcal{E},\mathcal{O}) = \mathcal{D}_X^{\leq n} \otimes \mathcal{E}^*$$

where the tensor product is formed w.r.t. the right-$\mathcal{O}_X$-module structure.

3.3.4. In the special case $X = G$, where G is an algebraic group, we have a natural isomorphism (compatible with the filtration by degree):

$$\mathcal{D}_G = \mathrm{colim}_n\,\mathcal{D}_G^{\leq n} = \mathcal{O}_G \otimes U(\mathfrak{g}).$$

Elements of $\mathfrak{g}$ are considered to be vector fields using the action by left-translation. They are invariant under the action of G on G by right-translation. The isomorphism is hence G-equivariant under right-translation, where G acts on the right hand side only on $\mathcal{O}_G$. It is G-equivariant under left-translation if G on the right hand side acts on $\mathcal{O}_G$ by left translation and via Ad on $\mathfrak{g}$.

3.3.5. We now consider the special case $X = Q\backslash P$, where Q is a quasi-parabolic subgroup of P. These are the generalized flag varieties, denoted $M_Y^\vee$ in the last section. Hence we assume that $M^\vee$ has a k-rational point Q in the sequel.

Proposition 3.3.6. *Let E be a P-representation and*

$$\mathcal{E} = Q\backslash(P \times E)$$

the corresponding P-equivariant vector bundle on $Q\backslash P$. Then we have

$$\mathcal{D}(\mathcal{E}^*,\mathcal{O}) = Q\backslash(P \times U(\mathfrak{p}) \otimes_{U(\mathfrak{q})} E)$$

where Q acts on $U(\mathfrak{p})$ via Ad *and on E via the given representation.*

Proof. A section on $U \subset Q\backslash P$ of the bundle $Q\backslash(P \times U(\mathfrak{p}) \otimes_{U(\mathfrak{q})} E)$ can be considered as a Q-invariant section s on $\pi^{-1}U$ of the constant bundle $U(\mathfrak{p}) \otimes_{U(\mathfrak{q})} E$. Such sections act on the space $H^0(U,\mathcal{E}^*) = H^0(\pi^{-1}U,E^*)^Q$ as follows: Let $f \in H^0(\pi^{-1}U,E^*)^Q$. A tensor $s = g(X \otimes v)$ acts as

$$f \mapsto g(Xv(f)).$$

In local coordinates one checks that this induces an isomorphism with the appropriate sheaf of differential operators. □

Definition 3.3.7. *We define*

$$J^n E := ((U(\mathfrak{p}) \otimes_{U(\mathfrak{q})} E^*)^{\leq n})^*.$$

Corollary 3.3.8 (to Proposition 3.3.6)**.** *The P_Y-equivariant sheaf on $M_Y^\vee$ associated with the representation $J^n E$ is $J^n \mathcal{E}$.*

3.3.9. There is a logarithmic version of the sheaves of differential operators defined in the last section. Let $X = \overline{M}$ be a smooth k-variety equipped with a divisor with normal crossings. We define

$$\mathcal{D}^{\leq n}(\mathcal{E}, \mathcal{F})(\log) \subset \mathcal{D}^{\leq n}(\mathcal{E}, \mathcal{F})$$

as the subsheaf of differential operators generated by vector fields in $\mathcal{T}_X(\log)$. Wet set

$$J^n_{\log}\mathcal{E} := \mathcal{D}^{\leq n}(\mathcal{E}, \mathcal{O}_X)(\log)^\vee.$$

The following theorem was shown in [4] for the case of Shimura varieties.

Theorem 3.3.10. *Let V be a representation of Q_M, and $\mathcal{V} := \Xi^* \widetilde{V}$ the corresponding automorphic vector bundle on $\overline{M}$. Then the automorphic vector bundle associated with $J^n V$ is precisely $J^n_{\log}\mathcal{V}$.*

Proof. Let $\widetilde{V}$ denote the bundle $Q\backslash(V \times P)$ on $Q\backslash P$. It suffices to show, dually, that the automorphic vector bundle associated with the P-equivariant vector bundle $\mathcal{D}^{\leq n}(\widetilde{V}^*, \mathcal{O})$ on $Q\backslash P$ is $\mathcal{D}^{\leq n}(\log)(\mathcal{V}^*, \mathcal{O})$.
Let Y be a stratum. For the proof it suffices to take $Y = M$, however, we will need the more refined discussion later. There are P_Y-equivariant homomorphisms of ring sheaves (which respect the filtrations by degree), cf. (3.1.2–3.1.3):

$$\begin{aligned} \mu : \pi^{-1}\mathcal{D}_{M_Y}(\log) &\to \mathcal{D}_{B_Y}(\log) \\ \nu : p^{-1}\mathcal{D}_{M_Y^\vee} &\to \mathcal{D}_{B_Y}(\log) \end{aligned}$$

given by the *flat* connection s_Y (and the Torelli axiom). They are compatible with the left- and right-module structure under $p^{-1}\mathcal{O}_{M_Y^\vee}$, resp. $\pi^{-1}\mathcal{O}_{M_Y}$. Furthermore, we have

$$\mathcal{O}_{B_Y} \cdot \nu(\mathcal{D}^{\leq n}_{M_Y^\vee}) = \mathcal{D}^{\text{horz}}_{B_Y} = \mathcal{O}_{B_Y} \cdot \mu(\mathcal{D}^{\leq n}_{M_Y}(\log)),$$

where $\mathcal{D}^{\text{horz}}_{B_Y}$ is the sub-ring sheaf of $\mathcal{D}_{B_Y}$ generated by $\mathcal{O}_{B_Y}$ and $\mathcal{T}^{\text{horz}}_{B_Y}$.
The bundle $\mathcal{D}^{\leq n}(\widetilde{V}, \mathcal{O})$ on $M_Y^\vee$ is isomorphic to

$$\mathcal{D}^{\leq n}_{M_Y^\vee} \otimes_{\mathcal{O}_{M_Y^\vee}} \widetilde{V}^*$$

where the tensor product has been formed w.r.t. the $\mathcal{O}_{M_Y^\vee}$-right-module structure on $\mathcal{D}^{\leq n}_{M_Y^\vee}$.
Furthermore, we have a P_Y-equivariant isomorphism:

$$p^*(\mathcal{D}^{\leq n}_{M_Y^\vee} \otimes_{\mathcal{O}_{M_Y^\vee}} \widetilde{V}) \cong \mathcal{O}_{B_Y} \cdot \mu(\mathcal{D}^{\leq n}_{M_Y}(\log)) \otimes_{\mathcal{O}_{B_Y}} p^*\widetilde{V}$$

(Lemma 3.3.11 below). Now, P_Y acts on $\mathcal{O}_{B_Y} \cdot \mu(\mathcal{D}^{\leq n}_{M_Y}(\log))$ exclusively on the first factor, i.e.

$$(\mathcal{O}_{B_Y} \cdot \mu(\mathcal{D}^{\leq n}_{M_Y}(\log)))^{P_Y} \cong \mathcal{D}^{\leq n}_{M_Y}(\log)$$

using the identification of P_Y-invariant sections of a P_Y-bundle on B_Y with the sections of a vector bundle on M_Y. Conclusion:

$$(p^*(\mathcal{D}^{\leq n}_{M_Y^\vee} \otimes_{\mathcal{O}_{M_Y^\vee}} \widetilde{V}))^{P_Y} \cong \mathcal{D}^{\leq n}_{M_Y}(\log) \otimes_{\mathcal{O}_{M_Y}} (p^*\widetilde{V})^{P_Y}.$$

□

Lemma 3.3.11. *The submodule $\mathcal{O}_{B_Y} \cdot \nu(\mathcal{D}^{\le n}_{M_Y^\vee})$ of $\mathcal{D}_{B_Y}(\log)$ is also a right-$\mathcal{O}_{B_Y}$-submodule, and we have:*

$$p^*(\mathcal{D}^{\le n}_{M_Y^\vee} \otimes_{\mathcal{O}_{M_Y^\vee}} \widetilde{V}) \cong (\mathcal{O}_{B_Y} \cdot \nu(\mathcal{D}^{\le n}_{M_Y^\vee})) \otimes_{\mathcal{O}_{B_Y}} p^*\widetilde{V}$$

where the tensor product in both cases is formed w.r.t. the right-module structure.

Proof. This follows by induction on the degree from the fact that ν is compatible with the right-$p^{-1}\mathcal{O}_{M_Y^\vee}$-module structure. □

3.4 Fourier-Jacobi categories

Definition 3.4.1. *Let $\overline{M}$ be a toroidal compactification equipped with automorphic data. We define the Fourier-Jacobi category $[\ \overline{M}\text{-}\mathbf{FJ}\ ]$ of $\overline{M}$. The objects are collections of functors*

$$F_Y : \mathbb{Z}^{n_Y} \to [\ [M_Y^\vee/P_Y]\text{-}\mathbf{Qcoh}\ ]$$

for each stratum Y, satisfying the following conditions:

1. *For each j there is an $N \in \mathbb{Z}$ such that for all v with $v_j \ge N$ the objects*

$$F_Y(v)$$

do not depend on v_j and for all $v \le v'$ with $v_j, v'_j \ge N$ the morphisms

$$F_Y(v \to v')$$

do not depend on v_j and are identities if $v_i = v'_i$ for all $i \ne j$. In other words, the F_Y are isomorphic to a left Kan extension of a functor $\mathbb{Z}^{n_Y}_{\le N} \to [\ [M_Y^\vee/P_Y]\text{-}\mathbf{Qcoh}\ ]$[4]*.*

We denote the respective constant value by $\lim_{\lambda\to\infty} F_Y(v+\lambda e_j)$. Note that also expressions like $\lim_{\lambda_1,\lambda_2\to\infty} F_Y(v+\lambda_1 e_j+\lambda_2 e_k)$ etc. make sense.

2. *For all $Z \le Y$ with corresponding map $\beta_{ZY} : [n_Y] \hookrightarrow [n_Z]$ and morphism $\alpha_{ZY} : P_Z \to P_Y$ there is an isomorphism:*

$$\mu_{ZY}(v) : \alpha_{ZY}^* F_Y(v) \xrightarrow{\sim} \lim_{\lambda_{k_1},\dots,\lambda_{k_l}\to\infty} F_Z(\beta_{ZY}(v) + \lambda_{k_1} e_{k_1} + \dots + \lambda_{k_l} e_{k_l})$$

for all $v \in \mathbb{Z}^{n_Y}$. Here $\{k_1,\dots,k_l\}$ is the complement of $\mathrm{im}(\beta_{ZY})$. These isomorphisms are supposed to be natural transformations of functors in v and to be functorial w.r.t. three strata $W \le Y \le Z$.

The morphisms in the category $[\ \overline{M}\text{-}\mathbf{FJ}\ ]$ are collections of morphisms of functors $\{F_Y \to F'_Y\}_Y$ for all strata which are compatible with the isomorphisms $\mu_{ZY}(v)$.
In the same way, we define categories $[\ \overline{Y}\text{-}\mathbf{FJ}\ ]$, where the objects only consist of functors F_Z for $Z \le Y$. We also define $[\ Y\text{-}\mathbf{FJ}\ ]$, whose objects are just functors F_Y satisfying property 1. All Fourier-Jacobi categories are Abelian categories.

[4]This would rather only say that the F_Y become constant *up to isomorphism*, but there is no harm in requiring that they are *actually* constant.

Definition 3.4.2. *We define the following full subcategories of the Fourier-Jacobi categories:*

1. $[\ \overline{M}$**-FJ-**$\geq\ ]$*: We ask in addition that for each stratum Y there is an $N \in \mathbb{Z}$ such that*

   $$F_Y(v) = 0$$

   if some $v_j < N$. *Such elements shall be called* **bounded below**. *It means that F_Y is actually a left Kan extension from a functor $\Delta_n^{n_Y} \to [\ [M_Y^\vee/P_Y]$*-**Qcoh** $]$ *for some $n \in \mathbb{N}$, where Δ_n is considered as an interval $[N, N+n] \subset \mathbb{Z}$.*

2. $[\ \overline{M}$**-FJ-coh** $]$*: As before but with the additional condition that $F_Y(v)$ is finite dimensional for all Y and v. Such elements shall be called* **coherent**.

3. $[\ \overline{M}$**-FJ-**$\geq N\ ]$, $[\ \overline{M}$**-FJ-**$\geq N$**-coh** $]$*: As before but with fixed N.*

4. $[\ \overline{M}$**-FJ-tf** $]$*: All bounded-below objects, such that in addition for all $v \leq w$, the morphism $F_Y(v) \to F_Y(w)$ is a monomorphism. Such elements shall be called* **torsion-free**.

5. $[\ \overline{M}$**-FJ-lf** $]$*: All torsions-free objects, such that for a diagram*

   $$\begin{array}{ccc} v & \longrightarrow & v + e_i \\ \downarrow & & \downarrow \\ v + e_j & \longrightarrow & v + e_i + e_j \end{array}$$

   the corresponding diagrams

   $$\begin{array}{ccc} F_Y(v) & \longrightarrow & F_Y(v + e_i) \\ \downarrow & & \downarrow \\ F_Y(v + e_j) & \longrightarrow & F_Y(v + e_i + e_j) \end{array}$$

   are Cartesian. Such elements shall be called **locally free**.

6. $[\ \overline{M}$**-FJ-lf-coh** $]$*: All locally free and coherent objects.*

3.4.3. Obviously the definition of Fourier-Jacobi category mimics the situation for vector bundles on toroidal compactifications and we now proceed to define an exact functor

$$\Xi^* : [\ \overline{M}\textbf{-FJ-coh}\ ] \to [\ \overline{M}\textbf{-TCoh}\ ]$$

as follows: For each $F_Y(v) \in [\ P_Y$**-Vect on** $M_Y^\vee\ ]$ we form $p^*(F_Y(v))^{P_Y}|_{\overline{Y}}$ which is a vector bundle on $\overline{Y}$. It carries an action of $\mathbb{M}_m^{n_Z - n_Y}$ on

$$C_{\overline{Z}}(p_Y^*(F_Y(v))^{P_Y}|_{\overline{Y}}) \cong (p_Z^*(\alpha_{ZY}^* F_Y(v))^{P_Z})|_{\overline{Y}}$$

which is a *canonical extension* (cf. 2.2.7).
The functors

$$F'_Y : \mathbb{Z}^{n_Y} \to [\ \overline{Y}\textbf{-TCoh-can}\ ]$$

(where $\overline{Y}$ is equipped with its structure as restricted toroidal compactification) together with the maps induced by the μ_{ZY} satisfy the requirements of Lemma 2.3.7. Hence we get a coherent sheaf $\Xi^*(\{F_Y\})$ on $\overline{M}$ which carries a $\mathbb{G}_m^{n_Y}$ action on $C_{\overline{Y}}(\Xi^*(\{F_Y\}))$.
We call the sheaves in the image of Ξ^* **generalized automorphic sheaves**.

Example 3.4.4. *The easiest case is just*

$$\Xi^* V := (p_M^* V)^{P_M}$$

where V is a bundle on $[\ M$**-FJ-coh** $] = [\ [M^\vee/P_M]$**-Coh** $]$. *It is a vector bundle which is a canonical extension itself and can be described by the collection of functors*

$$F_Y : v \mapsto \begin{cases} \alpha_{YM}^* V & v \in \mathbb{Z}_{\geq 0}^{n_Y} \\ 0 & \text{otherwise.} \end{cases}$$

Sheaves of this form are locally free and called **automorphic vector bundles**.

Remark 3.4.5. *The Fourier-Jacobi categories are related to the classical Fourier-Jacobi expansions as follows. For each* $F \in [\ \overline{M}$**-FJ** $]$ *and stratum Y there is a morphism* **Fourier-Jacobi expansion***:*

$$H^0(\overline{M}, \Xi^* F) \to \prod_{v \in \mathbb{Z}^{n_Y}} H^0(\overline{M}, \Xi^* F_v),$$

where F_v is the following element of $F \in [\ \overline{Y}$**-FJ** $]$. *On Y it is defined by*

$$F_{v,Y}(w) = \begin{cases} F_Y(v) & \text{for } w = v, \\ 0 & \text{otherwise.} \end{cases}$$

and is a similar restriction of F on strata $Z \leq Y$ and 0 on all other. Note that $\Xi^* F_v$ has support on $\overline{Y}$.

Definition 3.4.6. *For the category* $[\ \overline{M}$**-FJ-tf-coh** $]$ *we define a tensor product mimimicing the tensor product of 2.2.6. Let F and G be objects of* $[\ \overline{M}$**-FJ-tf-coh** $]$. *We define*

$$(F \otimes G)_Y : v \mapsto \sum_{v_1 + v_2 = v} F_Y(v_1) \otimes G_Y(v_2)$$

where the sum is formed in $(\lim_{v\to\infty} F_Y(v)) \otimes (\lim_{v\to\infty} G_Y(v))$.

Lemma 3.4.7. *The exact functor (cf. 3.4.3)*

$$\Xi^* : [\ \overline{M}\textbf{-FJ-coh}\ ] \to [\ \overline{M}\textbf{-TCoh}\ ]$$

preserves the tensor product when restricted to $[\ \overline{M}$**-FJ-tf-coh** $]$.

3.4.8. For each pair (Y, v) where Y is a stratum and $v \in \mathbb{Z}^{n_Y}$ there exist restriction functors:

$$\begin{array}{llcl} (v)_Y^* : & [\ \overline{M}\textbf{-FJ-}{\geq N}\textbf{-coh}\ ] & \to & [\ [M_Y^\vee/P_Y]\textbf{-Coh}\ ] \\ (v)_Y^* : & [\ \overline{M}\textbf{-FJ}\ ] & \to & [\ [M_Y^\vee/P_Y]\textbf{-Qcoh}\ ] \\ (v)_Y^* : & [\ \overline{M}\textbf{-FJ-}{\geq N}\ ] & \to & [\ [M_Y^\vee/P_Y]\textbf{-Qcoh}\ ] \end{array}$$

given by $F \mapsto F_Y(v)$. Those are exact and have each an *exact right-adjoint* $(v)_{Y,*}$ which is given as follows. $((v)_{Y,*}V)_Y$ is given by the right Kan-extension v_*, where $v : \{\cdot\} \hookrightarrow \mathbb{Z}^{n_Y}$, resp. $v : \{\cdot\} \hookrightarrow \mathbb{Z}_{\geq N}^{n_Y}$ also denotes the inclusion of v. In other words we have

$$((v)_{Y,*}V)_Y(w) = \begin{cases} V & \text{if } w \leq v \text{ (and p.r.n. } w_i \geq N \text{ for all } i) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $v \le w$ means that $v_i \le w_i$ for all i. For any stratum $Z \le Y$ we define

$$((v)_{Y,*}V)_Z(v) \coloneqq \alpha_{ZY}^*((v)_{Y,*}V)_Y(\mathrm{pr}(v))$$

where $\mathrm{pr} : \mathbb{Z}^{n_Z} \to \mathbb{Z}^{n_Y}$ is the projection induced by β_{ZY}. In the bounded case it is set identically zero if $v_i < N$ for any i. For all other strata Z the functor $((v)_{Y,*}V)_Z$ is set identically zero. The so defined object $(v)_{Y,*}V$ satisfies conditions 1. and 2. of the definition of the Fourier-Jacobi category 3.4.1.

3.4.9. For each stratum Y and each $N \in \mathbb{Z}$, there are exact restriction functors

$$\iota_N^* : [\ \overline{Y}\textbf{-FJ-coh}\ ] \to [\ \overline{Y}\textbf{-FJ-}{\ge}\, N\textbf{-coh}\ ]$$

which have an *exact left-adjoint*

$$\iota_{N,!} : [\ \overline{Y}\textbf{-FJ-}{\ge}\, N\textbf{-coh}\ ] \hookrightarrow [\ \overline{Y}\textbf{-FJ-coh}\ ]$$

which is given by the natural inclusion (or, in other words, by extension by zero or left Kan extension for the individual F_Z).

Corollary 3.4.10. *For Y and $v \in \mathbb{Z}^{n_Y}$ appropriate, there are* fully-faithful *functors of categories*

$$(v)_{Y,*} : D^{\star}([\ [M_Y^\vee/P_Y]\textbf{-Coh}\ ]) \hookrightarrow D^{\star}([\ \overline{M}\textbf{-FJ-}{\ge}\, N\textbf{-coh}\ ])$$

and

$$\iota_{N,!} : D^{\star}([\ \overline{M}\textbf{-FJ-}{\ge}\, N\textbf{-coh}\ ]) \hookrightarrow D^{\star}([\ \overline{M}\textbf{-FJ-coh}\ ])$$

for $\star \in \{b, +, -, \varnothing\}$.

Proof. We have in each case a pair of adjoint functors in which the unit, resp. the counit, is an isomorphism. Since all four functors are exact, they induce functors on the derived categories without modification, and form again pairs of adjoint functors (because the counit/unit-equations still hold). Since also the unit, resp. the counit, is still an isomorphism we get the requested fully-faithfulness of the left- (resp. right-) adjoint. □

In particular, for $Y = M$ and $N = 0$ we get that the canonical extension functor (cf. Example 3.4.4) is fully-faithful on the level of derived categories.

Remark 3.4.11. *The statement of Corollary 3.4.10 is also true for the functors*

$$(v)_{Y,*} : D^{\star}([\ [M_Y^\vee/P_Y]\textbf{-Qcoh}\ ]) \hookrightarrow D^{\star}([\ \overline{M}\textbf{-FJ-}{\ge}\, N\ ])$$

and

$$\iota_{N,!} : D^{\star}([\ \overline{M}\textbf{-FJ-}{\ge}\, N\ ]) \hookrightarrow D^{\star}([\ \overline{M}\textbf{-FJ}\ ]).$$

We also have the following two lemmas, which however will not be needed in the sequel.

Lemma 3.4.12. *The categories* $[\ \overline{M}\textbf{-FJ-}{\ge}\, N\ ]$ *and* $[\ \overline{M}\textbf{-FJ}\ ]$ *do have enough injectives (while* $[\ \overline{M}\textbf{-FJ-}{\ge}\ ]$ *does not).*

Proof. For any object $\{F_Y\}$ we define an injective resolution by

$$\prod_{(Y,v),v_i\le N_Y} (v)_{Y,*} I((v)_Y^* F)$$

where $I((v)_Y^* F)$ is an injective resolution of $(v)_Y^* F$ in the category $[\ [M_Y^\vee/P_Y]\text{-}\mathbf{Qcoh}\ ]$. Note that right-adjoints of exact functors and $\prod$ preserve injective objects. Here N_Y is some appropriate upper bound for the stratum Y. Note that because of the bound, the product exists (as opposed to general products in $[\ \overline{M}\text{-}\mathbf{FJ}\text{-}\ge N\ ]$ and $[\ \overline{M}\text{-}\mathbf{FJ}\ ]$). □

Lemma 3.4.13. *The functors*

$$D^{☆}([\ \overline{M}\text{-}\mathbf{FJ}\text{-}\ge N\text{-}\mathbf{coh}\ ]) \hookrightarrow D^{☆}([\ \overline{M}\text{-}\mathbf{FJ}\text{-}\ge N\ ])$$

$$D^{☆}([\ \overline{M}\text{-}\mathbf{FJ}\text{-}\mathbf{coh}\ ]) \hookrightarrow D^{☆}([\ \overline{M}\text{-}\mathbf{FJ}\text{-}\ge\ ])$$

are fully-faithful for $☆ \in \{b, -\}$.

Proof. Follows from (the dual of) [11, Theorem 13.2.8]. □

These two lemmas imply, in particular, that $D^b([\ \overline{M}\text{-}\mathbf{FJ}\text{-}\ge N\text{-}\mathbf{coh}\ ])$ and $D^b([\ \overline{M}\text{-}\mathbf{FJ}\text{-}\mathbf{coh}\ ])$ are locally small.

3.5 Jet bundles in the Fourier-Jacobi categories

3.5.1. We write $M_Y := C_{\overline{Y}}(\overline{M})$ and $M_Y|_Y$ for the formal open subscheme on Y. Recall the definition of the vector bundle $\Omega_{\overline{M}}(\log)$ on a variety with divisor of normal crossings. Locally the bundle $C_{\overline{Y}}(\Omega_{\overline{M}}(\log))|_Y$ is the bundle $\Omega_{M_Y|_Y}(\log)$ (defined in 2.2.8) on the formally toroidal formal scheme $M_Y|_Y$, but not on M_Y! Recall the description of the associated functor of $\Omega_{M_Y|_Y}(\log)$ on $M_Y|_Y$ from 2.2.8.
By Theorem 3.3.10 the vector bundle $\Omega_{\overline{M}}(\log)$ on $\overline{M}$ can therefore be obtained by glueing and is associated with the following element in $[\ \overline{M}\text{-}\mathbf{FJ}\text{-}\mathbf{lf}\text{-}\mathbf{coh}\ ]$:

$$F_Y : v \mapsto \begin{cases} \Omega_{M_Y^\vee} & \text{if } v \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $Z \le Y$ the restriction $\alpha_{ZY}^* \Omega_{M_Y^\vee}$ is canonically isomorphic to $\Omega_{M_Z^\vee}$ because α_{ZY} is supposed to be an open embedding by definition.
If the given automorphic data with flat logarithmic connection satisfies the unipotent monodromy axiom (M) then the subbundle $\Omega_{\overline{M}}$ can be described by the following functor

$$F_Y : v \mapsto \begin{cases} \{\xi \in \Omega_{M_Y^\vee} \mid p_i(\xi) = 0\ \forall i : v_i = 0\} & \text{if } v \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Here p_i is given as follows: We have by the unipotent monodromy axiom that there are subbundles $T^{(i)}_{M_Y^\vee} \subset T_{M_Y^\vee}$ given by the Lie algebras $\mathfrak{u}_i$ of 1-dimensional normal unipotent subgroups $U_i \subset G_Y$. (In other words, choosing a base point $Q_Y \in M_Y^\vee$, this is induced by the inclusion $\mathfrak{u}_i \to \mathfrak{p}_Y/\mathfrak{q}_Y$. Since $\mathfrak{u}_i$ is normal, the associated P_Y-equivariant bundles do not depend on the choice of base point.) p_i is then defined as the projection dual to this inclusion. By the unipotent monodromy axiom (M) we have $\mathcal{O}_{B_Y} \cdot \pi^{-1}(\mathrm{can}_{i,M_Y}) \cong p^*(T^{(i)}_{M_Y^\vee})$ under the natural P_Y-equivariant isomorphism

$$\pi^* \mathcal{T}_{M_Y}(\log) \cong p^* \mathcal{T}_{M_Y^\vee}.$$

It follows therefore from the proof of Theorem 3.3.10 that $\Omega_{\overline{M}}$ is associated with this subfunctor.

3.5.2. Assume for the rest of the section that there exists a k-valued point in $M^\vee$ and let Q_M be the corresponding quasi-parabolic subgroup of P_M. The discussion in the preceeding paragraph enables us to refine Theorem 3.3.10. Given a Q_M-representation V or equivalently a P_M-equivariant vector bundle $\widetilde{V}$ on $M^\vee$ we define the object $(J^n\widetilde{V})'$ in [$\overline{M}$**-FJ-lf-coh**] by

$$(J^n\widetilde{V})'_Y : v \mapsto J^n(\widetilde{V})^v$$

where we define a $\mathbb{Z}^{n_Y}$-indexed filtration on $J^n(\widetilde{V})$ induced by the dual of the filtration on $(U(\mathfrak{p}_Y) \otimes_{U(\mathfrak{q}_Y)} V^*)^{\le n}$ given by the trivial filtration on V^* and the filtration on $U(\mathfrak{p}_Y)$ which is the quotient of the induced filtration on $T(\mathfrak{p}_Y)$ (tensor algebra) of the following filtration on $\mathfrak{p}_Y$:

$$\mathfrak{p}_Y(v) = \begin{cases} \mathfrak{p}_Y & v \ge 0 \\ \mathfrak{u}_i & v_i = -1 \text{ and } v_j \ge 0 \ \forall j \ne i \\ 0 & \text{otherwise.} \end{cases}$$

(This is essentially the dual of (8).)

Theorem 3.5.3. *Let V be a representation of Q_M, and let $\mathcal{V} := \Xi^*\widetilde{V}$ be the corresponding automorphic vector bundle on $\overline{M}$. Then the generalized automorphic sheaf associated with the element $(J^n\widetilde{V})'$ in [$\overline{M}$**-FJ-lf-coh**] is precisely $J^n\mathcal{V}$.*

3.5.4. Define $\omega_{\overline{M}}(\log) := \Lambda^n(\Omega_{\overline{M}}(\log))$, where $n = \dim(M)$. By Proposition 3.3.10, this is an automorphic line bundle associated with $\omega_{M^\vee}$ and by the above discussion the subbundle $\omega_{\overline{M}} \subset \omega_{\overline{M}}(\log)$ is a generalized automorphic sheaf on $\overline{M}$ given by the functors

$$\omega_Y : v \mapsto \begin{cases} \omega_{M_Y^\vee} & \text{if } v_i \ge 1 \ \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words it is given by $\iota_{1,!}\ (0)_{M,*}\ \omega_{M^\vee}$, where $(0)_{Y,*}$ is considered as a functor with values in [$\overline{M}$**-FJ-**≥ 1**-coh**]. Note that $\omega_{M_Y^\vee}$ is associated with the Q_Y-representation $\Lambda^n(\mathfrak{p}_Y/\mathfrak{q}_Y)$. We also define the following generalized automorphic sheaves ω_Y associated with the functor in [Y**-FJ-coh**]:

$$(\omega_Y)_Y : v \mapsto \begin{cases} \omega_{M_Y^\vee} & \text{if } v = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It extends (as canonical extension along smaller strata) to an element $\omega_{\overline{Y}}$ in [$\overline{Y}$**-FJ-coh**] (cf. 3.4.8). In other words $\omega_{\overline{Y}}$ is given by $\iota_{0,!}\ (0)_{Y,*}\ \omega_{M_Y^\vee}$, where $(0)_{Y,*}$ is considered as a functor with values in [$\overline{M}$**-FJ-**≥ 0**-coh**].

Lemma 3.5.5. *There is an exact sequence in [$\overline{M}$**-FJ-coh**]*

$$0 \longrightarrow \omega \longrightarrow \omega_{M^\vee} \longrightarrow \bigoplus_{Y \text{ codim 1 strata}} \omega_{\overline{Y}} \longrightarrow \bigoplus_{Y \text{ codim 2 strata}} \omega_{\overline{Y}} \longrightarrow \cdots$$

where the sums go over certain multi-*sets of strata which we will not specify because we do not need them explicitly.*

3.6 Automorphic data on toroidal compactifications of (mixed) Shimura varieties

3.6.1. The toroidal compactifications of (mixed) Shimura varieties are naturally equipped with automorphic data with logarithmic connection in the sense of Definition 3.1.1. We only sketch the relation with the theory of mixed Shimura varieties and their toroidal compactifications in this section, hinting at the reasons for the axioms to be satisfied. The only exception is the boundary vanishing axiom which will be investigated more in detail.
Firstly we may fix the particular boundary component $\mathbf{B}_Y$ in its conjugacy class such that for $Z \leq Y$ we get a boundary map $\mathbf{B}_Z \to \mathbf{B}_Y$, i.e. a closed embedding $P_{\mathbf{B}_Z} \hookrightarrow P_{\mathbf{B}_Y}$ together with a compatible open embedding $\mathbb{D}_{\mathbf{B}_Y}$ into $\mathbb{D}_{\mathbf{B}_Z}$. By [8, Main Theorem 2.5.12] to each of these boundary components $\mathbf{B}_Y$ there exists a "compact" dual $\mathrm{M}^\vee(\mathbf{B}_Y)$ (which is only proper for $\mathbf{B}_Y = \mathbf{X}$, i.e. $Y = M$, if $\mathbf{X}$ is itself pure). It is of the form $M_Y^\vee$ as required in the definition of automorphic data, i.e. a $P_{\mathbf{B}}$-equivariant component in the classifying space of quasi-parabolics for $P_{\mathbf{B}}$. Except possibly in the case $Y = M$ if M is already proper (i.e. where there is nothing to compactify) $M^\vee$ is even defined over $\mathbb{Q}$ and there is a $\mathbb{Q}$-rational point in $M^\vee(\mathbf{B}_Y)$, i.e. a quasi-parabolic $Q_Y \subset P_{\mathbf{B}_Y}$ such that $M_Y^\vee = P_{\mathbf{B}_Y}/Q_Y$. For the definition of automorphic data, we will, however, consider all varieties and groups as schemes over the reflex field $E(\mathbf{X})$.

3.6.2. The following is a summary of [8, Main Theorem 2.5.14]. For each stratum Y there is a $P_{\mathbf{B}_Y}$-principal bundle $\mathrm{B}({}^{K_Y}_{\Delta_Y}\mathbf{B})$ over the mixed Shimura variety $\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y)$ together with an equivariant map to the "compact" dual:

$$\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y) \xleftarrow{p} \mathrm{B}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y) \xrightarrow{\pi} \mathrm{M}^\vee(\mathbf{B}_Y)$$

Because of the functoriality (the torus action comes from a morphism of mixed Shimura data) the morphism p is $\mathbb{M}_m^{n_Y}$-equivariant and the morphism π is $\mathbb{M}_m^{n_Y}$-invariant. These data are compatible in the sense that if we have strata $Z \leq Y$ then there is a commutative diagram

$$\begin{array}{ccccccc}
C_{\overline{Z}}\mathrm{M}({}^{K}_{\Delta}\mathbf{X}) & \xrightarrow{\sim} & C_{\overline{\widetilde{Z}}}\mathrm{M}({}^{K_Z}_{\Delta_Z}\mathbf{B}_Z) & \xleftarrow{p} & C_{p^{-1}\overline{\widetilde{Z}}}\mathrm{B}({}^{K_Z}_{\Delta_Z}\mathbf{B}_Z) & \longrightarrow & \mathrm{M}^\vee(\mathbf{B}_Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_{\overline{Y}}\mathrm{M}({}^{K}_{\Delta}\mathbf{X}) & \xrightarrow{\sim} & C_{\overline{\widetilde{Y}}}\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y) & \xleftarrow{p} & C_{p^{-1}\overline{\widetilde{Y}}}\mathrm{B}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y) & \longrightarrow & \mathrm{M}^\vee(\mathbf{B}_Y)
\end{array}$$

where the maps are functorial w.r.t. relations $W \leq Z \leq Y$ of strata.
The flat logarithmic connection can be defined analytically by means of the flat section ξ on the universal cover given as follows:

$$\begin{array}{ccccc}
\mathbb{D}_{\mathbf{B}_Y} \times P_{\mathbf{B}_Y}(\mathbb{A}^{(\infty)})/K_Y & \xrightarrow{\xi:[\tau,g]\mapsto[\tau,1,g]} & & & \\
\downarrow & & & & \\
P_{\mathbf{B}_Y}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{B}_Y} \times P_{\mathbf{B}_Y}(\mathbb{A}^{(\infty)})/K_Y & \longleftarrow & P_{\mathbf{B}_Y}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{B}_Y} \times P_{\mathbf{B}_Y}(\mathbb{C}) \times P_{\mathbf{B}_Y}(\mathbb{A}^{(\infty)})/K_Y & \longrightarrow & P_{\mathbf{B}_Y}(\mathbb{C})/Q_Y(\mathbb{C})
\end{array}$$

That the corresponding connection is defined over $E(G, X)$ can be deduced from [3, 3.4]. In purely algebraic constructions of Shimura varieties as moduli spaces it comes from the Gauss-Manin connection on the cohomology bundle.

3.6.3. The Torelli axiom (T) is immediately clear analytically from the picture above since the composition

$$\mathbb{D}_{\mathbf{B}_Y} \times P_{\mathbf{B}_Y}(\mathbb{A}^{(\infty)})/K_Y \to P_{\mathbf{B}_Y}(\mathbb{C})/Q_Y(\mathbb{C})$$

is the Borel *open embedding* (after projection to the first factor). In purely algebraic constructions of Shimura varieties it corresponds to infinitesimal Torelli theorems of the parametrized objects which can be proven purely algebraically.

3.6.4. The unipotent monodromy axiom (M) is satisfied because the cone σ describing a boundary component sits per definition in $U_{\mathbf{B}_Y,\mathbb{R}}(-1)$ and $U_{\mathbf{B}_Y} \cong \mathbb{G}_a^u$ is a normal subgroup of $P_{\mathbf{B}_Y}$. By construction the fundamental vector fields can_i of the action of $\mathbb{G}_m^{n_Y}$ on $\mathrm{M}({}^{K_Y}_{\Delta_Y}\mathbf{B}_Y)$ lifted to the universal cover correspond to the basis-vectors of $(U_{\mathbf{B}_Y} \cap K_Y)(-1)$ spanning σ. In cases, in which the mixed Shimura variety is constructed using a moduli problem of 1-motives as in [8, 2.7], the unipotent monodromy axiom can be read off from the construction.

Proposition 3.6.5 (Boundary vanishing condition (B))**.** *Let* $\mathbf{Y}$ *be a mixed Shimura datum (e.g. one of the boundary components* $\mathbf{B}_Y$*), let* n *be the dimension of* $\mathrm{M}^\vee(\mathbf{Y})$*, let* Q *be one of the quasi-parabolics parametrized by* $\mathrm{M}^\vee(\mathbf{Y})$*, let* ω *be the* Q*-representation corresponding to the* $P_{\mathbf{X}}$*-equivariant bundle* $\Omega^n_{\mathrm{M}^\vee(\mathbf{Y})}$ *on* $\mathrm{M}^\vee(\mathbf{Y})$*, and let* u *be the dimension of* $U_{\mathbf{Y}}$*. Then we have:*

$$H^i([\cdot/Q], \omega) = 0$$

for all $i \geq n - u$ *provided* $u \neq 0$*.*

Note that all boundary strata Y which come from rational polyhedral cones in the unipotent cone satisfy $\dim(Y) \geq n - u$.

Proof. W.l.o.g. we may assume that the base field of the category of Q-representations is $\mathbb{C}$ and that all algebraic groups involved are defined over $\mathbb{C}$. We have the following zoo of connected linear algebraic groups (cf. [8, 2.2] or [13]):

$\mathbb{S}$	$=$	$\mathbb{G}_m^2$
$P = P_{\mathbf{Y}}$	$=$	$G \cdot V \cdot U$
$G = G_{\mathbf{Y}}$		is a maximal reductive subgroup
$V = V_{\mathbf{Y}}$	$\cong$	$\mathbb{G}_a^{2v}$
$U = U_{\mathbf{Y}}$	$\cong$	$\mathbb{G}_a^u$
$h : \mathbb{S} \to G$		any homomorphism in $h_{\mathbf{Y}}(\mathbb{D}_{\mathbf{Y}})$
R	$=$	$K \cdot R^+ = G \cap Q$
		is a parabolic in G (with its Levi decomposition)
R^+, R^-	$\cong$	$\mathbb{G}_a^{n_0}$
V	$=$	$V^+ \cdot V^-$
V^+	$=$	$Q \cap V$
Q	$=$	$R \cdot V^+$ is the quasi-parabolic defining $M^\vee(\mathbf{Y})$

By definition of a mixed Shimura datum the Lie algebras of these groups have the following weights under $\mathbb{S}$:

$\mathrm{Lie}(U)$	$(-1,-1)$

$\mathrm{Lie}(V^+)$	$(-1,0)$
$\mathrm{Lie}(V^-)$	$(0,-1)$

$\mathrm{Lie}(R^+)$	$(-1,1)$
$\mathrm{Lie}(K)$	$(0,0)$
$\mathrm{Lie}(R^-)$	$(1,-1)$

We have the following sequence of affine morphisms

$$M^\vee(\mathbf{Y}) = P/(R\cdot V^+) \to G\cdot V/(R\cdot V^+) \to G/R$$

of relative dimensions $u = \dim(U)$, and $v = \dim(V^-)$, respectively. G/R is a projective flag variety of dimension $n_0 = \dim(R^+)$.
STEP 1: We have

$$H^i([\cdot/P],\omega) = H^i([\cdot/V^+\cdot R^+],\omega)^K$$

because K is reductive. Furthermore since ω is 1-dimensional and hence trivial as a V^+ and R^+ representation, we have as K-representations

$$H^i([\cdot/V^+\cdot R^+],\omega) = H^i([\cdot/V^+\cdot R^+],\mathbb{C})\otimes\omega.$$

STEP 2: V^+ and R^+ commute (because there is no part of the Lie algebra of weight $(-2,1)$). Hence $H^i([\cdot/V^+\cdot R^+],\mathbb{C})$ is just the cohomology of $\mathbb{G}_a^{n_0+v}$ w.r.t. the trivial representation. Hence $H^i([\cdot/V^+\cdot R^+],\mathbb{C}) = \Lambda^i(\mathrm{Lie}(V^+)^*\oplus\mathrm{Lie}(R^+)^*)$ as natural $\mathrm{Aut}(V^+\cdot R^+)$-modules [10, p.64, Remark 2)]. Therefore $H^i([\cdot/V^+\cdot R^+],\mathbb{C}) = 0$ for $i > n_0+v$ and

$$H^{n_0+v}([\cdot/V^+\cdot R^+],\mathbb{C}) = \Lambda^{n_0+v}(\mathrm{Lie}(V^+)^*\oplus\mathrm{Lie}(R^+)^*)\cong\mathbb{C}.$$

STEP 3: Since the last isomorphism is compatible w.r.t. the natural $\mathrm{Aut}(V^+\cdot R^+)$-actions, we see that $H^{n_0+v}([\cdot/V^+\cdot R^+],\mathbb{C})$ is one-dimensional of weight

$$(v+n_0,-n_0)$$

under $\mathbb{S}$. Also ω is one-dimensional of weight

$$(-u-v-n_0,n_0-u).$$

Therefore

$$H^{n_0+v}([\cdot/V^+\cdot R^+],\mathbb{C})\otimes\omega \quad \text{has weight} \quad (-u,-u)$$

and therefore cannot have any K-invariants as long as $u\neq 0$. □

4 Hirzebruch-Mumford proportionality

4.1 Chern classes

4.1.1. Let X be a smooth projective complex variety of dimension n. There are several ways of constructing the Chern classes of vector bundles on X. We will use the following, cf. [2]. Let $\mathcal{E}$ be a vector bundle on X. It defines an Atiyah extension (where J^1 is the first jet bundle (cf. Section 3.3))

$$0\longrightarrow\Omega^1_X\otimes\mathcal{E}\longrightarrow J^1\mathcal{E}\longrightarrow\mathcal{E}\longrightarrow 0.$$

Tensoring with $\mathcal{E}^*$ and pulling back along the unit $\mathcal{O}_X\to\mathcal{E}^*\otimes\mathcal{E}$ we get an extension

$$0\longrightarrow\Omega^1_X\otimes\mathrm{End}(\mathcal{E})\longrightarrow\mathcal{A}\longrightarrow\mathcal{O}_X\longrightarrow 0.$$

This might be seen as a morphism

$$\mathcal{O}_X\to\Omega^1_X\otimes\mathrm{End}(\mathcal{E})[1]$$

in $D^b([\ \mathcal{O}_X\textbf{-Coh}\ ])$. The coefficients of the characteristic polynomial of this "endomorphism" give morphisms

$$c_i(\mathcal{E}) : \mathcal{O}_X \to \Omega_X^i[i].$$

Furthermore, any polynomial p in the graded polynomial ring $\mathbb{Q}[c_1, c_2, \dots, c_n]$ (where $\deg(c_i) = i$) of degree n gives a morphism

$$p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) : \mathcal{O}_X \to \Omega_X^n[n] =: \omega_X[n].$$

The corresponding extension $p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) \in \operatorname{Ext}^n(\mathcal{O}_X, \omega_X)$ can be constructed explicitly using only locally free sheaves. Using the trace map $\operatorname{tr} : \operatorname{Ext}^n(\mathcal{O}_X, \omega_X) \to k$ of Serre duality, we get elements $\operatorname{tr}(p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E}))) \in k$. The compatibility with other constructions of Chern classes using algebraic cycles shows that even $\operatorname{tr}(p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E}))) \in \mathbb{Q}$.

4.2 Proportionality

Theorem 4.2.1 (Hirzebruch-Mumford proportionality)**.** *Let $\overline{M}$ be a toroidal compactification of dimension n equipped with automorphic data with logarithmic connection which satisfies the axioms (F, T, M, B) and such that P_M is reductive. There is $c \in \mathbb{Q}$ such that for all polynomials p of degree n in the graded polynomial ring $\mathbb{Q}[c_1, c_2, \dots, c_n]$ and all P_M-equivariant vector bundles $\mathcal{E}$ in $[\ [M^\vee/P_M]\textbf{-Coh}\ ]$ the proportionality*

$$p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) = c \cdot p(c_1(\Xi^*\mathcal{E}), \dots, c_n(\Xi^*\mathcal{E}))$$

holds true.

Proof. Starting from the sequence in $[\ \overline{M}\textbf{-FJ-coh}\ ]$ (cf. 3.5.2 for the definition of $J^1(\mathcal{E})'$):

$$0 \longrightarrow (\Omega^1)' \otimes \mathcal{E} \longrightarrow J^1(\mathcal{E})' \longrightarrow \mathcal{E} \longrightarrow 0$$

by the procedure described in the last section we can construct an element

$$\widetilde{p}(\mathcal{E}) \in \operatorname{Ext}^n_{[\ \overline{M}\textbf{-FJ-coh}\ ]}(\mathcal{O}, \omega).$$

Note that in the construction only the tensor product of locally free objects is involved and the exactness of $\otimes$ on sequences involving those.
Consider the following two compositions of functors

$$D^b([\ \overline{M}\textbf{-FJ-coh}\ ]) \xrightarrow{\mathcal{D}^b(\Xi^*)} D^b([\ \mathcal{O}_{\overline{M}}\textbf{-Coh}\ ])$$

$$D^b([\ \overline{M}\textbf{-FJ-coh}\ ]) \longrightarrow D^b([\ M\textbf{-FJ-coh}\ ]) = D^b([\ [M^\vee/P_M]\textbf{-Coh}\ ]) \longrightarrow D^b([\ \mathcal{O}_{M^\vee}\textbf{-Coh}\ ]).$$

The images of the morphism $\widetilde{p}(\mathcal{E}) : \mathcal{O} \to \omega[n]$ give

$$p(c_1(\mathcal{E}), \dots, c_n(\mathcal{E})) \quad \text{and} \quad p(c_1(\Xi^*\mathcal{E}), \dots, c_n(\Xi^*\mathcal{E}))$$

respectively. Here it is used that Ξ^* is an exact functor that is compatible with the tensor product when restricted to locally free (or even torsion-free) objects, that by Theorem 3.5.3 the image of $J^1(\mathcal{E})'$ under Ξ^* is precisely $J^1(\Xi^*\mathcal{E})$, and that the image under the second functor is $J^1(\mathcal{E})$ where the P_M-action on $\mathcal{E}$ is forgotten (by definition of $J^1(\mathcal{E})'$).
The theorem therefore follows from Proposition 4.2.2 below. In the compact case, i.e. if $M = \overline{M}$, and P_M is reductive, this is easier and Lemma 4.2.3 can be applied. □

Proposition 4.2.2. *If P_M is reductive, we have*

$$\dim(\mathrm{Ext}^n_{[\ \overline{M}\textbf{-FJ-coh}\ ]}(\mathcal{O},\omega)) \leq 1.$$

Proof. By Proposition 3.5.5 we have an exact sequence

$$0 \longrightarrow \omega \longrightarrow \omega_{M^\vee} \longrightarrow \mathcal{D} \longrightarrow 0$$

and a finite resolution

$$0 \longrightarrow \mathcal{D} \longrightarrow \bigoplus_{Y \text{ codim 1 strata}} \omega_{\overline{Y}} \longrightarrow \bigoplus_{Y \text{ codim 2 strata}} \omega_{\overline{Y}} \longrightarrow \cdots \tag{9}$$

We get the long exact sequence

$$\mathrm{Ext}^{n-1}(\mathcal{O},\mathcal{D}) \longrightarrow \mathrm{Ext}^n(\mathcal{O},\omega) \longrightarrow \mathrm{Ext}^n(\mathcal{O},\omega_{M^\vee}) \longrightarrow \mathrm{Ext}^n(\mathcal{O},\mathcal{D})$$

(all Ext-groups are computed in the category $[\ \overline{M}\textbf{-FJ-coh}\ ]$). By Lemma 4.2.3 below the dimension of $\mathrm{Ext}^n(\mathcal{O},\omega_{M^\vee})$ is at most one. Hence it suffices to show that $\mathrm{Ext}^{n-1}(\mathcal{O},\mathcal{D}) = 0$. Splitting up the exact sequence (9) into short exact sequences one sees that it suffices to show that $\mathrm{Ext}^i(\mathcal{O},\omega_{\overline{Y}}) = 0$ for $i \leq \dim(Y)$ and for $Y \neq M$. We have fully-faithful embeddings (cf. Corollary 3.4.10)

$$D^b([\ [\cdot/P_Y]\textbf{-Coh}\ ]) \hookrightarrow D^b([\ \overline{M}\textbf{-FJ-}{\geq 0}\textbf{-coh}\ ])) \hookrightarrow D^b([\ \overline{M}\textbf{-FJ-coh}\ ])$$

such that the image of $\omega_{M_Y^\vee} = \Lambda^n \mathfrak{p}_Y/\mathfrak{q}_Y$ under the composition is $\omega_{\overline{Y}}$.
Furthermore we have

$$\mathcal{O} = \iota_{0,!}\,\iota_0^*\,\mathcal{O}.$$

Hence

$$\begin{array}{lll} & \mathrm{Hom}_{\mathcal{D}^b([\ \overline{M}\textbf{-FJ-coh}\ ])}(\iota_{0,!}\,\iota_0^*\,\mathcal{O}, \iota_{0,!}\,(0)_*\,\omega_{M_Y^\vee}[i]) & \\ = & \mathrm{Hom}_{\mathcal{D}^b([\ \overline{M}\textbf{-FJ-}\geq 0\textbf{-coh}\ ])}(\iota_0^*\,\mathcal{O}, (0)_*\,\omega_{M_Y^\vee}[i]) & \text{(fully-faithfulness)} \\ = & \mathrm{Hom}_{\mathcal{D}^b([\ [M_Y^\vee/P_Y]\textbf{-Coh}\ ])}(\mathcal{O}_{M_Y^\vee}, \omega_{M_Y^\vee}[i]) & \text{(adjunction)} \end{array}$$

Therefore the Proposition follows from boundary vanishing condition (axiom B):

$$H^i([M_Y^\vee/P_Y], \omega_{M_Y^\vee}) = 0 \text{ for } i \geq \dim(Y).$$

□

Lemma 4.2.3. *If P_M is reductive, we have*

$$\dim(\mathrm{Ext}^n_{[\ \overline{M}\textbf{-FJ-coh}\ ]}(\mathcal{O},\omega_{M^\vee})) \leq 1.$$

Proof. We have a fully-faithful embedding (cf. Corollary 3.4.10)

$$D^b([\ [M^\vee/P_M]\textbf{-Coh}\ ]) \hookrightarrow D^b([\ \overline{M}\textbf{-FJ-coh}\ ]).$$

The functor $R\,\mathrm{Hom}(\mathcal{O},-)$ is the same as the composion

$$D^b([\ [M^\vee/P_M]\textbf{-Coh}\ ]) \to D^b([\ [\cdot/P_M]\textbf{-Coh}\ ]) \to D^b([\ \mathrm{spec}(k)\textbf{-Coh}\ ])$$

where the first functor is the right derived functor of taking global sections and the second is the functor of P_M-invariants. However, the last functor is exact (because P_M is reductive) and therefore we have

$$\mathrm{Ext}^n_{[\ \overline{M}\textbf{-FJ-coh}\ ]}(\mathcal{O},\omega_{M^\vee}) = H^n(M^\vee,\omega_{M^\vee})^{P_M}.$$

Since $H^n(M^\vee,\omega_{M^\vee})$ is one-dimensional by Serre duality, the Lemma follows. □

Remark 4.2.4. *For the compact dual associated with a pure Shimura datum this can also be seen from the explicit calculation in the proof of Proposition 3.6.5, which shows that the dimension is in fact one.*

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