Derivator Six-Functor-Formalisms — Definition and Construction I

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Abstract

A theory of a *derivator version* of six-functor-formalisms is developed, using an extension of the notion of fibered multiderivator due to the author. Using the language of (op)fibrations of 2-multicategories this has (like a usual fibered multiderivator) a very neat definition. This definition not only encodes all compatibilities among the six functors but also their interplay with homotopy Kan extensions. One could say: a nine-functor-formalism. This is essential, for instance, to deal with (co)descent questions. Finally, it is shown that every fibered multiderivator (for example encoding any kind of derived four-functor formalism $(f_*, f^*, \otimes, \mathcal{HOM})$ occurring in nature) satisfying base-change and projection formula *formally* gives rise to such a derivator six-functor-formalism in which "$f_! = f_*$", i.e. a derivator Grothendieck context.

Contents

1 Introduction

A formalism of the “six functors” lies at the core of many different theories in mathematics, as for example the theory of Abelian sheaves on topological spaces, etale, l-adic, or coherent sheaves on schemes, D-modules, representations of (pro-)finite groups, motives, and many more. Given a base category of “spaces” $\mathcal{S}$, for instance, the category of schemes, topological spaces, analytic manifolds, etc. such a formalism roughly consists of a collection of (derived) categories $\mathcal{D}_S$ of “sheaves”, one for each “base space” S in $\mathcal{S}$, and the following six types of functors between those categories:

$$
\begin{array}{ccl}
f^* & f_* & \textit{for each } f \textit{ in } \mathrm{Mor}(\mathcal{S}) \\[1em]
f_! & f^! & \textit{for each } f \textit{ in } \mathrm{Mor}(\mathcal{S}) \\[1em]
\otimes & \mathcal{HOM} & \textit{in each fiber } \mathcal{D}_S
\end{array}
$$

The functors on the left hand side are left adjoints of the functors on the right hand side. The functor $f_!$ is “the dual of f_*” and is called **push-forward with proper support**, because in the topological setting (Abelian sheaves over topological spaces) this is what it is derived from. Its right adjoint $f^!$ is called the **exceptional pull-back**. These functors come along with a bunch of compatibilities between them.

1.1. More precisely, part of the datum of the six functors are the following natural isomorphisms in the “left adjoints” column:

	isomorphisms between left adjoints	isomorphisms between right adjoints
$(*,*)$	$(fg)^* \xrightarrow{\sim} g^* f^*$	$(fg)_* \xrightarrow{\sim} f_* g_*$
$(!,!)$	$(fg)_! \xrightarrow{\sim} f_! g_!$	$(fg)^! \xrightarrow{\sim} g^! f^!$
$(!,*)$	$g^* f_! \xrightarrow{\sim} F_! G^*$	$G_* F^! \xrightarrow{\sim} f^! g_*$
$(\otimes,*)$	$f^*(- \otimes -) \xrightarrow{\sim} f^* - \otimes f^* -$	$f_* \mathcal{HOM}(f^* -, -) \xrightarrow{\sim} \mathcal{HOM}(-, f_* -)$
$(\otimes,!)$	$f_!(- \otimes f^* -) \xrightarrow{\sim} (f_! -) \otimes -$	$f_* \mathcal{HOM}(-, f^! -) \xrightarrow{\sim} \mathcal{HOM}(f_! -, -)$
		$f^! \mathcal{HOM}(-, -) \xrightarrow{\sim} \mathcal{HOM}(f^* -, f^! -)$
$(\otimes,\otimes)$	$(- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$	$\mathcal{HOM}(- \otimes -, -) \xrightarrow{\sim} \mathcal{HOM}(-, \mathcal{HOM}(-, -))$

Here f, g, F, G are morphisms in $\mathcal{S}$, which in the $(!,*)$-row, are related by a *Cartesian* diagram:

$$
\begin{tikzcd}
\cdot \arrow[r, "G"] \arrow[d, "F"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}
$$

In the right hand side column the corresponding adjoint natural transformations have been inserted. In each case the left hand side natural isomorphism uniquely determines the right hand side one and vice versa. (In the $(\otimes,!)$-case there are two versions of the commutation between the right adjoints; in this case any of the three isomorphisms determines the other two). The $(!,*)$-isomorphism (between left adjoints) is called **base change**, the $(\otimes,!)$-isomorphism is called the

projection formula, and the $(*, \otimes)$-isomorphism is usually part of the definition of a **monoidal functor**. The $(\otimes, \otimes)$-isomorphism is the associativity of the tensor product and part of the definition of a monoidal category. The $(*, *)$-isomorphism, and the $(!, !)$-isomorphism, express that the corresponding functors arrange as a pseudo-functor with values in categories.
Furthermore part of the datum are isomorphisms

$$f^* \xrightarrow{\sim} f^!$$

for all isomorphisms f[1]. Of course, there have to be compatibilities among those natural isomorphisms, and it is not easy to give a complete list of them. In [4] we explained how to proceed in a more abstract way (like in the ideas of fibered category or multicategory) and get a *precise* definition of a **six-functor-formalism** without having to specify any of these compatibilities explicitly. The natural isomorphisms of 1.1 are derived from a composition law in a **2-multicategory** and all compatibilities will be just a consequence of the associativity of this composition law, see below.
The six functors are the right framework to study duality theorems like Serre duality, Poincaré duality, various (Tate) dualities for the (co)homology of groups, etc.

Example 1.2 (Serre duality)**.** *Let k be a field. If $\mathcal{S}$ is the category of k-varieties, we have a six-functor-formalism in which $\mathcal{D}_S$ is the derived category of (quasi-)coherent sheaves*[2] *on S. Let $\pi : S \to \mathrm{spec}(k)$ be a proper and smooth k-scheme of dimension n. Consider a locally free sheaf $\mathcal{E}$ on S and consider the following isomorphism (one of the two adjoints of the projection formula):*

$$\pi_* \mathcal{HOM}(\mathcal{E}, \pi^! k) \xrightarrow{\sim} \mathcal{HOM}(\pi_! \mathcal{E}, k)$$

In this case, we have $\pi_! \mathcal{E} = \pi_* \mathcal{E}$ because π is proper, and $\pi^! k = \Omega^n_S[n]$. Taking i-th homology of complexes we arrive at

$$H^{i+n}(S, \mathcal{E}^\vee \otimes \Omega^n_S) \cong H^{-i}(S, \mathcal{E})^*.$$

This is the classical formula of Serre duality.

Example 1.3 (Poincaré duality)**.** *Let k be a field. If $\mathcal{S}$ is a category of nice topological spaces, we have a six-functor-formalism in which $\mathcal{D}_S$ is the derived category of sheaves of k-vector spaces on S. Let X be an n-dimensional topological manifold. Consider a local system $\mathcal{E}$ of k-vector spaces on X and consider the isomorphism (again one of the two adjoints of the projection formula):*

$$\pi_* \mathcal{HOM}(\mathcal{E}, \pi^! k) \xrightarrow{\sim} \mathcal{HOM}(\pi_! \mathcal{E}, k)$$

We have $\pi^! k = \mathcal{L}_{or}[n]$, where $\mathcal{L}_{or}$ is the orientation sheaf of X over k. Taking i-th homology of complexes we arrive at

$$H^{i+n}(X, \mathcal{E}^\vee \otimes \mathcal{L}_{or}) \cong H^{-i}_c(X, \mathcal{E})^*.$$

This is the classical formula of Poincaré duality.

Example 1.4 (Group (co)homology)**.** *The six-functor-formalism of Example 1.3 extends to stacks. Let G be a group and consider the classifying stack $[\cdot/G]$ and the projection $\pi : [\cdot/G] \to \cdot$. Note: Abelian sheaves on $[\cdot/G]$ = G-representations in Abelian groups. The extension of the six-functor-formalism encodes duality theorems like Tate duality. In this case π_* yields group cohomology and $\pi_!$ yields group homology. If G is finite, we also have a natural morphism $\pi_! \to \pi_*$ whose cone (homotopy cokernel) is Tate cohomology.*

[1] There are more general formalisms, which we call proper or etale six-functor-formalisms where there is a morphism $f^* \to f^!$ or a morphism $f_! \to f_*$ for *certain* morphisms f (cf. Section **??**)

[2] Neglecting here for a moment the fact that $f_!$ exists in general only after passing to pro-coherent sheaves.

We now recall the precise definition of six-functor-formalisms from [4]. Let $\mathcal{S}$ be a (base) category with fiber products as above. Recall the symmetric 2-multicategory $\mathcal{S}^{\mathrm{cor}}$, whose objects are the same as those of $\mathcal{S}$, and whose 1-multimorphisms $S_1, \dots, S_n \to T$ are the multicorrespondences

$$\begin{tikzcd} & & & A \arrow[llld, "g_1"'] \arrow[ld, "g_n"] \arrow[rd, "f"] & \\ S_1 & \cdots & S_n & ; & T \end{tikzcd} \tag{1}$$

with composition given by forming fiber products, and whose 2-morphisms are the isomorphisms of these multicorrespondences[3]. In [4], we explained the following formal definition of a six-functor-formalism.

Definition. *A symmetric six-functor-formalism on $\mathcal{S}$ is a 1-bifibration and 2-bifibration of symmetric 2-multicategories with 1-categorical fibers*

$$p : \mathcal{D} \to \mathcal{S}^{\mathrm{cor}}.$$

Such a fibration can also be seen as a pseudo-functor of 2-multicategories

$$\mathcal{S}^{\mathrm{cor}} \to \mathcal{CAT}$$

with the property that all multivalued functors in the image have right adjoints w.r.t. all slots. Note that $\mathcal{CAT}$, the "category"[4] of categories, has naturally the structure of a 2-"multicategory" where the 1-multimorphisms are functors of several variables.
This pseudo-functor maps the correspondence (1) to a functor isomorphic to

$$f_!((g_1^* -) \otimes_A \cdots \otimes_A (g_n^* -))$$

where $\otimes_A$, $f_!$, and g_i^*, are the images of the following correspondences

$$\begin{tikzcd} & & A \arrow[lld, equal] \arrow[ld, equal] \arrow[rd, equal] & \\ A & A & & A \end{tikzcd} \quad \begin{tikzcd} & A \arrow[ld, equal] \arrow[rd, "f"] & \\ A & & T \end{tikzcd} \quad \begin{tikzcd} & A \arrow[ld, "g_i"'] \arrow[rd, equal] & \\ S_i & & A \end{tikzcd}$$

As was explained in [4], the definition of six-functor-formalisms using $\mathcal{S}^{\mathrm{cor}}$ has the advantage that all the 6 types of isomorphisms between those functors (cf. 1.1) and all compatibilities between those isomorphisms (not easy to give a complete list!) are already encoded in this simple definition.

Definition of derivator six-functor-formalisms

In most cases occurring in nature, the values of the six-functor-formalism, i.e. the fibers of the fibration $\mathcal{D} \to \mathcal{S}^{\mathrm{cor}}$, are *derived categories*. It is therefore natural to seek to enhance the situation to a *fibered multiderivator*. We will not give a detailled account on fibered multiderivators here but say only that (in the stable case) they enhance fibrations of triangulated monoidal categories in a similar way that a usual derivator enhances triangulated categories. See [3] for a detailed introduction.

[3] Later more general definitions of 2-multicategories $\mathcal{S}^{\mathrm{cor},0}$, and $\mathcal{S}^{\mathrm{cor},G}$, where 2-morphisms can be more general morphisms between multicorrespondences will become important.

[4] having, of course, a *higher class* of objects.

Enhancements of this sort are essential to deal with (co)descent questions [loc. cit.]. The notion of a fibered multiderivator given in [loc. cit.], however, is not sufficient because $\mathcal{S}^{\mathrm{cor}}$ is a 2-multicategory (as opposed to a usual multicategory). Although the 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ gives rise to a usual (not-represented) pre-multiderivator, by identifying 2-isomorphic 1-morphisms in the diagram categories $\mathcal{S}^{\mathrm{cor}}(I)$, a fibered multiderivator over *that* pre-multiderivator would not encode what we want[5]. It turns out that the theory of fibered multiderivators over pre-multiderivators has a straightforward extension to 2-pre-multiderivators in which the knowledge of the 2-morphisms of the base is preserved.

In the first half of this article, we thus develop the theory of fibered multiderivators over 2-pre-multiderivators. This allows to consider the symmetric 2-pre-multiderivator $\mathbb{S}^{\mathrm{cor}}$ represented by the symmetric 2-multicategory $\mathcal{S}^{\mathrm{cor}}$ and to define:

Definition 6.1. *A* **(symmetric) derivator six-functor-formalism** *is a left and right fibered (symmetric) multiderivator*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}.$$

It becomes important to have notions of (symmetric) *(op)lax* fibered multiderivators as well. Those are useful to enhance to derivators the definition of a proper or etale six-functor-formalism which arises, for instance, whenever for some class of morphisms one has isomorphisms $f^! \cong f^*$ or $f_! \cong f_*$ which are part of the formalism. If this is the case for all morphisms, one speaks respectively of a Wirthmüller, or Grothendieck context.

The second half of the article is devoted to the *construction* of derivator six-functor-formalisms. There, we concentrate on the case in which $f_! = f_*$ for all morphisms f in $\mathcal{S}$, i.e. to Grothendieck contexts. (The case in which $f_! \neq f_*$ is much more involved and will be discussed in a subsequent article [5].) In the classical case this construction is almost tautological:

1. One starts with a four-functor-formalism $(f_*, f^*, \otimes, \mathcal{HOM})$ encoded by a bifibration of usual (symmetric) multicategories $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ (where $\mathcal{S}^{\mathrm{op}}$ becomes a multicategory via the product), or equivalently by a pseudo-functor of (2-)multicategories $\mathcal{S}^{\mathrm{op}} \to \mathcal{CAT}$. Then one simply defines a pseudo-functor $\mathcal{S}^{\mathrm{cor}} \to \mathcal{CAT}$ by mapping a multicorrespondence (1) to the functor

$$f_*((g_1^* -) \otimes \cdots \otimes (g_n^* -)).$$

   It is straightforward (but slightly tedious) to check that this defines a pseudo-functor if and only if base-change and projection formula hold [4, Proposition 3.13].

2. In the derived world, using theorems on Brown representability, one gets formally that the f_* functors have right adjoints $f^!$ (provided that f_* commutes with infinite coproducts as well), hence the 1-opfibration (and 2-bifibration) with 1-categorical fibers $\mathcal{E} \to \mathcal{S}^{\mathrm{cor}}$, which corresponds to the pseudo-functor in 1., is also a 1-fibration.

It is surprising, however, that constructions 1. and 2. are also possible in the world of fibered multiderivators, although they become more involved. The first is however still completely formal. We will describe the results in detail now. It turns out that one can relax the condition that $\mathcal{S}^{\mathrm{op}}$ is a multicategory coming from a usual category $\mathcal{S}$ via the categorical product in $\mathcal{S}$. One can start with any opmulticategory $\mathcal{S}$. The definition of $\mathcal{S}^{\mathrm{cor}}$ generalizes readily to this situation.

[5] E.g. the push-forward along a correspondence of the form $\{\cdot\} \leftarrow X \to \{\cdot\}$ should be something like the cohomology with compact support of X with constant coefficients. Identifying 2-isomorphic 1-morphisms in $\mathcal{S}^{\mathrm{cor}}$ would force this to become the invariant part (in a derived sense) under automorphisms of X.

Construction of derivator six-functor-formalisms

Let $\mathcal{S}$ be an opmulticategory with multipullbacks. As before, $\mathcal{S}$ will mostly be a usual category with the structure of opmulticategory given by the product, i.e.

$$\mathrm{Hom}_{\mathcal{S}}(Y; X_1, \dots, X_n) = \mathrm{Hom}_{\mathcal{S}}(Y, X_1) \times \cdots \times \mathrm{Hom}_{\mathcal{S}}(Y, X_n). \tag{2}$$

However, for the sequel $\mathcal{S}$ may be arbitrary (it also does not need to be representable). It may also be equipped with the structure of symmetric or braided opmulticategory. (Of course the structure mentioned before is canonically symmetric.) All other multicategories and 2-multicategories occurring, e.g. $\mathcal{S}^{\mathrm{cor}}$, will also be symmetric, resp. braided, and all functors have to be compatible with the corresponding actions.

Definition 1.5. *Let $\mathcal{S}$ be an opmulticategory with multipullbacks. Let $\mathcal{D} \to \mathcal{S}^{\mathrm{op}}$ be a bifibration of usual multicategories. We say that it satisfies* **multi-base-change***, if for every multipullback in $\mathcal{S}$*

$$\begin{tikzcd} X_1, \dots, X_i, \dots, X_n & Z \arrow[l, "g"'] \\ X_1, \dots, X_i', \dots, X_n \arrow[u, "{(\mathrm{id},\dots,f,\dots,\mathrm{id})}"] & Z' \arrow[l, "G"] \arrow[u, "F"'] \end{tikzcd}$$

the natural transformation

$$g_\bullet(-, \dots, \underbrace{f^\bullet -}_{at\ i}, \dots, -) \longrightarrow F^\bullet G_\bullet(-, \dots, -)$$

is an isomorphism. In the case that $\mathcal{S}$ is a usual category equipped with the opmulticategory structure (2) this encodes projection formula *and* base change*.*

In this definition, $f^\bullet$ denotes the pull-back along f^{op} in $\mathcal{S}^{\mathrm{op}}$, that is, the usual push-forward f_* along f in $\mathcal{S}$. The reason for this notation is that we stick to the convention that $f^\bullet$ is always right adjoint to $f_\bullet$ and, at the same time, we want to avoid the notation $f_*, f^*, f^!, f_!$ because of the possible confusion with the left and right Kan extension functors which will be denoted by $\alpha_!$, and α_*, respectively.

Theorem 7.1. *Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}^{\mathrm{op}}$ be the (symmetric) pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a (symmetric) left and right fibered multiderivator such that the following holds:*

1. *The pullback along 1-ary morphisms (i.e. pushforward along 1-ary morphisms in $\mathcal{S}$) commutes also with homotopy* colimits *(of shape in* Dia*).*

2. *In the underlying bifibration $\mathbb{D}(\cdot) \to \mathbb{S}(\cdot)$ multi-base-change holds in the sense of Definition 1.5.*

Then there exists a (symmetric) oplax left fibered multiderivator

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$$

satisfying the following properties

a) The corresponding (symmetric) 1-opfibration, and 2-opfibration of 2-multicategories with 1-categorical fibers

$$\mathbb{E}(\cdot) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(\cdot) = \mathcal{S}^{\mathrm{cor},G}$$

is just (up to equivalence) obtained from $\mathbb{D}(\cdot) \to \mathcal{S}^{\mathrm{op}}$ by the procedure described in [4, Definition 3.12].

b) For every $S \in \mathcal{S}$ there is a canonical equivalence between the fibers (which are usual left and right derivators):

$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Using standard theorems on Brown representability etc. [3, Section 3.1] we can refine this.

Theorem 7.2. *Let* Dia *be an infinite diagram category [3, Definition 1.1.1] which contains all finite posets. Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}$ be the corresponding represented (symmetric) pre-multiderivator. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be an* infinite *(symmetric) left and right fibered multiderivator satisfying conditions 1. and 2. of Theorem 7.1,* with stable, perfectly generated fibers *(cf. Definition 4.4 and [3, Section 3.1]).*
Then the restriction of the left fibered multiderivator $\mathbb{E}$ from Theorem 7.1 is a (symmetric) left and right *fibered multiderivator*

$$\mathbb{E}|_{\mathbb{S}^{\mathrm{cor}}} \to \mathbb{S}^{\mathrm{cor}}$$

and has an extension as a (symmetric) lax *right fibered multiderivator*

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}.$$

We call $\mathbb{E}|_{\mathbb{S}^{\mathrm{cor}}}$ together with the extensions to $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}$ and $\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$, respectively, a **derivator Grothendieck context**, cf. Definition 6.1.
The construction in [4, Definition 3.12] recalled above is quite tautological. Why is Theorem 7.1 not similarly tautological? To understand this point, let us look at the following simple example: A fibered derivator $\mathbb{D}$ over Δ_1, the (represented pre-derivator of the) usual category with one arrow, encodes an enhancement of an adjunction between derivators (the two fibers of $\mathbb{D}$). Think about the case, where this is the derived adjunction coming from an adjunction of underived functors f_*, f^*. This includes, for instance, as fiber of $\mathbb{D}(\Delta_1)$ over the identity in $\mathrm{Fun}(\Delta_1, \Delta_1)$, the category of *coherent* diagrams of the form $X \to f_*Y$, or equivalently $f^*X \to Y$, up to quasi-isomorphisms between such diagrams. Here f_*, f^* are the *underived* functors. The extension $\mathbb{E}$ in the theorem allows to consider *coherent* diagrams of the form $f_*X \to Y$, or equivalently $X \to f^!Y$, if f_* has a right adjoint $f^!$, the point being that, however, $f^!$ may not exist before passing to the derived categories. Nevertheless, we are now allowed to speak about "coherent diagrams of the form $X \to f^!Y$" although this does not make literally sense. In particular, the theorem yields a *coherent enhancement* in $\mathbb{D}(\Delta_1)_{p^*e_i}$ for $i = 0$, and $i = 1$, respectively, of the unit and counit

$$Rf_*f^!\mathcal{E} \to \mathcal{E} \qquad \mathcal{E} \to f^!Rf_*\mathcal{E}$$

where $f^!$ now denotes a right adjoint of the derived functor Rf_*.
In this particular case, the *construction* boils down to the following. The fiber of $\mathbb{E}(\Delta_1)$ over the correspondence

$$\begin{tikzcd} & e_0 \arrow[dl] \arrow[dr, equal] & \\ e_1 & & e_0 \end{tikzcd}$$

will consist of coherent diagrams of the form $f_*X \leftarrow Z \to Y$ *in the original fibered derivator* $\mathbb{D}$ with the property that the induced morphism $Rf_*X \leftarrow Z$ is a quasi-isomorphism, i.e. the morphism $X \leftarrow Z$ in $\mathbb{D}(\Delta_1)$ becomes *coCartesian*, when considered as a morphism in $\mathbb{D}(\cdot)$. The purpose of the second part of this article is thus to make this construction work in the full generality of Theorem 7.1. Although the idea is still very simple, the construction of $\mathbb{E}$ (cf. Definiton 7.17) and the proof that it really is a (left) fibered multiderivator over $\mathbb{S}^{\mathrm{cor}}$, becomes quite technical.

Localization triangles

As an application of the general definition of derivator six-functor-formalisms, in Section 9 we explain that the appearance of distinguished triangles like

$$j_! j^! \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \overline{j}_* \overline{j}^* \mathcal{E} \xrightarrow{[1]}$$

for an "open immersion" j and its complementary "closed immersion" $\overline{j}$ can be treated elegantly. Actually there are four flavours of these sequences, two for proper derivator six-functor-formalisms, and two for etale derivator six-functor-formalisms. More generally, a sequence of "open embeddings"

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n$$

leads immediately to so called $(n+1)$-angles in the sense of [2, §13] in the fiber over X_n which is a usual stable derivator.

2 2-pre-Multiderivators

We fix a diagram category Dia [3, Definition 1.1.1] once and for all. From Section 7 on, we assume that in Dia, in addition to the axioms of [loc. cit.], the construction of the diagrams ${}^{\Xi}I$ of 7.3 is permitted for any $I \in$ Dia. If one wants to specify Dia, one would speak about e.g. 2-pre-multiderivators, or fibered multiderivators, **with domain** Dia. For better readability we omit this. This is justified because all arguments of this article are completely formal, not depending on the choice of Dia at all. An exception is Theorem 7.2 where Brown representability type results are applied.

Definition 2.1. *A* **2-pre-multiderivator** *is a functor* $\mathbb{S} : \mathrm{Dia}^{1-\mathrm{op}} \to 2\text{-}\mathcal{MCAT}$ *which is strict in 1-morphisms (functors) and pseudo-functorial in 2-morphisms (natural transformations). More precisely, it associates with a diagram I a 2-multicategory $\mathbb{S}(I)$, with a functor $\alpha : I \to J$ a strict functor*

$$\mathbb{S}(\alpha) : \mathbb{S}(J) \to \mathbb{S}(I)$$

denoted also α^*, if $\mathbb{S}$ is understood, and with a natural transformation $\mu : \alpha \Rightarrow \alpha'$ a pseudo-natural transformation

$$\mathbb{S}(\eta) : \alpha^* \Rightarrow (\alpha')^*$$

such that the following holds:

1. *The association*

$$\mathrm{Fun}(I, J) \to \mathrm{Fun}(\mathbb{S}(J), \mathbb{S}(I))$$

given by $\alpha \mapsto \alpha^*$, resp. $\mu \mapsto \mathbb{S}(\mu)$, is a pseudo-functor (this involves, of course, the choice of further data). Here $\mathrm{Fun}(\mathbb{S}(J), \mathbb{S}(I))$ *is the 2-category of strict 2-functors, pseudo-natural transformations, and modifications.*

2. *(Strict functoriality w.r.t. compositons of 1-morphisms) For functors $\alpha : I \to J$ and $\beta : J \to K$, we have an* equality *of pseudo-functors* $\mathrm{Fun}(I,J) \to \mathrm{Fun}(\mathbb{S}(I), \mathbb{S}(K))$

$$\beta^* \circ \mathbb{S}(-) = \mathbb{S}(\beta \circ -).$$

A **symmetric, resp. braided 2-pre-multiderivator** *is given by the structure of strictly symmetric (resp. braided) 2-multicategory on $\mathbb{S}(I)$ such that the strict functors α^* are equivariant w.r.t. the action of the symmetric groups (resp. braid groups).*
Similarly we define a **lax, resp. oplax, 2-pre-multiderivator** *where the same as before holds but where the*

$$\mathbb{S}(\eta) : \alpha^* \Rightarrow (\alpha')^*$$

are lax (resp. oplax) natural transformations and in 1. "pseudo-natural transformations" is replaced by "lax (resp. oplax) natural transformations".

Definition 2.2. *A strict morphism $p : \mathbb{D} \to \mathbb{S}$ of 2-pre-multiderivators (resp. lax/oplax 2-pre-multiderivators) is given by a collection of strict 2-functors*

$$p(I) : \mathbb{D}(I) \to \mathbb{S}(I)$$

for each $I \in \mathrm{Dia}$ such that we have $\mathbb{S}(\alpha) \circ p(J) = p(I) \circ \mathbb{D}(\alpha)$ and $\mathbb{S}(\mu) * p(J) = p(I) * \mathbb{D}(\mu)$ for all functors $\alpha : I \to J$, $\alpha' : I \to J$ and natural transformations $\mu : \alpha \Rightarrow \alpha'$ as illustrated by the following diagram:

$$\begin{tikzcd}
\mathbb{D}(J) \arrow[rr, "p(J)"] \arrow[d, bend right, "\mathbb{D}(\alpha)"'] \arrow[d, bend left, "\mathbb{D}(\alpha')"] \arrow[d, phantom, "\overset{\mathbb{D}(\mu)}{\Rightarrow}"] & & \mathbb{S}(J) \arrow[d, bend right, "\mathbb{S}(\alpha)"'] \arrow[d, bend left, "\mathbb{S}(\alpha')"] \arrow[d, phantom, "\overset{\mathbb{S}(\mu)}{\Rightarrow}"] \\
\mathbb{D}(I) \arrow[rr, "p(I)"] & & \mathbb{S}(I)
\end{tikzcd}$$

Definition 2.3. *Given a (lax/oplax) 2-pre-derivator $\mathbb{S}$, we define*

$$\mathbb{S}^{1-\mathrm{op}} : I \mapsto \mathbb{S}(I^{\mathrm{op}})^{1-\mathrm{op}}$$

and given a (lax/oplax) 2-pre-multiderivator $\mathbb{S}$, we define

$$\mathbb{S}^{2-\mathrm{op}} : I \mapsto \mathbb{S}(I)^{2-\mathrm{op}}$$

reversing the arrow in the (lax/oplax) pseudo-natural transformations. I.e. the second operation interchanges lax and oplax 2-pre-multiderivators.

2.4. As with usual pre-multiderivators we consider the following axioms:

(Der1) For $I, J \in \mathrm{Dia}$, the natural functor $\mathbb{D}(I \coprod J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ is an equivalence of 2-multicategories. Moreover $\mathbb{D}(\varnothing)$ is not empty.

(Der2) For $I \in \mathrm{Dia}$ the 'underlying diagram' functor

$$\mathrm{dia} : \mathbb{D}(I) \to \mathrm{Fun}(I, \mathbb{D}(\cdot)) \quad \text{resp. } \mathrm{Fun}^{\mathrm{lax}}(I, \mathbb{D}(\cdot)) \quad \text{resp. } \mathrm{Fun}^{\mathrm{oplax}}(I, \mathbb{D}(\cdot))$$

is 2-conservative (this means that it is conservative on 2-morphisms and that a 1-morphism α is an equivalence if $\mathrm{dia}(\alpha)$ is an equivalence).

2.5. Let $\mathcal{D}$ be a 2-multicategory. We define some (lax/oplax) 2-pre-multiderivators which are called **representable**.
We define a 2-pre-multiderivator associated with $\mathcal{D}$ as

$$\begin{array}{rcl} \mathbb{D} : \mathrm{Dia} & \to & 2\text{-}\mathcal{MCAT} \\ I & \mapsto & \mathrm{Fun}(I, \mathcal{D}) \end{array}$$

where $\mathrm{Fun}(I, \mathcal{D})$ is the 2-multicategory of pseudo-functors, pseudo-natural transformations, and modifications. This is usually considered only if all 2-morphisms in $\mathcal{D}$ are invertible.
We define a lax 2-pre-multiderivator as

$$\begin{array}{rcl} \mathbb{D}^{\mathrm{lax}} : \mathrm{Dia} & \to & 2\text{-}\mathcal{MCAT} \\ I & \mapsto & \mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{D}) \end{array}$$

where $\mathrm{Fun}^{\mathrm{lax}}(I, \mathcal{D})$ is the 2-multicategory of pseudo-functors, *lax* natural transformations, and modifications.
We similarly define an oplax 2-pre-multiderivator $\mathbb{D}^{\mathrm{oplax}}$.

Proposition 2.6. *1. If $\mathcal{D} \to \mathcal{S}$ is a 1-fibration (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration) of 2-categories then $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-fibration (resp. 1-opfibration, resp. 2-fibration, resp. 2-opfibration) of 2-categories.*

2. If $\mathcal{D} \to \mathcal{S}$ is a 1-fibration and 2-opfibration of 2-categories then $\mathbb{D}^{\mathrm{lax}}(I) \to \mathbb{S}^{\mathrm{lax}}(I)$ is a 1-fibration and 2-opfibration of 2-categories. If $\mathcal{D} \to \mathcal{S}$ is a 1-opfibration and 2-fibration of 2-multicategories then $\mathbb{D}^{\mathrm{lax}}(I) \to \mathbb{S}^{\mathrm{lax}}(I)$ is a 1-opfibration and 2-fibration of 2-multicategories.

If $\mathcal{D} \to \mathcal{S}$ is a 1-fibration and 2-fibration of 2-categories then $\mathbb{D}^{\mathrm{oplax}}(I) \to \mathbb{S}^{\mathrm{oplax}}(I)$ is a 1-fibration and 2-fibration of 2-categories. If $\mathcal{D} \to \mathcal{S}$ is a 1-opfibration and 2-opfibration of 2-multicategories then $\mathbb{D}^{\mathrm{oplax}}(I) \to \mathbb{S}^{\mathrm{oplax}}(I)$ is a 1-opfibration and 2-opfibration of 2-multicategories.

3. If $\mathcal{D} \to \mathcal{S}$ is a 1-bifibration and 2-isofibration of 2-multicategories with complete *1-categorical fibers then $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-bifibration and 2-isofibration of 2-multicategories.*

The proof will be sketched in appendix A.

3 Correspondences of diagrams in a 2-pre-multiderivator

As explained in the introduction, the first goal of this article is to extend the notion of *fibered multiderivator* to bases which are 2-pre-multiderivators instead of pre-multiderivators. A definition as in [3], specifying axioms (FDer0) and (FDer3–5) for a morphism $p : \mathbb{D} \to \mathbb{S}$ of 2-pre-multiderivators, is possible (cf. Theorem 4.2). However, as was explained in [4] for usual fibered multiderivators, a much neater definition involving a certain category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ [4, Definition 5.6] of correspondences of diagrams in $\mathbb{S}$ is possible. In this article we take this as our principal approach. We therefore have to extend the definition of $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ for pre-multiderivators to 2-pre-multiderivators (resp. to lax/oplax 2-pre-multiderivators). For this, we first have to extend the definition of $\mathrm{Cor}_{\mathbb{S}}(I_1, \dots, I_n; J)$ to 2-pre-multiderivators $\mathbb{S}$.

Definition 3.1. *Let $\mathbb{S}$ be a (lax/oplax) 2-pre-multiderivator.*

For each collection $(I_1, S_1), \dots, (I_n, S_n); (J, T)$, where $I_1, \dots, I_n, J$ are diagrams in Dia *and $S_i \in \mathbb{S}(I_i), T \in \mathbb{S}(J)$ are objects, we define a pseudo-functor*

$$\mathrm{Cor}_{\mathbb{S}} : \mathrm{Cor}(I_1, \dots, I_n; J)^{1-\mathrm{op}} \to \mathcal{CAT}$$

in the oplax case and

$$\mathrm{Cor}_{\mathbb{S}} : \mathrm{Cor}(I_1, \dots, I_n; J)^{1-\mathrm{op}, 2-\mathrm{op}} \to \mathcal{CAT}$$

in the lax case. $\mathrm{Cor}_{\mathbb{S}}$ *maps a multicorrespondence of diagrams in* Dia

$$\begin{tikzcd} & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[rd, "\beta"] & \\ I_1 & \cdots & I_n & & J \end{tikzcd}$$

to the category

$$\mathrm{Hom}_{\mathbb{S}(A)}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T),$$

maps a 1-morphism $(\gamma, \nu_1, \dots, \nu_n, \mu)$ to the functor

$$\rho \mapsto \mathbb{S}(\mu)(T) \circ (\gamma^* \rho) \circ (\mathbb{S}(\nu_1)(S_1), \dots, \mathbb{S}(\nu_n)(S_n))$$

and maps a 2-morphism represented by $\eta : \gamma \Rightarrow \gamma'$ (and such that $(\alpha_i' * \eta) \circ \nu_i = \nu_i'$ and $\mu' \circ (\beta' * \eta) = \mu$) to the morphism

$$\mathbb{S}(\mu)(T) \circ (\gamma^* \rho) \circ (\mathbb{S}(\nu_1)(S_1), \dots, \mathbb{S}(\nu_n)(S_n)) \leftrightarrow \mathbb{S}(\mu') \circ ((\gamma')^* \rho) \circ (\mathbb{S}(\nu_1)(S_1), \dots, \mathbb{S}(\nu_n)(S_n)) \quad (3)$$

given as the composition of the isomorphisms

$$\mathbb{S}(\mu)(T) \xrightarrow{\sim} \mathbb{S}(\mu')(T) \circ \underbrace{\mathbb{S}(\beta' * \eta)(T)}_{= \mathbb{S}(\eta)((\beta')^* T)}$$

$$\mathbb{S}(\nu_i)(S_i) \xrightarrow{\sim} \underbrace{\mathbb{S}(\alpha_i' * \eta)(S_i)}_{= \mathbb{S}(\eta)((\alpha_i')^* S_i)} \circ \mathbb{S}(\nu_i)(S_i)$$

with the morphism

$$\mathbb{S}(\eta)((\beta')^* T) \circ (\gamma^* \rho) \leftrightarrow ((\gamma')^* \rho) \circ (\mathbb{S}(\eta)((\alpha_1')^* S_1), \dots, \mathbb{S}(\eta)((\alpha_n')^* S_n)) \quad (4)$$

coming from the fact that $\mathbb{S}(\eta)$ is a (lax/oplax) pseudo-natural transformation $\gamma^* \Rightarrow (\gamma')^*$. The morphisms (3) and (4) point to the left in the lax case and to the right in the oplax case.

Definition 3.2. *Let $\mathbb{S}$ be a (lax/oplax) 2-pre-multiderivator. Let $S_i \in \mathbb{S}(I_i)$, for $i = 1, \dots, n$ and $T \in \mathbb{S}(J)$ be objects. Let*

$$\mathrm{Cor}_{\mathbb{S}}((I_1, S_1), \dots, (I_n, S_n); (J, T))$$

be the strict 2-category obtained from the pseudo-functor $\mathrm{Cor}_{\mathbb{S}}$ *defined in 3.1 by the 2-categorical Grothendieck construction [4, Definition 2.14].*

Both definitions depend on the choice of Dia, but we do not specify it explicitly.

3.3. The category $\mathrm{Cor}_{\mathbb{S}}((I_1, S_1), \dots, (I_n, S_n); (J, T))$ defined in 3.2 is very important to understand fibered multiderivators. Therefore we explicitly spell out the definition in detail:

1. Objects are a multicorrespondence of diagrams in Dia

$$\begin{tikzcd} & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[rd, "\beta"] & \\ I_1 & \cdots & I_n & & J \end{tikzcd}$$

together with a 1-morphism

$$\rho \in \mathrm{Hom}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T)$$

in $\mathbb{S}(A)$.

2. The 1-morphisms $(A, \alpha_1, \dots, \alpha_n, \beta, \rho) \to (A', \alpha_1', \dots, \alpha_n', \beta', \rho')$ are tuples $(\gamma, \nu_1, \dots, \nu_n, \mu, \Xi)$, where $\gamma : A \to A'$ is a functor, ν_i is a natural transformation in

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "\alpha_i"'] & \arrow[d, phantom, "\overset{\nu_i}{\Rightarrow}"] & A' \arrow[dl, "\alpha_i'"] \\ & I_i & \end{tikzcd}$$

and μ is a natural transformation in

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\overset{\mu}{\Leftarrow}"] & A' \arrow[dl, "\beta'"] \\ & J & \end{tikzcd}$$

and Ξ is a 2-morphism in

$$\begin{tikzcd} \gamma^*(\alpha')^* S \arrow[rr, "\gamma^*\rho'"] & & \gamma^*(\beta')^* T \arrow[dd, "\mathbb{S}(\mu)"] \\ & \arrow[phantom, "\Uparrow\Xi"] & \\ \alpha^* S \arrow[uu, "\mathbb{S}(\nu)"] \arrow[rr, "\rho"] & & \beta^* T \end{tikzcd}$$

3. The 2-morphisms are the natural transformations $\eta : \gamma \Rightarrow \gamma'$ such that $(\alpha_i' * \eta) \circ \nu_i = \nu_i'$ and $(\beta' * \eta) \circ \mu' = \mu$ and such that following prism-shaped diagram

$$\begin{tikzcd} \gamma^*(\alpha')^* S \arrow[rrr, "\gamma^*\rho'"] \arrow[dr, "\mathbb{S}(\alpha' * \eta)(S) = \mathbb{S}(\eta)((\alpha')^* S)"] & \arrow[d, phantom, "\Updownarrow \mathbb{S}(\alpha' * \eta)(\rho')"] & & \gamma^*(\beta')^* T \arrow[dl, "\mathbb{S}(\eta)((\beta')^* T) = \mathbb{S}(\beta' * \eta)(T)"'] \arrow[dd, "\mathbb{S}(\mu)(T)"] \\ \arrow[r, phantom, "\overset{\mathbb{S}_{\eta,\nu}(S)}{\Rightarrow}"] & (\gamma')^*(\alpha')^* S \arrow[r, "(\gamma')^*\rho'"] & (\gamma')^*(\beta')^* T \arrow[dr, "\mathbb{S}(\mu')(T)"'] & \arrow[l, phantom, "\overset{\mathbb{S}_{\mu',\eta}(T)}{\Rightarrow}"] \\ \alpha^* S \arrow[uu, "\mathbb{S}(\nu)(S)"] \arrow[ur, "\mathbb{S}(\nu')(S)"'] \arrow[rrr, "\rho"] & \arrow[r, phantom, "\Uparrow \Xi'"] & & \beta^* T \end{tikzcd}$$

is 2-commutative, where the 2-morphism in the front face (not depicted) points upwards and is Ξ. We assumed here $n = 1$ for simplicity. Note that we have $\mathbb{S}(\alpha' * \eta)(S) = \mathbb{S}(\eta)((\alpha')^* S)$ because $\mathbb{S}$ is strictly compatible with composition of 1-morphisms (cf. Definition 2.1). Note that the 2-morphism denoted $\Updownarrow$ goes up in the lax case and down in the oplax (and plain) case while the two 'horizontal' 2-morphisms are invertible.

We again define the full subcategory $\mathrm{Cor}_{\mathbb{S}}^F$ insisting that $\alpha_1 \times \cdots \times \alpha_n : A \to I_1 \times \cdots \times I_n$ is a Grothendieck fibration and β is an opfibration.

Lemma 3.4. *Let $p : \mathbb{D} \to \mathbb{S}$ be a strict morphism of (lax/oplax) 2-pre-multiderivators (cf. Definition 2.2).*
Consider the strictly commuting diagram of 2-categories and strict 2-functors

$$\begin{tikzcd}
\mathrm{Cor}_{\mathbb{D}}^F((I_1,\mathcal{E}_1),\dots,(I_n,\mathcal{E}_n);(J,\mathcal{F})) \arrow[r, hook] \arrow[d] & \mathrm{Cor}_{\mathbb{D}}((I_1,\mathcal{E}_1),\dots,(I_n,\mathcal{E}_n);(J,\mathcal{F})) \arrow[d] \\
\mathrm{Cor}_{\mathbb{S}}^F((I_1,S_1),\dots,(I_n,S_n);(J,T))) \arrow[r, hook] \arrow[d] & \mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T)) \arrow[d] \\
\mathrm{Cor}^F(I_1,\dots,I_n;J) \arrow[r, hook] & \mathrm{Cor}(I_1,\dots,I_n;J)
\end{tikzcd}$$

1. *If the functors $\mathrm{Hom}_{\mathbb{D}(I)}(-,-) \to \mathrm{Hom}_{\mathbb{S}(I)}(-,-)$ induced by p are fibrations, the vertical 2-functors are 1-fibrations with 1-categorical fibers. They are 2-fibrations in the lax case and 2-opfibrations in the oplax case.*

2. *If the functors $\mathrm{Hom}_{\mathbb{D}(I)}(-,-) \to \mathrm{Hom}_{\mathbb{S}(I)}(-,-)$ induced by p are fibrations* with discrete fibers, *then the upper vertical 2-functors have discrete fibers.*

3. *Every object in a 2-category on the right hand side is in the image of the corresponding horizontal 2-functor* up to a chain of adjunctions.

Proof. This Lemma is a straightforward generalization of [4, Lemma 5.3]. 1. and 2. follow directly from the definition. 3. We first embed the left hand side category, say $\mathrm{Cor}_{\mathbb{S}}^F((I_1,S_1),\dots,(I_n,S_n);(J,T))$, into the full subcategory of $\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$ consisting of objects $(A,\alpha_1,\dots,\alpha_n,\beta,\rho)$, in which β is an opfibration but the α_i are arbitrary. We will show that every object is connected by an adjunction with an object of this bigger subcategory. By a similar argument one shows that this holds also for the second inclusion.
Consider an arbitrary correspondence ξ' of diagrams in Dia

$$\begin{tikzcd}
 & & & A \arrow[llld, "\alpha_1"'] \arrow[ld, "\alpha_n"] \arrow[rd, "\beta"] & \\
I_1 & \cdots & I_n & & J
\end{tikzcd}$$

and the 1-morphisms in $\mathrm{Cor}(I_1,\dots,I_n;J)$

$$\begin{tikzcd}
 & & & A \times_{/J} J \arrow[dlll, "\alpha_1 \mathrm{pr}_1"'] \arrow[dl, "\alpha_n \mathrm{pr}_1"] \arrow[dd, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & \\
I_1 & \cdots & I_n & \arrow[r, phantom, "\Rightarrow^{\mu}"] & J \\
 & & & A \arrow[ulll, "\alpha_1"] \arrow[ul, "\alpha_n"'] \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & A \arrow[dlll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dd, "\Delta"] \arrow[dr, "\beta"] & \\
I_1 & \cdots & I_n & & J \\
 & & & A \times_{/J} J \arrow[ulll, "\alpha_1 \mathrm{pr}_1"] \arrow[ul, "\alpha_n \mathrm{pr}_1"'] \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

One easily checks that $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_A$ and that the obvious 2-morphism $\Delta \circ \mathrm{pr}_1 \Rightarrow \mathrm{id}_{A \times_{/J} J}$ induced by μ define an adjunction in the 2-category $\mathrm{Cor}(I_1, \dots, I_n; J)$. Using [4, Lemma 5.4], we get a corresponding adjunction also in the 2-category $\mathrm{Cor}_{\mathbb{S}}((I_1, S_1), \dots, (I_n, S_n); (J, T))$. $\square$

Lemma 3.5. *Let $p : \mathbb{D} \to \mathbb{S}$ be a morphism of (lax/oplax) 2-pre-multiderivators. Consider the following strictly commuting diagram of functors obtained from the one of Lemma 3.4 by 1-truncation [4, 4.2]:*

$$\begin{tikzcd}
\tau_1(\mathrm{Cor}^F_{\mathbb{D}}((I_1, \mathcal{E}_1), \dots, (I_n, \mathcal{E}_n); (J, \mathcal{F}))) \arrow[r, hook] \arrow[d] & \tau_1(\mathrm{Cor}_{\mathbb{D}}((I_1, \mathcal{E}_1), \dots, (I_n, \mathcal{E}_n); (J, \mathcal{F}))) \arrow[d] \\
\tau_1(\mathrm{Cor}^F_{\mathbb{S}}((I_1, S_1), \dots, (I_n, S_n); (J, T))) \arrow[r, hook] \arrow[d] & \tau_1(\mathrm{Cor}_{\mathbb{S}}((I_1, S_1), \dots, (I_n, S_n); (J, T))) \arrow[d] \\
\tau_1(\mathrm{Cor}^F(I_1, \dots, I_n; J)) \arrow[r, hook] & \tau_1(\mathrm{Cor}(I_1, \dots, I_n; J))
\end{tikzcd}$$

1. *The horizontal functors are equivalences.*
2. *If the functors $\mathrm{Hom}_{\mathbb{D}(I)}(-,-) \to \mathrm{Hom}_{\mathbb{S}(I)}(-,-)$ induced by p are fibrations* with discrete fibers*, then the upper vertical morphisms are fibrations with discrete fibers. Furthermore the top-most horizontal functor maps Cartesian morphisms to Cartesian morphisms.*

Proof. This Lemma is a straightforward generalization of [4, Lemma 5.5]. That the horizontal morphisms are equivalences follows from the definition of the truncation and Lemma 3.4, 3. If we have a 1-fibration and 2-fibration of 2-categories $\mathcal{D} \to \mathcal{C}$ with *discrete* fibers then the truncation $\tau_1(\mathcal{D}) \to \tau_1(\mathcal{C})$ is again fibered (in the 1-categorical sense). Hence the second assertion follows from Lemma 3.4, 2. $\square$

Definition 3.6. *Let $\mathbb{S}$ be a 2-pre-multiderivator. We define a 2-multicategory $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ equipped with a strict functor*

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathrm{Dia}^{\mathrm{cor}}$$

as follows

1. *The objects of $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ are pairs (I, S) consisting of $I \in \mathrm{Dia}$ and $S \in \mathbb{S}(I)$.*
2. *The category $\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I_1, S_1), \dots, (I_n, S_n); (J, T))$ of 1-morphisms of $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is the truncated category $\tau_1(\mathrm{Cor}^F_{\mathbb{S}}((I_1, S_1), \dots, (I_n, S_n); (J, T)))$.*

Composition is given by the composition of correspondences of diagrams

$$\begin{tikzcd}
 & & & & & A \times_{J_i} B \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & & & \\
 & & & A \arrow[dlll] \arrow[dl] \arrow[drrr, "\beta_A"' near start] & & & & B \arrow[dlll] \arrow[dl, "\alpha_{B,i}"] \arrow[dr] \arrow[drrr] & & & \\
I_1 & \cdots & I_n & ; & J_1 & \cdots & J_i & \cdots & J_m & ; & K
\end{tikzcd}$$

and composing $\rho_A \in \mathrm{Hom}(\alpha_{A,1}^* S_1, \dots, \alpha_{A,n}^* S_n; \beta_A^* T_i)$ *with* $\rho_B \in \mathrm{Hom}(\alpha_{B,1}^* T_1, \dots, \alpha_{B,m}^* T_m; \beta_B^* U)$ *to*

$$(\mathrm{pr}_2^* \rho_B) \circ_i (\mathrm{pr}_1^* \rho_A).$$

If $\mathbb{S}$ *is symmetric or braided, then there is a natural action of the symmetric, resp. braid groups:*

$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I_1, S_1), \dots, (I_n, S_n); (J, T)) \to \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I_{\sigma(1)}, S_{\sigma(1)}), \dots, (I_{\sigma(n)}, S_{\sigma(n)}); (J, T))$$

involving the corresponding action in $\mathbb{S}$. *This turns* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *into a symmetric, resp. braided 2-multicategory.*

Note that because of the brute-force truncation this category is in general not 2-fibered anymore over Dia^{cor}.
For any strict morphism of 2-pre-multiderivators $p : \mathbb{D} \to \mathbb{S}$ we get an induced strict functor

$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}).$$

4 Fibered multiderivators over 2-pre-multiderivators

The definition of a fibered multiderivator over 2-pre-multiderivators is a straightforward generalization of the notion of *fibered multiderivator* from [3]. In [4] it was shown that this can, in a very neat way, alternatively be defined using the language of fibrations of 2-multicategories. It is also true for fibered multiderivators over 2-pre-multiderivators. In this article we choose the slicker formulation as our definition:

Definition 4.1. *A strict morphism* $\mathbb{D} \to \mathbb{S}$ *of (lax/oplax) 2-pre-multiderivators (Definition 2.2) such that* $\mathbb{D}$ *and* $\mathbb{S}$ *each satisfy (Der1) and (Der2) (cf. 2.4) is a*

1. **lax left (resp. oplax right) fibered multiderivator** *if the corresponding strict functor of 2-multicategories*
$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$$
of Definition 3.6 is a 1-opfibration (resp. 1-fibration) and 2-fibration with 1-categorical fibers.

2. **oplax left (resp. lax right) fibered multiderivator** *if the corresponding strict functor of 2-multicategories*
$$\mathrm{Dia}^{\mathrm{cor}}(p) : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$
of Definition 3.6 is a 1-opfibration (resp. 1-fibration) and 2-fibration with 1-categorical fibers.

Similarly, we define **symmetric**, *resp.* **braided** *fibered multiderivators where everything is, in addition, equipped in a compatible way with the action of the symmetric, resp. braid groups.*

If in $\mathbb{S}$ all 2-morphisms are invertible then left oplax=left lax and right oplax=right lax. In that case we omit the adjectives "lax" and "oplax".
It seems that, in the definition, one could release the assumption on 1-categorical fibers, to get an apparently more general definition. However, then the 1-truncation involved in the definition of $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is probably not the right thing to work with. In particular one does not get any generalized definition of a 2-derivator (or monoidal 2-derivator) as 2-fibered (multi-)derivator over $\{\cdot\}$.
The following Theorem 4.2 gives an alternative definition of a left/right fibered multiderivator over a 2-pre-multiderivator $\mathbb{S}$ more in the spirit of the original (1-categorical) definition of [3].

Theorem 4.2. *A strict morphism $p : \mathbb{D} \to \mathbb{S}$ of (lax/oplax) 2-pre-multiderivators such that $\mathbb{D}$ and $\mathbb{S}$ both satisfy (Der1) and (Der2) is a* left (resp. right) fibered multiderivator *if and only if the following axioms (FDer0 left/right) and (FDer3–5 left/right) hold true*[6].
Here (FDer3–4 left/right) can be replaced by the weaker (FDer3–4 left/right').

(FDer0 left) For each I in Dia the morphism p specializes to an 1-opfibered 2-multicategory with 1-categorical fibers. It is, in addition, 2-fibered in the lax case and 2-opfibered in the oplax case. Moreover any functor $\alpha : I \to J$ in Dia induces a diagram

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of 1-opfibered and 2-(op)fibered 2-multicategories, i.e. the top horizontal functor maps coCartesian 1-morphisms to coCartesian 1-morphisms and (co)Cartesian 2-morphisms to (co)Cartesian 2-morphisms.

(FDer3 left) For each functor $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 left))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left adjoint $\alpha_!^{(S)}$.

(FDer4 left) For each functor $\alpha : I \to J$ in Dia, and for any object $j \in J$, and for the 2-commutative square

$$\begin{tikzcd} I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\swArrow^{\mu}"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors $\alpha_{j!}(\mathbb{S}(\mu))_\bullet \iota^* \to j^* \alpha_!$ is an isomorphism[7].

(FDer5 left) For any opfibration $\alpha : I \to J$ in Dia, and for any 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors

$$\alpha_!(\alpha^*\xi)_\bullet(\alpha^*-, \cdots, \alpha^*-, \underbrace{-}_{\text{at } i}, \alpha^*-, \cdots, \alpha^*-) \cong \xi_\bullet(-, \cdots, -, \underbrace{\alpha_!-}_{\text{at } i}, -, \cdots, -)$$

are isomorphisms for all $i = 1, \dots, n$.

[6]where (FDer3–5 left), resp. (FDer3–5 right), only make sense in the presence of (FDer3–5 left), resp. (FDer0 right)
[7]This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.

Instead of (FDer3/4 left) the following axioms are sufficient:

(FDer3 left') For each *opfibration* $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 left))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a left-adjoint $\alpha_!^{(S)}$.

(FDer4 left') For each *opfibration* $\alpha : I \to J$ in Dia, and for any object $j \in J$, the induced natural transformation of functors $\mathrm{pr}_{2,!} \mathrm{pr}_1^* \to j^* \alpha_!$ is an isomorphism for any base. Here pr_1 and pr_2 are defined by the Cartesian square

$$\begin{tikzcd} I \times_J j \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J. \end{tikzcd}$$

We use the same notation for the axioms as in the case of usual fibered multiderivators because, in case that $\mathbb{S}$ is a usual 1-pre-multiderivator they specialize to the familiar ones. Dually, we have the following axioms:

(FDer0 right) For each I in Dia the morphism p specializes to a 1-fibered multicategory with 1-categorical fibers. It is, in addition, 2-fibered in the lax case and 2-opfibered in the oplax case. Furthermore, any *opfibration* $\alpha : I \to J$ in Dia induces a diagram

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, "\alpha^*"] \arrow[d] & \mathbb{D}(I) \arrow[d] \\ \mathbb{S}(J) \arrow[r, "\alpha^*"] & \mathbb{S}(I) \end{tikzcd}$$

of 1-fibered and 2-(op)fibered multicategories, i.e. the top horizontal functor maps Cartesian 1-morphisms w.r.t. the i-th slot to Cartesian 1-morphisms w.r.t. the i-th slot for any i and maps (co)Cartesian 2-morphisms to (co)Cartesian 2-morphisms.

(FDer3 right) For each functor $\alpha : I \to J$ in Dia and $S \in \mathbb{S}(J)$ the functor α^* between fibers (which are 1-categories by (FDer0 right))

$$\mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$$

has a right adjoint $\alpha_*^{(S)}$.

(FDer4 right) For each morphism $\alpha : I \to J$ in Dia, and for any object $j \in J$, and for the 2-commutative square

$$\begin{tikzcd} j \times_{/J} I \arrow[r, "\iota"] \arrow[d, "\alpha_j"'] \arrow[dr, phantom, "\nearrow \mu"] & I \arrow[d, "\alpha"] \\ \{j\} \arrow[r, hook, "j"] & J \end{tikzcd}$$

the induced natural transformation of functors $j^* \alpha_* \to {\alpha_j}_* (\mathbb{S}(\mu))^\bullet \iota^*$ is an isomorphism[8].

[8] This is meant to hold w.r.t. all bases $S \in \mathbb{S}(J)$.

(FDer5 right) For *any* functor $\alpha : I \to J$ in Dia, and for any 1-morphism $\xi \in \mathrm{Hom}(S_1, \dots, S_n; T)$ in $\mathbb{S}(\cdot)$ for some $n \geq 1$, the natural transformations of functors

$$\alpha_*(\alpha^*\xi)^{\bullet,i}(\alpha^*-, \overset{\widehat{i}}{\dots}, \alpha^*- \,;\, -) \cong \xi^{\bullet,i}(-, \overset{\widehat{i}}{\dots}, - \,;\, \alpha_*-)$$

are isomorphisms for all $i = 1, \dots, n$.

There is similarly a weaker version of (FDer3/4 right) in which α has to be a fibration.
For *representable* 2-pre-multiderivators we get the following:

Proposition 4.3. *If $\mathcal{D} \to \mathcal{S}$ is a 1-bifibration and 2-fibration of 2-multicategories with 1-categorical and bicomplete fibers then*

1. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{lax}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{lax}})$ *is a 1-opfibration and 2-fibration,*
2. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{oplax}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{oplax}})$ *is a 1-fibration and 2-fibration.*

If $\mathcal{D} \to \mathcal{S}$ is a 1-bifibration and 2-opfibration of 2-multicategories with 1-categorical and bicomplete fibers then

1. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{lax},2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{lax},2-\mathrm{op}})$ *is a 1-fibration and 2-fibration.*
2. $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{\mathrm{oplax},2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{oplax},2-\mathrm{op}})$ *is a 1-opfibration and 2-fibration.*

Proof. This follows from Proposition 2.6 doing the same constructions as in [3, Proposition 4.1.26]. □

Definition 4.4. *For (lax/oplax) fibered derivators over an (lax/oplax) 2-pre-derivator $p : \mathbb{D} \to \mathbb{S}$ and an object $S \in \mathbb{S}(I)$ we have that*

$$\mathbb{D}_{I,S} : J \mapsto \mathbb{D}(I \times J)_{\mathrm{pr}_2^* S}$$

is a usual derivator. We call p **stable** *if $\mathbb{D}_{I,S}$ is stable for all $S \in \mathbb{S}(I)$ and for all I.*

5 Yoga of correspondences of diagrams in a 2-pre-multiderivator

Let $\mathbb{S}$ be a 2-pre-multiderivator. To prove Theorem 4.2 we need some preparation to improve our understanding of the category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$. This section is almost word for word the same as [4, Section 6]. There instead a usual pre-multiderivator $\mathbb{S}$ is considered. Since the transition from a pre-multiderivator to a 2-pre-multiderivator slightly changes every detail of the constructions, we decided for the convenience of the reader to duplicate the whole section making necessary changes everywhere instead of just referring to [loc. cit.].
We will define three types of generating 1-morphisms in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$. We first define them as objects in the categories $\mathrm{Cor}_{\mathbb{S}}(\dots)$ (without the restriction ^{F}).

$[\beta^{(S)}]$ for a functor $\beta : I \to J$ in Dia and an object $S \in \mathbb{S}(J)$, consists of the correspondence of diagrams

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, "\beta"] & \\ I & & J \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((I, \beta^*S); (J, S)))$ the canonical correspondence given by the identity id_{β^*S}.

$[\alpha^{(S)}]'$ for a functor $\alpha : I \to J$ in Dia and an object $S \in \mathbb{S}(J)$, consists of the correspondence of diagrams

$$\begin{tikzcd} & I \arrow[dl, "\alpha"'] \arrow[dr, equal] & \\ J & & I \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((J,S);(I,\alpha^*S)))$ the canonical correspondence given by the identity id_{α^*S}.

$[f]$ for a morphism $f \in \mathrm{Hom}_{\mathbb{S}(A)}(S_1,\dots,S_n;T)$, where A is any diagram in Dia, and $S_1,\dots,S_n,T$ are objects in $\mathbb{S}(A)$, is defined by the trivial correspondence of diagrams

$$\begin{tikzcd} & & A \arrow[dll, equal] \arrow[dl, equal] \arrow[dr, equal] & \\ A & A & & A \end{tikzcd}$$

together with f.

5.1. Note that the correspondences of the last paragraph do not define 1-morphisms in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ yet, as we defined it, because they are not always objects in the Cor^F subcategory ($[\alpha^{(S)}]'$ is already, if α is a fibration; $[\beta^{(S)}]$ is, if β is an opfibration; and $[f]$ is, if $n = 0,1$, respectively).
From now on, we denote by the same symbols $[\alpha^{(S)}], [\beta^{(S)}]', [f]$ morphisms in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ which are isomorphic to those defined above in the τ_1-categories (cf. Lemma 3.5). Those are are determined only up to 2-isomorphism in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$.
For definiteness, we choose $[\beta^{(S)}]$ to be the correspondence

$$\begin{tikzcd} & I \times_{/J} J \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ I & & J \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((I,\beta^*S);(J,S)))$ the 1-morphism $\mathrm{pr}_1^*\beta^*S \to \mathrm{pr}_2^*S$ given by the natural transformation $\mu_\beta : \beta \circ \mathrm{pr}_1 \Rightarrow \mathrm{pr}_2$. Similarly, we choose $[\alpha^{(S)}]'$ to be the correspondence

$$\begin{tikzcd} & J \times_{/J} I \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ J & & I \end{tikzcd}$$

and over it in $\tau_1(\mathrm{Cor}_{\mathbb{S}}((J,S);(I,\alpha^*S)))$ the 1-morphism $\mathrm{pr}_1^*S \to \mathrm{pr}_2^*\alpha^*S$ given by the natural transformation $\mu_\alpha : \mathrm{pr}_1 \Rightarrow \alpha \circ \mathrm{pr}_2$.

5.2. For any $\alpha : I \to J$, we define a 2-morphism

$$\epsilon : \mathrm{id} \Rightarrow [\alpha^{(S)}] \circ [\alpha^{(S)}]'$$

given by the diagrams

$$
\begin{tikzcd}
& I \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\Delta"] & \\
I & & I \\
& I \times_{/J} J \times_{/J} I \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_3"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^* \mathrm{pr}_1^* \alpha^* S \arrow[r, equal, "\Delta^*(\mathbb{S}(\mu_2 \circ \mu_1)(S)) = \mathrm{id}_{\alpha^* S}"] \arrow[d, equal] & \Delta^* \mathrm{pr}_3^* \alpha^* S \arrow[d, equal] \\
\alpha^* S \arrow[r, equal] & \alpha^* S
\end{tikzcd}
$$

and we define a 2-morphism

$$\mu : [\alpha^{(S)}]' \circ [\alpha^{(S)}] \Rightarrow \mathrm{id}$$

given by the diagrams

$$
\begin{tikzcd}
& J \times_{/J} I \times_{/J} J \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_3"] \arrow[dd, "\alpha\, \mathrm{pr}_2"] & \\
J \arrow[dr, equal] \arrow[r, phantom, "\overset{\mu_2}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\mu_1}{\Rightarrow}"] & J \arrow[dl, equal] \\
& J &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pr}_2^* \alpha^* S \arrow[r, equal] & \mathrm{pr}_2^* \alpha^* S \arrow[d, "\mathbb{S}(\mu_1)(S)"] \\
\mathrm{pr}_1^* S \arrow[u, "\mathbb{S}(\mu_2)(S)"] \arrow[r, "\mathbb{S}(\mu_2 \circ \mu_1)(S)"'] \arrow[ur, phantom, "\Uparrow"] & \mathrm{pr}_3^* S
\end{tikzcd}
$$

where the 2-isomorphism from the pseudo-functoriality of $\mathbb{S}$ is taken.

5.3. A natural transformation $\nu : \alpha \Rightarrow \beta$ establishes a morphism

$$[\nu] : [\mathbb{S}(\nu)(S)] \circ [\alpha^{(S)}] \Rightarrow [\beta^{(S)}]$$

given by the diagrams:

$$
\begin{tikzcd}
& J \times_{/J,\beta} I \arrow[dl, "\mathrm{pr}_1'"'] \arrow[dr, "\mathrm{pr}_2'"] & \\
J & & I \\
& J \times_{/J,\alpha} I \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_2"'] \arrow[uu, leftarrow, "\widetilde{\nu}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\widetilde{\nu})^* \mathrm{pr}_1^* S \arrow[r, "\widetilde{\nu}^* \mathbb{S}(\mu_\alpha)"] \arrow[d, equal] & (\widetilde{\nu})^* \mathrm{pr}_2^* \alpha^* S \arrow[r, "\widetilde{\nu}^* \mathrm{pr}_2^* \mathbb{S}(\nu)"] \arrow[d, phantom, "\Uparrow"] & \widetilde{\nu}^* \mathrm{pr}_2^* \beta^* S \arrow[d, equal] \\
(\mathrm{pr}_1')^* S \arrow[rr, "\mathbb{S}(\mu_\beta)(S)"'] & & (\mathrm{pr}_2')^* \beta^* S
\end{tikzcd}
$$

where the 2-isomorphism from the pseudo-functoriality of $\mathbb{S}$ is taken. Note that we have the equation of natural transformations $(\nu * \mathrm{pr}_2') \circ (\mu_\alpha * \widetilde{\nu}) = \mu_\beta$. Here μ_α and μ_β are as in 5.1.
Similarly, a natural transformation $\nu : \alpha \Rightarrow \beta$ establishes a morphism

$$[\nu] : [\beta^{(S)}]' \circ [\mathbb{S}(\nu)(S)] \Rightarrow [\alpha^{(S)}]'.$$

5.4. Consider the diagrams from axiom (FDer3 left/right)

$$
\begin{tikzcd}
I \times_{/J} j \arrow[r, "\iota"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow^{\mu}"] & I \arrow[d, "\alpha"] \\
j \arrow[r, hook] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
j \times_{/J} I \arrow[r, "\iota"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow^{\mu}"] & I \arrow[d, "\alpha"] \\
j \arrow[r, hook] & J
\end{tikzcd}
$$

By the constructions in 5.3, we get a canonical 2-morphism

$$[\mathbb{S}(\mu)(S)] \circ [\iota^{(\alpha^*S)}] \circ [\alpha^{(S)}] \Rightarrow [p^{(S_j)}] \circ [j^{(S)}]. \tag{5}$$

and a canonical 2-morphism

$$[\alpha^{(S)}]' \circ [\iota^{(\alpha^*S)}]' \circ [\mathbb{S}(\mu)(S)] \Rightarrow [j^{(S)}]' \circ [p^{(S_j)}]'. \tag{6}$$

respectively. Here S_j denotes j^*S where j, by abuse of notation, also denotes the inclusion of the one-element category j into J.

5.5. Let ξ be any 1-morphism $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ given by

$$\begin{tikzcd}
 & & & A \arrow[lll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] & \\
I_1 & \cdots & I_n & & J
\end{tikzcd}$$

and a 1-morphism

$$f_\xi \in \mathrm{Hom}_{\mathbb{S}(A)}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T).$$

We define a 1-morphism $\xi \times K$ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ by

$$\begin{tikzcd}
 & & & A \times K \arrow[lll, "\alpha_1 \times \mathrm{id}"'] \arrow[dl, "\alpha_n \times \mathrm{id}"] \arrow[dr, "\beta \times \mathrm{id}"] & \\
I_1 \times K & \cdots & I_n \times K & & J \times K
\end{tikzcd}$$

and

$$f_{\xi \times K} := \mathrm{pr}_1^* f_\xi \in \mathrm{Hom}_{\mathbb{S}(A)}(\mathrm{pr}_1^* \alpha_1^* S_1, \dots, \mathrm{pr}_1^* \alpha_n^* S_n; \mathrm{pr}_1^* \beta^* T).$$

Note that the here defined $\xi \times K$ does not necessarily lie in the category $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$. Hence we denote by $\xi \times K$ any isomorphic correspondence which does lie in $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$. We also define a correspondence $\xi \times_j K$ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ by

$$\begin{tikzcd}
 & & & & & A \times K \arrow[lllll, "\alpha_1 \mathrm{pr}_1"'] \arrow[lll, "\alpha_j \times \mathrm{id}"] \arrow[dl, "\alpha_n \mathrm{pr}_1"] \arrow[dr, "\beta \times \mathrm{id}"] & \\
I_1 & \cdots & I_j \times K & \cdots & I_n & & J \times K
\end{tikzcd}$$

and

$$f_{\xi \times_j K} := \mathrm{pr}_1^* \xi \in \mathrm{Hom}_{\mathbb{S}(A)}(\mathrm{pr}_1^* \alpha_1^* S_1, \dots, \mathrm{pr}_1^* \alpha_n^* S_n; \mathrm{pr}_1^* \beta^* T).$$

The here defined $\xi \times_j K$ does already lie in the category $\mathrm{Cor}_{\mathbb{S}}^F(\dots)$.

Lemma 5.6. *1. The 2-morphisms of 5.2*

$$\epsilon : \mathrm{id} \Rightarrow [\alpha^{(S)}] \circ [\alpha^{(S)}]' \qquad \mu : [\alpha^{(S)}]' \circ [\alpha^{(S)}] \Rightarrow \mathrm{id}$$

establish an adjunction between $[\alpha^{(S)}]$ *and* $[\alpha^{(S)}]'$ *in the 2-category* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$.

2. *The exchange 2-morphisms of (5) and and of (6) w.r.t. the adjunction of 1., namely*

$$[p^{(S_j)}]' \circ [\mathbb{S}(\mu)(S)] \circ [\iota^{(\alpha^* S)}] \Rightarrow [j^{(S)}] \circ [\alpha^{(S)}]'$$

and

$$[\iota^{(\alpha^* S)}]' \circ [\mathbb{S}(\mu)(S)] \circ [p^{(S_j)}] \Rightarrow [\alpha^{(S)}] \circ [j^{(S)}]'$$

are 2-isomorphisms.

3. *For any $\alpha : K \to L$ there are natural isomorphisms*

$$[\alpha^{(\mathrm{pr}_1^* T)}] \circ (\xi \times L) \cong (\xi \times K) \circ ([\alpha^{(\mathrm{pr}_1^* S_1)}], \dots, [\alpha^{(\mathrm{pr}_1^* S_n)}]) \tag{7}$$

and

$$[\alpha^{(\mathrm{pr}_1^* T)}] \circ (\xi \times_j L) \cong (\xi \times_j K) \circ_j [\alpha^{(\mathrm{pr}_1^* S_j)}] \tag{8}$$

4. *The exchange of (7) w.r.t. the adjunction of 1., namely*

$$[\alpha^{(\mathrm{pr}_1^* T)}]' \circ (\xi \times K) \circ ([\alpha^{(\mathrm{pr}_1^* S_1)}], \dots, \mathrm{id}, \dots, [\alpha^{(\mathrm{pr}_1^* S_n)}]) \cong (\xi \times L) \circ_j [\alpha^{(\mathrm{pr}_1^* S_j)}]'$$

is an isomorphism if α is an opfibration. The exchange of (8) w.r.t. the adjunction of 1., namely

$$[\alpha^{(\mathrm{pr}_1^* T)}]' \circ (\xi \times_j K) \cong (\xi \times_j L) \circ_j [\alpha^{(\mathrm{pr}_1^* S_j)}]'$$

is an isomorphism for any α.

5. *For any $f \in \mathrm{Hom}_{\mathbb{S}(J)}(S_1, \dots, S_n; T)$ and $\alpha : I \to J$ there is a natural isomorphism*

$$[\alpha^{(T)}] \circ [f] \cong [\alpha^* f] \circ ([\alpha^{(S_1)}], \dots, [\alpha^{(S_n)}]) \tag{9}$$

6. *The exchange of (9) w.r.t. the adjunction of 1., namely*

$$[\alpha^{(T)}]' \circ [\alpha^* f] \circ ([\alpha^{(S_1)}], \dots, \mathrm{id}, \dots, [\alpha^{(S_n)}]) \cong [f] \circ_j [\alpha^{(S_j)}]'$$

is an isomorphism if α is an opfibration.

Proof. A purely algebraic manipulation that we leave to the reader. □

5.7. Let $\mathbb{D} \to \mathbb{S}$ be a morphism of (lax/oplax) 2-pre-multiderivators satisfying (Der1) and (Der2). Consider the strict 2-functor

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \text{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$

and assume that it is a 1-opfibration, and 2-fibration with 1-categorical fibers. The fiber over a pair (I, S) is just the fiber $\mathbb{D}(I)_S$ of the strict 2-functor $\mathbb{D}(I) \to \mathbb{S}(I)$ over S and hence this is a 1-category. The 1-opfibration and 2-fibration can be seen (via the construction of [4, Proposition 2.16]) as a pseudo-functor of 2-multicategories

$$\Psi : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{(2-\mathrm{op})}) \to \mathcal{CAT}.$$

5.8. If
$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \text{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$
is a *1-fibration*, and 2-fibration with 1-categorical fibers there is still an associated pseudo-functor of 2-categories (not 2-multicategories)
$$\Psi' : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{(2-\mathrm{op})})^{1-\mathrm{op},2-\mathrm{op}} \to \mathcal{CAT}.$$

Proposition 5.9. *1. Assume that*
$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \textit{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$
is a 1-opfibration*, and 2-fibration with 1-categorical fibers. Then the functor Ψ of 5.7 maps (up to isomorphism of functors)*
$$\begin{array}{rcl} [\alpha^{(S)}] & \mapsto & (\alpha^{(S)})^* \\ [\beta^{(S)}]' & \mapsto & \beta_!^{(S)} \\ [f] & \mapsto & f_\bullet \end{array}$$
where $\beta_!^{(S)}$ is a left adjoint of $(\beta^{(S)})^*$ and $f_\bullet$ is a functor determined by $\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \cong \mathrm{Hom}_{\mathbb{D}(I)_T}(f_\bullet(\mathcal{E}_1, \dots, \mathcal{E}_n), \mathcal{F})$.

2. Assume that
$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \qquad \textit{resp.} \qquad \mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}^{2-\mathrm{op}}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{2-\mathrm{op}})$$
is a 1-fibration*, and 2-fibration with 1-categorical fibers.*
Then pullback functors[9] *w.r.t. the following 1-morphisms in* $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ *are given by*
$$\begin{array}{rcl} [\alpha^{(S)}] & \mapsto & \alpha_*^{(S)} \\ [\beta^{(S)}]' & \mapsto & (\beta^{(S)})^* \\ [f] & \mapsto & f^{\bullet,j} \qquad \textit{pullback w.r.t. the } j\textit{-th slot.} \end{array}$$
where $\alpha_*^{(S)}$ is a right adjoint of $(\alpha^{(S)})^*$ and $f^{\bullet,j}$ is a functor determined by $\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}) \cong \mathrm{Hom}_{\mathbb{D}(I)_T}(\mathcal{E}_j, f^{\bullet,j}(\mathcal{E}_1, .\widehat{.j}., \mathcal{E}_n; \mathcal{F}))$.

Proof. 1. We have an isomorphism of sets[10]
$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),[\alpha^{(S)}]}((J,\mathcal{E}),(I,\mathcal{F})) \cong \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(I,S)}}(\Psi([\alpha^{(S)}])\mathcal{E}, \mathcal{F}).$$
On the other hand, by definition and by Lemma 3.5, the left hand side is isomorphic to the set
$$\mathrm{Hom}_{\mathbb{D}(I)_S}(\alpha^*\mathcal{E}, \mathcal{F}).$$
The first assertion follows from the fact that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(I,S)} = \mathbb{D}(I)_S$.
The second assertion follows from the first because by Lemma 5.6, 1. the 1-morphisms $[\alpha^{(S)}]$ and $[\alpha^{(S)}]'$ are adjoint in the 2-category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$. Note that a pseudo-functor like Ψ preserves adjunctions.

[9] In the case of $[\alpha^{(S)}]$ and $[\beta^{(S)}]'$ these are $\Psi'([\alpha^{(S)}])$ and $\Psi'([\beta^{(S)}]')$.
[10] We identify a discrete category with its set of isomorphism classes.

We have an isomorphism of sets

$$\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),[f]}((A,\mathcal{E}_1),\dots,(A,\mathcal{E}_n);(A,\mathcal{F})) \cong \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(A,T)}}(\Psi([f])(\mathcal{E}_1,\dots,\mathcal{E}_n),\mathcal{F}).$$

On the other hand, by definition and by Lemma 3.5, the left hand side is isomorphic to the set

$$\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$$

and the third assertion follows from the fact that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})_{(A,T)} = \mathbb{D}(A)_T$.
The proof of 2. is completely analogous. □

Corollary 5.10. *Assuming the conditions of 5.7, consider any correspondence*

$$\xi' \in \mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T))$$

consisting of

$$\begin{tikzcd} & & A \arrow[dll, "\alpha_1"'] \arrow[dl, "\alpha_n"] \arrow[dr, "\beta"] & \\ I_1 & \cdots\ I_n & & J \end{tikzcd}$$

and a 1-morphism

$$f \in \mathrm{Hom}(\alpha_1^* S_1,\dots,\alpha_n^* S_n;\beta^* T)$$

in $\mathbb{S}(A)$.

1. *Over any 1-morphism ξ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$, which is isomorphic to ξ', a corresponding push-forward functor between fibers (which is $\Psi(\xi')$ in the discussion 5.7) is given (up to isomorphism) by the composition:*
$$\beta_!^{(T)} \circ f_\bullet \circ (\alpha_1^*,\dots,\alpha_n^*).$$

2. *Over any 1-morphism ξ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$, which is isomorphic to ξ', a pull-back functor w.r.t. any slot j between fibers (which is $\Psi'(\xi')$ in the discussion 5.8 if ξ is a 1-ary 1-morphism) is given (up to isomorphism) by the composition:*
$$\alpha_{j,*}^{(S_j)} \circ f^{\bullet,j} \circ (\alpha_1^*, \overset{\widehat{j}}{\dots}, \alpha_n^*;\beta^*).$$

Proof. Because of Proposition 5.9, in both cases, we only have to show that there is an isomorphism

$$\xi \cong [\beta^{(T)}]' \circ [f] \circ ([\alpha_1^{(S_1)}],\dots,[\alpha_n^{(S_n)}])$$

in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$, which is an easy and purely algebraic manipulation. □

We are now ready to give the

Proof of Theorem 4.2. We concentrate on the lax left case, the other cases are shown completely analogously.
We first show that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is a 1-opfibration and a 2-fibration, if $\mathbb{D} \to \mathbb{S}$ satisfies (FDer0 left), (FDer3–4 left'), and (FDer5 left). By (FDer0 left) $\mathbb{D}(I) \to \mathbb{S}(I)$ is a 1-opfibration and 2-fibration with 1-categorical fibers and we have already seen that this implies that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is 2-fibered as well (cf. Lemma 3.5).

Let $x = (A; \alpha_{A,1}, \dots, \alpha_{A,n}; \beta_A)$ be a correspondence in $\mathrm{Cor}^F(I_1, \dots, I_n; J)$ and let

$$f \in \mathrm{Hom}(\alpha_1^* S_1, \dots, \alpha_n^* S_n; \beta^* T)$$

be a 1-morphism in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ lying over x. In $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ we have the following composition of isomorphisms of sets (because of Lemma 3.5, 2.)[11]:

$$\begin{aligned}
& \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}((I_1, \mathcal{E}_1), \dots, (I_n, \mathcal{E}_n); (J, \mathcal{F})) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}(A),f}(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n; \beta^* \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}(A),\mathrm{id}_{\beta^* T}}(f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n); \beta^* \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}(A),\mathrm{id}_T}(\beta_! f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n); \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),\mathrm{id}_{(J,T)}}((J, \beta_! f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n)); (J, \mathcal{F}))
\end{aligned}$$

using (FDer0 left) and (FDer3 left'). One checks that this composition is induced by the composition in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ with a 1-morphism in

$$\mathrm{Hom}_f\,((I_1, \mathcal{E}_1), \dots, (I_n, \mathcal{E}_n); (J, \beta_! f_\bullet(\alpha_1^* \mathcal{E}_1, \dots, \alpha_n^* \mathcal{E}_n)))$$

which is thus weakly coCartesian.
Note that we write $\mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}$ for the category of 1-morphisms which map to f in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ and those 2-morphisms that map to id_f in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$.
It remains to be shown that the composition of weakly coCartesian 1-morphisms is weakly coCartesian [4, Proposition 2.6]. Let

$$g \in \mathrm{Hom}(\alpha_{B,1}^* T_1, \dots, \alpha_{B,m}^* T_m; \beta_B^* U)$$

be another 1-morphism in $\mathbb{S}(B)$, composable with f, and lying over a correspondence $y = (B; \alpha_{B,1}, \dots, \alpha_{B,m}; \beta_B)$ in $\mathrm{Cor}^F(J_1, \dots, J_m; K)$. Setting $J_i \coloneqq J$ and $T_i \coloneqq T$, the composition of x and y w.r.t. the i-th slot is the correspondence

$$\begin{tikzcd}
 & & & & & A \times_{J_i} B \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & & & \\
 & & & A \arrow[dlll] \arrow[dl] \arrow[drrr, "\beta_A"] & & & & B \arrow[dlll] \arrow[dl, "\alpha_{B,i}"'] \arrow[dr] \arrow[drrr] & & & \\
I_1 & \cdots & I_n & ; & J_1 & \cdots & J_i & \cdots & J_m & ; & K
\end{tikzcd}$$

The composition of g and f is determined by the morphism

$$\mathrm{pr}_2^*\, g \circ_i \mathrm{pr}_1^*\, f$$

lying in

$$\mathrm{Hom}(\mathrm{pr}_2^*\, \alpha_{B,1}^* T_1, \dots, \underbrace{\mathrm{pr}_1^*\, \alpha_{A,1}^* S_1, \dots, \mathrm{pr}_1^*\, \alpha_{A,n}^* S_n}_{\text{at } i}, \dots, \mathrm{pr}_2^*\, \alpha_{B,m}^* T_m; \mathrm{pr}_2^*\, \beta_B^* U).$$

[11] We identify a discrete category with its set of isomorphism classes.

We have to show that the natural map

$$\begin{aligned}
&\beta_{B,!}g_\bullet(\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{\alpha_{B,i}^*\beta_{A,!}f_\bullet(\alpha_{A,1}^*\mathcal{E}_1,\dots,\alpha_{A,n}^*\mathcal{E}_n)}_{\text{at } i},\dots\alpha_{B,m}^*\mathcal{F}_m)\\
\to\quad &\beta_{B,!}\,\mathrm{pr}_{2,!}(\mathrm{pr}_2^*\,g\circ_i\mathrm{pr}_1^*\,f)_\bullet(\mathrm{pr}_2^*\,\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{\mathrm{pr}_1^*\,\alpha_{A,1}^*\mathcal{E}_1,\dots,\mathrm{pr}_1^*\,\alpha_{A,n}^*\mathcal{E}_n}_{\text{at } i},\dots\mathrm{pr}_2^*\,\alpha_{B,m}^*\mathcal{F}_m)
\end{aligned}$$

is an isomorphism. It is the composition of the following morphisms which are all isomorphisms respectively by (FDer4 left'), (FDer5 left) observing that pr_2 is a Grothendieck opfibration, the second part of (FDer0 left) for pr_1, and the first part of (FDer0 left) in the form that the composition of coCartesian morphisms is coCartesian:

$$\begin{aligned}
&\beta_{B,!}g_\bullet(\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{\alpha_{B,i}^*\beta_{A,!}f_\bullet(\alpha_{A,1}^*\mathcal{E}_1,\dots,\alpha_{A,n}^*\mathcal{E}_n)}_{\text{at } i},\dots,\alpha_{B,m}^*\mathcal{F}_m)\\
\to\quad &\beta_{B,!}g_\bullet(\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{\mathrm{pr}_{2,!}\,\mathrm{pr}_1^*\,f_\bullet(\alpha_{A,1}^*\mathcal{E}_1,\dots,\alpha_{A,n}^*\mathcal{E}_n)}_{\text{at } i},\dots,\alpha_{B,m}^*\mathcal{F}_m)\\
\to\quad &\beta_{B,!}\,\mathrm{pr}_{2,!}(\mathrm{pr}_2^*\,g)_\bullet(\mathrm{pr}_2^*\,\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{\mathrm{pr}_1^*\,f_\bullet(\alpha_{A,1}^*\mathcal{E}_1,\dots,\alpha_{A,n}^*\mathcal{E}_n)}_{\text{at } i},\dots,\mathrm{pr}_2^*\,\alpha_{B,m}^*\mathcal{F}_m)\\
\to\quad &\beta_{B,!}\,\mathrm{pr}_{2,!}(\mathrm{pr}_2^*\,g)_\bullet(\mathrm{pr}_2^*\,\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{(\mathrm{pr}_1^*\,f)_\bullet(\mathrm{pr}_1^*\,\alpha_{A,1}^*\mathcal{E}_1,\dots,\mathrm{pr}_1^*\,\alpha_{A,n}^*\mathcal{E}_n)}_{\text{at } i},\dots,\mathrm{pr}_2^*\,\alpha_{B,m}^*\mathcal{F}_m)\\
\to\quad &\beta_{B,!}\,\mathrm{pr}_{2,!}(\mathrm{pr}_2^*\,g\circ_i\mathrm{pr}_1^*\,f)_\bullet(\mathrm{pr}_2^*\,\alpha_{B,1}^*\mathcal{F}_1,\dots,\underbrace{\mathrm{pr}_1^*\,\alpha_{A,1}^*\mathcal{E}_1,\dots,\mathrm{pr}_1^*\,\alpha_{A,n}^*\mathcal{E}_n}_{\text{at } i},\dots,\mathrm{pr}_2^*\,\alpha_{B,m}^*\mathcal{F}_m).
\end{aligned}$$

Now we proceed to prove the converse, hence we assume that $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})\to\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is a 1-opfibration and show that the axioms (FDer0 left, FDer3–5 left) are satisfied.
(FDer0 left) First we have an obvious pseudo-functor of 2-multicategories

$$\begin{aligned}
F:\mathbb{S}(A) &\hookrightarrow \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})\\
S &\mapsto (A,S)\\
f &\mapsto [f]
\end{aligned}$$

By [4, Proposition 2.24] the pull-back $F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})\to\mathbb{S}(A)$ in the sense of [4, Definition 2.23] is 1-opfibered and 2-fibered if $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})\to\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ is 1-opfibered and 2-fibered. To show that $\mathbb{D}(I)\to\mathbb{S}(I)$ is a 1-opfibration and 2-fibration of multicategories, it thus suffices to show that the pull-back $F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ is equivalent to $\mathbb{D}(I)$ over $\mathbb{S}(I)$. The class of objects of $F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ is by definition isomorphic to the class of objects of $\mathbb{D}(I)$. Therefore we are left to show that there are equivalences of categories (compatible with composition)

$$\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})\to\mathrm{Hom}_{F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$$

for any 1-morphism $f\in\mathrm{Hom}_{\mathbb{S}(I)}(S_1,\dots,S_n;T)$, where $\mathcal{E}_i$ is an object of $\mathbb{D}(I)$ over S_i and $\mathcal{F}$ is an object over T. Note that the left-hand side is a discrete category. We have a 2-Cartesian diagram of categories

$$\begin{tikzcd}
\mathrm{Hom}_{F^*\,\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D}),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})}((I,\mathcal{E}_1),\dots,(I,\mathcal{E}_n);(I,\mathcal{F})) \arrow[d] \\
\{f\} \arrow[r, "F"] & \mathrm{Hom}_{\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})}((I,S_1),\dots,(I,S_n);(I,T))
\end{tikzcd}$$

Since the right vertical morphism is a fibration (cf. Lemma 3.5) the diagram is also Cartesian (cf. [4, Lemma 2.2]). Futhermore by Lemma 3.5 the right vertical morphism is equivalent to

$$\begin{array}{c} \tau_1(\mathrm{Cor}_{\mathbb{D}}((I,\mathcal{E}_1),\dots,(I,\mathcal{E}_n);(I,\mathcal{F}))) \\ \downarrow \\ \tau_1(\mathrm{Cor}_{\mathbb{S}}((I,S_1),\dots,(I,S_n);(I,T))) \end{array}$$

(Here $\mathrm{Cor}^F_{\mathbb{D}}(\dots)$ was changed to $\mathrm{Cor}_{\mathbb{D}}(\dots)$ and similarly for $\mathrm{Cor}^F_{\mathbb{S}}(\dots)$.)
In the category $\tau_1(\mathrm{Cor}_{\mathbb{S}}((I_1,S_1),\dots,(I_n,S_n);(J,T)))$, the object $F(f)$ is isomorphic to f over the trivial correspondence $(\mathrm{id}_I,\dots,\mathrm{id}_I;\mathrm{id}_I)$ whose fiber in $\tau_1(\mathrm{Cor}_{\mathbb{D}}((I,\mathcal{E}_1),\dots,(I,\mathcal{E}_n);(I,\mathcal{F})))$ is precisely the discrete category $\mathrm{Hom}_{\mathbb{D}(I),f}(\mathcal{E}_1,\dots,\mathcal{E}_n;\mathcal{F})$. The remaining part of (FDer0 left) will be shown below.
Since we have a 1-opfibration and 2-fibration we can equivalently see the given datum as a pseudo-functor

$$\Psi : \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathcal{CAT}$$

and we have seen by Proposition 5.9 that this morphism maps $[\alpha^{(S)}]$ to a functor which is isomorphic to the functor $\alpha^* : \mathbb{D}(J)_S \to \mathbb{D}(I)_{\alpha^* S}$. We have the freedom to choose Ψ in such a way that it maps $[\alpha^{(S)}]$ precisely to α^*.
Axiom (FDer3 left) follows from Lemma 5.6, 1. stating that $[\alpha^{(S)}]$ has a left adjoint $[\alpha^{(S)}]'$ in the category $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ (cf. also Proposition 5.9).
Axiom (FDer4 left) follows by applying Ψ to the (first) 2-isomorphism of Lemma 5.6, 2.
Axiom (FDer5 left) follows by applying Ψ to the 2-isomorphism of Lemma 5.6, 4.
The remaining part of (FDer0 left), i.e. that α^* maps coCartesian arrows to coCartesian arrows follows by applying Ψ to the 2-isomorphism of Lemma 5.6, 3. □

6 Derivator six-functor-formalisms

Our main purpose for introducing the more general notion of fibered multiderivator over 2-pre-multiderivators (as opposed to those over usual pre-multiderivators) is that it provides the right framework to think about any kind of *derived* six-functor-formalism:

Definition 6.1. *Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks*[12]*. Recall from [4, Section 3] the definition of the (symmetric) 2-multicategory $\mathcal{S}^{cor}$ (resp. $\mathcal{S}^{cor,0}$ with choice of classes of proper or etale morphisms). Denote its associated represented 2-pre-multiderivator by $\mathbb{S}^{\mathrm{cor}}$, $\mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$, and $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$, respectively (cf. 2.5).*

1. *We define a* **(symmetric) derivator six-functor-formalism** *as a left and right fibered (symmetric) multiderivator*
$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}.$$

2. *We define a* **(symmetric) proper derivator six-functor-formalism** *as before which has an extension as oplax left fibered (symmetric) multiderivator*
$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}},$$
and an extension as lax right fibered (symmetric) multiderivator
$$\mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}.$$

[12] e.g. a category $\mathcal{S}$ with fiber products made into a symmetric opmulticategory like in (2)

3. *We define a* **(symmetric) etale derivator six-functor-formalism** *as before which has an extension as lax left fibered (symmetric) multiderivator*

$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}},$$

and an extension as oplax right fibered (symmetric) multiderivator

$$\mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}.$$

In particular, and in view of [4, Section 8], if $\mathcal{S}^{cor,0} = \mathcal{S}^{cor,G}$ is formed w.r.t. the choice of *all morphisms*, we call a proper derivator six-functor-formalism a **derivator Grothendieck context** and an etale derivator six-functor-formalism a **derivator Wirthmüller context**.

6.2. As mentioned, if $\mathbb{S}$ is really a 2-pre-multiderivator, as opposed to a usual pre-multiderivator, the functor

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \mathrm{Dia}^{\mathrm{cor}},$$

has hardly ever any fibration properties, because of the truncation involved in the definition of the categories of 1-morphisms. Nevertheless the composition

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}) \to \{\cdot\}$$

is often 1-bifibered, i.e. there exists an absolute monoidal product on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S})$ extending the one on $\mathrm{Dia}^{\mathrm{cor}}$. For example, if $\mathcal{S}$ is a usual 1-category with fiber products and final object equipped with the opmulticategory structure (2) then for the 2-pre-multiderivator $\mathbb{S}^{\mathrm{cor}}$ represented by $\mathcal{S}^{\mathrm{cor}}$, we have on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ the monoidal product

$$(I, F) \boxtimes (J, G) = (I \times J, F \times G)$$

where $F \times G$ is the diagram of correspondences in $\mathcal{S}$ formed by applying $\times$ point-wise. Similarly we have

$$\mathbf{HOM}((I, F), (J, G)) = (I^{\mathrm{op}} \times J, F^{\mathrm{op}} \times G)$$

where in F^{op} all correspondences are flipped. In particular any object (I, F) in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ is dualizable with duality explicitly given by

$$\mathbf{HOM}((I, F), (\cdot, \cdot)) = (I^{\mathrm{op}}, F^{\mathrm{op}}).$$

Given a derivator six-functor-formalism $\mathbb{D} \to \mathbb{S}^{\mathrm{cor}}$ we get an external monoidal product even on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ which prolongs the one on diagrams of correspondences, and in many concrete situations all objects will be dualizable.

7 Construction — Part I

In the remaining part of the article we formally *construct* a (symmetric) derivator six-functor-formalism in which $f_! = f_*$, i.e. a derivator Grothendieck context, starting from a (symmetric) fibered multiderivator $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$. The precise statement is as follows:

Main Theorem 7.1. *Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}^{\mathrm{op}}$ be the (symmetric) pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a (symmetric) left and right fibered multiderivator such that the following holds:*

1. *The pullback along 1-ary morphisms (i.e. pushforward along 1-ary morphisms in $\mathcal{S}$) commutes also with homotopy* colimits *(of shape in* Dia*).*

2. *In the underlying bifibration $\mathbb{D}(\cdot) \to \mathbb{S}(\cdot)$ multi-base-change holds in the sense of Definition 1.5.*

Then there exists a (symmetric) oplax left fibered multiderivator

$$\mathbb{E} \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$$

satisfying the following properties

a) *The corresponding (symmetric) 1-opfibration, and 2-opfibration of 2-multicategories with 1-categorical fibers*
$$\mathbb{E}(\cdot) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(\cdot) = \mathcal{S}^{\mathrm{cor},G}$$
is just (up to equivalence) obtained from $\mathbb{D}(\cdot) \to \mathcal{S}^{\mathrm{op}}$ by the procedure described in [4, Definition 3.12].

b) *For every $S \in \mathcal{S}$ there is a canonical equivalence between the fibers (which are usual left and right derivators):*
$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Using standard theorems on Brown representability [3, Section 3.1] we can refine this.

Main Theorem 7.2. *Let* Dia *be an infinite diagram category [3, Definition 1.1.1] which contains all finite posets. Let $\mathcal{S}$ be a (symmetric) opmulticategory with multipullbacks and let $\mathbb{S}$ be the corresponding represented (symmetric) pre-multiderivator. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be an* infinite *(symmetric) left and right fibered multiderivator satisfying conditions 1. and 2. of Theorem 7.1,* with stable, perfectly generated fibers *(cf. Definition 4.4 and [3, Section 3.1]).*
Then the restriction of the left fibered multiderivator $\mathbb{E}$ from Theorem 7.1 is a (symmetric) left and right *fibered multiderivator*
$$\mathbb{E}|_{\mathbb{S}^{\mathrm{cor}}} \to \mathbb{S}^{\mathrm{cor}}$$
and has an extension as a (symmetric) lax *right fibered multiderivator*

$$\mathbb{E}' \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}.$$

In other words, we get a (symmetric) derivator Grothendieck context in the sense of Section 6.

We begin by explaining the construction of $\mathbb{E}$. We need some preparation:

7.3. Let I be a diagram, n a natural number and $\Xi = (\Xi_1, \dots, \Xi_n) \in \{\uparrow, \downarrow\}^n$ be a sequence of arrow directions. We define a diagram
$${}^{\Xi}I$$

whose objects are sequences of $n-1$ morphisms in I

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_n$$

and whose morphisms are commutative diagrams

$$\begin{tikzcd} i_1 \arrow[r] \arrow[d, leftrightarrow] & i_2 \arrow[r] \arrow[d, leftrightarrow] & \cdots \arrow[r] & i_n \arrow[d, leftrightarrow] \\ i_1' \arrow[r] & i_2' \arrow[r] & \cdots \arrow[r] & i_n' \end{tikzcd}$$

in which the j-th vertical arrow goes in the direction indicated by Ξ_j. We call such morphisms **of type** j if at most the morphism $i_j \to i'_j$ is *not* an identity. *From now on we assume that* Dia *permits this construction for any* $I \in \mathrm{Dia}$*, i.e. if* $I \in \mathrm{Dia}$ *then also* ${}^{\Xi}I \in \mathrm{Dia}$ *for every finite* Ξ*.*

Example 7.4.

$$\begin{aligned} {}^{\downarrow}I &= I \\ {}^{\uparrow}I &= I^{\mathrm{op}} \\ {}^{\downarrow\downarrow}I &= I \times_{/I} I \\ {}^{\downarrow\uparrow}I &= \mathrm{tw}(I) \end{aligned}$$

where $I \times_{/I} I$ *is the comma category (or the arrow category of* I*) and* $\mathrm{tw}(I)$ *is called the* **twisted arrow category***.*

7.5. For any ordered subset $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, denoting Ξ' the restriction of Ξ to the subset, we get an obvious restriction functor

$$\pi_{i_1,\dots,i_m} : {}^{\Xi}I \to {}^{\Xi'}I.$$

If $\Xi = \Xi' \circ \Xi'' \circ \Xi'''$, where $\circ$ means concatenation, then the projection

$$\pi_{1,\dots,n'} : {}^{\Xi}I \to {}^{\Xi'}I$$

is a *fibration* if the last arrow of Ξ' is $\downarrow$ and an *opfibration* if the last arrow of Ξ' is $\uparrow$ while the projection

$$\pi_{n-n'''+1,\dots,n} : {}^{\Xi}I \to {}^{\Xi'''}I$$

is an *opfibration* if the first arrow of Ξ''' is $\downarrow$ and a *fibration* if the first arrow of Ξ''' is $\uparrow$.

7.6. A functor $\alpha : I \to J$ induces an obvious functor

$${}^{\Xi}\alpha : {}^{\Xi}I \to {}^{\Xi}J.$$

A natural transfomation $\mu : \alpha \Rightarrow \beta$ induces functors

$$({}^{\Xi}\mu)_0, \dots, ({}^{\Xi}\mu)_n : {}^{\Xi}I \to {}^{\Xi}J$$

with $({}^{\Xi}\mu)_0 = {}^{\Xi}\alpha$, and $({}^{\Xi}\mu)_n = {}^{\Xi}\beta$, defined by mapping an object $i_1 \xrightarrow{\nu_1} i_2 \longrightarrow \dots \xrightarrow{\nu_{n-1}} i_n$ of ${}^{\Xi}I$ to the sequence:

$$\begin{tikzcd} \alpha(i_1) \arrow[r] & \cdots \arrow[r] & \alpha(i_{n-j}) \arrow[dr, "\beta(\nu_{n-j})\circ\mu(i_{n-j})"] & & & \\ & & & \beta(i_{n-j+1}) \arrow[r] & \cdots \arrow[r] & \beta(i_n) \end{tikzcd}$$

There is a sequence of natural transformations

$${}^{\Xi}\alpha = ({}^{\Xi}\mu)_0 \Leftrightarrow \cdots \Leftrightarrow ({}^{\Xi}\mu)_n = {}^{\Xi}\beta$$

where the natural transformations at position i (the count starting with 0) goes to the right if $\Xi_{n-i} = \downarrow$ and to the left if $\Xi_{n-i} = \uparrow$. Furthermore, the natural transformation at position i consists element-wise of morphisms of type $n-i$.

7.7. If $\alpha : I \to J$ is an opfibration and we form the pull-back

$$\begin{tikzcd} {}^{\downarrow\uparrow}J \times_J I \arrow[r] \arrow[d] & I \arrow[d, "\alpha"] \\ {}^{\downarrow\uparrow}J \arrow[r, "\pi_1"'] & J \end{tikzcd}$$

then obviously the left vertical functor is an opfibration as well.

7.8. Let $S : I \to \mathcal{S}^{\mathrm{cor}}$ be a pseudo-functor. We can associate to it a natural functor $S' : {}^{\downarrow\uparrow}I \to \mathcal{S}$ such that for each composition of three morphisms $\gamma\beta\alpha$ the commutative diagram

$$\begin{tikzcd} \gamma\beta\alpha \arrow[r] \arrow[d] & \beta\alpha \arrow[d] \\ \gamma\beta \arrow[r] & \beta \end{tikzcd} \tag{10}$$

in ${}^{\downarrow\uparrow}I$ is mapped to a Cartesian square in $\mathcal{S}$. We call such diagrams **admissible**.
Note that the horizontal morphisms are of type 2 and the vertical ones of type 1. Conversely every square in ${}^{\downarrow\uparrow}I$ with these properties has the above form.
The construction of S' is as follows. S maps a morphism ν in I to a correspondence in $\mathcal{S}$

$$X_\nu \longleftarrow A_\nu \longrightarrow Y_\nu,$$

and we define $S'(\nu) := A_\nu$. A morphism $\xi : \nu \to \mu$ defined by

$$\begin{tikzcd} i \arrow[r, "\nu"] \arrow[d, "\alpha"'] & j \\ k \arrow[r, "\mu"'] & l \arrow[u, "\beta"'] \end{tikzcd}$$

induces, by definition of the composition in $\mathcal{S}^{\mathrm{cor}}$, a commutative diagram in which all squares are Cartesian:

$$\begin{tikzcd}[column sep=small] & & & A_\nu \arrow[dl] \arrow[dr] & & & \\ & & A_{\mu\alpha} \arrow[dl] \arrow[dr] & & A_{\beta\mu} \arrow[dl] \arrow[dr] & & \\ & A_\alpha \arrow[dl] \arrow[dr] & & A_\mu \arrow[dl] \arrow[dr] & & A_\beta \arrow[dl] \arrow[dr] & \\ X & & Y & & Z & & W \end{tikzcd}$$

We define $S'(\xi)$ to be the induced morphism $A_\nu \to A_\mu$. Note that the square of the form (10) is just mapped to the upper square in the above diagram, thus to a Cartesian square. Hence the so defined functor S' is admissible.

7.9. A multi-morphism

$$T \longrightarrow S_1, \ldots, S_n$$

of admissible diagrams in $\mathbb{S}({}^{\downarrow\uparrow}I)$ is called **type i admissible** ($i = 1, 2$), if for any morphism $\xi : \nu \to \mu$ in ${}^{\downarrow\uparrow}I$ of type i the diagram

$$\begin{tikzcd} T(\nu) \arrow[r] \arrow[d] & (S_1(\nu), \dots, S_n(\nu)) \arrow[d] \\ T(\mu) \arrow[r] & (S_1(\mu), \dots, S_n(\mu)) \end{tikzcd}$$

is a multipullback.
A multimorphism $(X_1, \dots, X_n) \to Y$ in $\mathcal{S}^{\mathrm{cor}}(I)$ can be seen equivalently as a multicorrespondence of admissible diagrams in $\mathcal{S}({}^{\downarrow\uparrow}I)$

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (X_1, \dots, X_n) & & Y \end{tikzcd}$$

where f is type 2 admissible and g is type 1 admissible. In this description, the 2-morphisms are the commutative diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\ (X_1, \dots, X_n) & & Y \\ & A' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where the morphism h is an isomorphism.
In this way, we see that the 2-multicategory $\mathcal{S}^{\mathrm{cor}}(I)$ is equivalent to the 2-multicategory having as objects admissible diagrams ${}^{\downarrow\uparrow}I \to \mathcal{S}$ with the 1-multimorphisms and 2-morphisms described above.

Lemma 7.10. *Type i admissible morphisms $S \to T$ between admissible diagrams $S, T \in \mathbb{S}({}^{\downarrow\uparrow}I)$ satisfy the following property:*
If $h_3 = h_2 \circ h_1$ and h_2 is type i-admissible then h_1 is type i admissible if and only if h_3 is type i admissible.

Proof. This follows immediately from the corresponding property of Cartesian squares. □

7.11. The discussion in 7.9 has an (op)lax variant. Recall the definition of the category (value of the represented (op)lax 2-pre-multiderivator) $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$ (resp. $\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(I)$), of pseudo-functors, *(op)lax* natural transformations, and modifications. A *lax* multimorphism of pseudo-functors

$$(X_1, \dots, X_n) \longrightarrow Y$$

can be equivalently seen as a multicorrespondence of admissible diagrams in $\mathbb{S}({}^{\downarrow\uparrow}I)$

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (X_1, \dots, X_n) & & Y \end{tikzcd}$$

where g is type 1 admissible and f is *arbitrary*. Similarly an *oplax* multimorphism can be seen as such a multicorrespondence in which g is *arbitrary* and f is type 2 admissible. In the 2-morphisms the morphism h can be an arbitrary morphism, which is automatically type 1 admissible in the lax case and type 2 admissible in the oplax case (cf. Lemma 7.10).

7.12. We can therefore describe the represented 2-pre-multiderivator $\mathbb{S}^{\mathrm{cor}}$, $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}$, and $\mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}$, respectively, in a different way: A diagram I is mapped to the 2-multicategory of admissible diagrams ${}^{\downarrow\uparrow}I \to \mathcal{S}$ where multimorphisms are multicorrespondences

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (X_1, \dots, X_n) & & Y \end{tikzcd}$$

of admissible diagrams with the corresponding conditions discussed above and where 2-morphisms are the isomorphisms (resp. arbitrary morphisms) between these multicorrespondences.
A functor $\alpha : I \to J$ is mapped to the composition $- \circ ({}^{\downarrow\uparrow}\alpha)$. This is a strict functor and the association is strictly functorial. A natural transformation $\mu : \alpha \Rightarrow \beta$ is mapped to the following natural transformation. First of all it gives rise (cf. 7.6) to a sequence of natural transformations

$$({}^{\downarrow\uparrow}\alpha) \Leftarrow ({}^{\downarrow\uparrow}\mu)_1 \Rightarrow ({}^{\downarrow\uparrow}\beta).$$

For any admissible diagram $S : {}^{\downarrow\uparrow}I \to \mathcal{S}$ this defines a diagram

$$\begin{tikzcd} & ({}^{\downarrow\uparrow}\mu)_1^* S \arrow[dl, "g_S"'] \arrow[dr, "f_S"] & \\ ({}^{\downarrow\uparrow}\alpha)^* S & & ({}^{\downarrow\uparrow}\beta)^* S \end{tikzcd}$$

in which the morphism f_S is type 2 admissible and the morphism g_S is type 1 admissible. This defines a 1-morphism

$$({}^{\downarrow\uparrow}\alpha)^* S \to ({}^{\downarrow\uparrow}\beta)^* S$$

in the alternative description (cf. 7.9) of $\mathbb{S}^{\mathrm{cor}}(I)$. For any admissible diagram S this defines a pseudo-functor $\alpha \mapsto \alpha^* S$ from the category of functors $\mathrm{Hom}(I, J)$ to the 2-category $\mathbb{S}^{\mathrm{cor}}(I)$.

7.13. Let I be a diagram. Consider the category ${}^{\downarrow\uparrow\downarrow}I$ defined in 7.3. Recall that its objects are compositions of two morphisms in I and its morphisms $\nu \to \mu$ are commutative diagrams

$$\begin{tikzcd} i \arrow[r, "\nu_1"] \arrow[d] & j \arrow[r, "\nu_2"] & k \arrow[d] \\ i' \arrow[r, "\mu_1"'] & j' \arrow[u] \arrow[r, "\mu_2"'] & k' \end{tikzcd}$$

7.14. If $\alpha : I \to J$ is an opfibration and we form the pull-back

$$\begin{tikzcd} {}^{\downarrow\uparrow\downarrow}J \times_J I \arrow[r] \arrow[d] & I \arrow[d, "\alpha"] \\ {}^{\downarrow\uparrow\downarrow}J \arrow[r, "\pi_1"'] & J \end{tikzcd}$$

and

$$\begin{tikzcd} {}^{\downarrow\uparrow\downarrow}J \times_{{}^{\downarrow\uparrow}J} {}^{\downarrow\uparrow}I \arrow[r] \arrow[d] & {}^{\downarrow\uparrow}I \arrow[d, "{}^{\downarrow\uparrow}\alpha"] \\ {}^{\downarrow\uparrow\downarrow}J \arrow[r, "\pi_{12}"'] & {}^{\downarrow\uparrow}J \end{tikzcd}$$

then obviously the left vertical functors are opfibrations as well.

Lemma 7.15. *Let $\alpha : I \to J$ be an opfibration, and consider the sequence defined by the universal property of pull-backs*

$$ {}^{\downarrow\uparrow\downarrow}I \xrightarrow{q_1} {}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I \xrightarrow{q_2} {}^{\downarrow\uparrow\downarrow}J \times_J I. $$

1. *The functor q_1 is an opfibration. The fiber of q_1 over a pair $j_1 \to j_2 \to j_3$ and $i_1 \to i_2$ is*

$$ i_3 \times_{/I_{j_3}} I_{j_3} $$

where i_3 is the target of a coCartesian arrow over $j_2 \to j_3$ with source i_2.

2. *The functor q_2 is a fibration. The fiber of q_2 over a pair $j_1 \to j_2 \to j_3$ and i_1 is*

$$ (i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}} $$

where i_2 is the target of a coCartesian arrow over $j_1 \to j_2$ with source i_1.

Proof. Straightforward. □

Recall the following definition from [3, Definition 2.4.1], in which $\mathbb{S}$ (generalizing slightly the definition of [loc. cit.]) can be any 2-pre-multiderivator.

Definition 7.16. *Let $\mathbb{D} \to \mathbb{S}$ be a right (resp. left) fibered (multi-)derivator of domain* Dia. *Let $I, E \in$* Dia *be diagrams and let $\alpha : I \to E$ be a functor in* Dia. *We say that an object*

$$ \mathcal{E} \in \mathbb{D}(I) $$

*is E-***(co-)Cartesian***, if for any morphism $\mu : i \to j$ in I mapping to an identity in E, the corresponding morphism $\mathbb{D}(\mu) : i^*\mathcal{E} \to j^*\mathcal{E}$ is (co-)Cartesian.*
If E is the trivial category, we omit it from the notation, and talk about (co-)Cartesian objects.

These notions define full subcategories $\mathbb{D}(I)^{E-\mathrm{cart}}$ (resp. $\mathbb{D}(I)^{E-\mathrm{cocart}}$) of $\mathbb{D}(I)$, and $\mathbb{D}(I)_S^{E-\mathrm{cart}}$ (resp. $\mathbb{D}(I)_S^{E-\mathrm{cocart}}$) of $\mathbb{D}(I)_S$ for any $S \in \mathbb{S}(I)$. If we want to specify the functor α, we speak about α-(co)Cartesian objects and denote these e.g. by $\mathbb{D}(I)_S^{\alpha-\mathrm{cart}}$.

Definition 7.17. *Let $\mathcal{S}$ be an opmulticategory with multipullbacks and let $\mathbb{S}^{\mathrm{op}}$ be the pre-multiderivator represented by $\mathcal{S}^{\mathrm{op}}$. Let $\mathbb{D} \to \mathbb{S}^{\mathrm{op}}$ be a (left and right) fibered multiderivator such that conditions 1. and 2. of Theorem 7.1 hold true.*
We define the morphism of 2-pre-multiderivators $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ of Theorem 7.1. The 2-pre-multiderivator $\mathbb{E}$ is defined as follows: A diagram I is mapped to a 1-opfibered, and 2-opfibered multicategory with 1-categorical fibers $\mathbb{E}(I) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(I)$. We will specify this by giving the pseudo-functor of 2-multicategories

$$ \mathbb{S}^{\mathrm{cor},G,\mathrm{oplax}}(I)^{2-\mathrm{op}} \to \mathcal{CAT} $$

where we understand $\mathbb{S}^{\mathrm{cor}}(I)$ (resp. $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$) in the form described in 7.12. An admissible diagram $S : {}^{\downarrow\uparrow}I \to \mathcal{S}$ is mapped to the category

$$ \boxed{\mathbb{E}(I)_S := \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}} $$

(cf. Definition 7.16). Note that $({}^{\downarrow\uparrow}I)^{\mathrm{op}} = {}^{\uparrow\downarrow}I$.

A multicorrespondence

$$\begin{tikzcd}
 & & A \arrow[dll, "g"'] \arrow[dr, "f"] & \\
(S_1, \dots, S_n) & & & T
\end{tikzcd}$$

where f is type 2 admissible and g is type 1 admissible is mapped to the functor

$$\boxed{(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet : \mathbb{E}(I)_{S_1} \times \cdots \times \mathbb{E}(I)_{S_n} \to \mathbb{E}(I)_T}$$

Note that, by Lemma 7.18, $(\pi_{23}^* g)_\bullet$ preserves the subcategory of π_{12}-Cartesian objects and, by Lemma 7.19, $(\pi_{23}^* f)^\bullet$ preserves the subcategory of π_{13}-coCartesian objects. In the oplax case, the condition on f is repealed and the multicorrespondence is mapped to

$$\Box_* (\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet$$

where $\Box_*$ is the right coCartesian projection *defined and discussed in Section 8.*
A 2-morphism, given by a morphism of multicorrespondences

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dd, "h"] \arrow[dr, "g"] & \\
X_1, \dots, X_n & & Y \\
 & A' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

where h is an isomorphism, is mapped to the natural transformation given by the unit

$$\boxed{(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet \cong (\pi_{23}^* f')^\bullet (\pi_{23}^* h)^\bullet (\pi_{23}^* h)_\bullet (\pi_{23}^* g')_\bullet \leftarrow (\pi_{23}^* f')^\bullet (\pi_{23}^* g')_\bullet}$$

In the oplax case, h can be an arbitrary morphism (which will be automatically type 1 admissible). The 2-morphism is then mapped to the natural transformation given by the unit

$$\Box_* (\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet \cong \Box_* (\pi_{23}^* f')^\bullet \, \Box_* \, (\pi_{23}^* h)^\bullet (\pi_{23}^* h)_\bullet (\pi_{23}^* g')_\bullet \leftarrow \Box_* (\pi_{23}^* f')^\bullet (\pi_{23}^* g')_\bullet.$$

A functor $\alpha : I \to J$ is mapped to the functor

$$\boxed{({}^{\downarrow\uparrow\downarrow}\alpha)^*}$$

which obviously preserves the (co)Cartesianity conditions. This is strictly compatible with composition of functors between diagrams. A natural transformation $\mu : \alpha \to \beta$ is mapped to the following natural transformation $({}^{\downarrow\uparrow\downarrow}\alpha)^* \to ({}^{\downarrow\uparrow\downarrow}\beta)^*$: We have the correspondence (cf. 7.12)

$$\begin{tikzcd}
 & ({}^{\downarrow\uparrow}\mu)_1^* S \arrow[dl, "g_\mu"'] \arrow[dr, "f_\mu"] & \\
({}^{\downarrow\uparrow}\alpha)^* S & & ({}^{\downarrow\uparrow}\beta)^* S
\end{tikzcd}$$

where f_μ is type 2 admissible and g_μ is type 1 admissible by the definition of admissible diagram. On the other hand, there are natural transformations (cf. 7.6)

$${}^{\downarrow\uparrow\downarrow}\alpha \Rightarrow ({}^{\downarrow\uparrow\downarrow}\mu)_1 \Leftarrow ({}^{\downarrow\uparrow\downarrow}\mu)_2 \Rightarrow {}^{\downarrow\uparrow\downarrow}\beta.$$

Inserting $\pi_{23}^*(S^{\mathrm{op}})$ into this, we get

$$\pi_{23}^*({}^{\downarrow\uparrow}\alpha)^*(S^{\mathrm{op}}) \xrightarrow{\pi_{23}^* g_\mu} \pi_{23}^*({}^{\downarrow\uparrow}\mu)_1^*(S^{\mathrm{op}}) \xleftarrow{\pi_{23}^* f_\mu} \pi_{23}^*({}^{\downarrow\uparrow}\beta)^*(S^{\mathrm{op}}) = \pi_{23}^*({}^{\downarrow\uparrow}\beta)^*(S^{\mathrm{op}}). \tag{11}$$

The natural transformation $\mu : \alpha \to \beta$ may be seen as a functor $\Delta_1 \times I \to J$ and therefore we get a functor

$${}^{\downarrow\uparrow\downarrow}\mu : {}^{\downarrow\uparrow\downarrow}\Delta_1 \times {}^{\downarrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow\downarrow}J.$$

Applying the (pre-)derivator $\mathbb{D}$ and partially evaluating at the objects and morphisms of ${}^{\downarrow\uparrow\downarrow}\Delta_1$ we get natural transformations

$$\begin{array}{rcl} (\pi_{23}^* g_\mu)_\bullet ({}^{\downarrow\uparrow\downarrow}\alpha)^* & \to & ({}^{\downarrow\uparrow\downarrow}\mu)_1^* \\ ({}^{\downarrow\uparrow\downarrow}\mu)_2^* & \to & (\pi_{23}^* f_\mu)^\bullet ({}^{\downarrow\uparrow\downarrow}\mu)_1^* \\ ({}^{\downarrow\uparrow\downarrow}\mu)_2^* & \to & ({}^{\downarrow\uparrow\downarrow}\beta)^* \end{array}$$

where the $(-)^*$-functors are now considered to be functors between the respective fibers over the objects of (11). Clearly the first two morphisms (in particular the second) are isomorphisms when restricted to the respective categories of (co)Cartesian objects. Therefore we can form their composition:

$$\boxed{(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet ({}^{\downarrow\uparrow\downarrow}\alpha)^* \to ({}^{\downarrow\uparrow\downarrow}\beta)^*}$$

which will be the image of μ under the 2-pre-multiderivator $\mathbb{E}$. One checks that for any admissible diagram $S \in \mathbb{S}({}^{\downarrow\uparrow}I)$, this defines a pseudo-functor from the category of functors $\mathrm{Hom}(I,J)$ *to the 2-category of functors of the 2-category $\mathbb{E}(I)$ to the 2-category $\mathbb{E}(J)$, pseudo-natural transformations and modifications.*

Lemma 7.18. *Under the conditions of Theorem 7.1, let $S, T : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be admissible diagrams and let $f : S \to T$ be any morphism in $\mathbb{S}({}^{\downarrow\uparrow}I)$. Then the functor*

$$(\pi_{23}^* f)^\bullet : \mathbb{D}(I)_{\pi_{23}^* T^{\mathrm{op}}} \to \mathbb{D}(I)_{\pi_{23}^* S^{\mathrm{op}}}$$

maps always π_{13}-Cartesian objects to π_{13}-Cartesian objects, and maps π_{12}-coCartesian objects to π_{12}-coCartesian if f is type 2 admissible.

Proof. This follows immediately from base-change and from the definition of type 2 admissible. □

Lemma 7.19. *Under the conditions of Theorem 7.1, let $S_1, \dots, S_n, T : {}^{\downarrow\uparrow}I \to \mathcal{S}$ be admissible diagrams and let $g : S_1, \dots, S_n \to T$ be any multimorphism in $\mathbb{S}({}^{\downarrow\uparrow}I)$. Then the functor*

$$(\pi_{23}^* g)_\bullet : \mathbb{D}(I)_{\pi_{23}^* S_1^{\mathrm{op}}} \times \cdots \times \mathbb{D}(I)_{\pi_{23}^* S_n^{\mathrm{op}}} \to \mathbb{D}(I)_{\pi_{23}^* T^{\mathrm{op}}}$$

maps always π_{12}-coCartesian objects to π_{12}-coCartesian objects, and maps π_{13}-Cartesian objects to π_{13}-Cartesian objects if g is type 1 admissible.

Proof. This follows immediately from multi-base-change and from the definition of type 1 admissible. □

7.20. Recall that a diagram I is called contractible, if

$$\mathrm{id} \Rightarrow p_{I,*}(p_I)^*,$$

or equivalently

$$p_{I,!}(p_I)^* \Rightarrow \mathrm{id},$$

is an isomorphism for all derivators. Cisinski showed that this is the case if and only if $N(I)$ is weakly contractible in the sense of simplicial sets. For instance, any diagram possessing a final or initial object is contractible. The following lemma was shown in [3] for the case of all contractible diagrams for a restricted class of stable derivators. We will only need the mentioned special case which is easy to prove in full generality:

Lemma 7.21. *If $\mathbb{D}$ is a left derivator and I has a final object, or $\mathbb{D}$ is a right derivator and I has an initial object, then the functor*

$$p_I^* : \mathbb{D}(\cdot) \to \mathbb{D}(I)^{\mathrm{cart}} = \mathbb{D}(I)^{\mathrm{cocart}}$$

is an equivalence.

Note that Cartesian=coCartesian here only means that all morphisms in the underlying diagram in $\mathrm{Hom}(I, \mathbb{D}(\cdot))$ are isomorphisms.

Proof. Assume we have a left derivator and I has a final object (the other statement is dual). It suffices to show that the counit

$$p_{I,!}p_I^* \Rightarrow \mathrm{id}$$

is an isomorphism and that the unit

$$\mathrm{id} \Rightarrow p_I^* p_{I,!}$$

is an isomorphism when restricted to the subcategory of Cartesian objects. Since I has a final object i we have an isomorphism

$$p_{I,!} \cong i^*$$

and the unit and counit become the morphisms induced by the natural transformations $p_I \circ i = \mathrm{id}$ and $\mathrm{id} \Rightarrow i \circ p_I$. Hence we have

$$i^* p_I^* = \mathrm{id}$$

and the morphism

$$\mathrm{id} \Rightarrow p_I^* i^*$$

is an isomorphism on (co)Cartesian objects by definition of (co)Cartesian. □

Corollary 7.22. *If $\mathbb{D}$ is a left and right derivator and I has a final* or *initial object then*

$$p_I^* : \mathbb{D}(\cdot) \to \mathbb{D}(I)^{\mathrm{cart}} = \mathbb{D}(I)^{\mathrm{cocart}}$$

is an equivalence, whose inverse is given by $p_{I,!}$ or equivalently by $p_{I,*}$.

Proof. The first part is just restating the above lemma. The fact that both the restriction of $p_{I,!}$, and the restriction of $p_{I,*}$, to the subcategory $\mathbb{D}(I)^{\mathrm{cart}}$ are an inverse to p_I^* follows because these restrictions are obviously still left, resp. right, adjoints to the equivalence p_I^*, hence both inverses, because of the uniqueness of adjoints (up to unique isomorphism). □

Lemma 7.23. *Under the assumptions of Theorem 7.1, if $\alpha : I \to J$ is an opfibration then the functors*

$$\mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_J I)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\mathrm{op}})} \xrightarrow{q_2^*} \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\mathrm{op}})} \xrightarrow{q_1^*} \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$$

are equivalences. In particular (applying this to $J = \cdot$ and variable I) we have an equivalence of fibers:

$$\mathbb{E}_S \cong \mathbb{D}_S.$$

Proof. We first treat the case of q_1^*. We know by Lemma 7.15 that q_1 is an opfibration with fibers of the form $i_3 \times_{/I_{j_3}} I_{j_3}$. Neglecting the conditions of being (co)Cartesian, we know that q_1^* has a left adjoint:

$$q_{1,!} : \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I)_{\pi_{23}^*(S^{\mathrm{op}})}$$

We will show that the unit and counit

$$\mathrm{id} \Rightarrow q_1^* q_{1,!} \qquad q_{1,!} q_1^* \Rightarrow \mathrm{id}$$

are isomorphisms *when restricted to the subcategory of π_{12}-coCartesian objects.* Since the conditions of being π_{13}-Cartesian match under q_1^* this shows the first assertion. Since q_1 is an opfibration this is the same as to show that for any $\gamma \in {}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I$ with fiber $F = i_3 \times_{/I_{j_3}} I_{j_3}$ the unit and counit

$$\mathrm{id} \Rightarrow p_F^* p_{F,!} \qquad p_{F,!} p_F^* \Rightarrow \mathrm{id} \tag{12}$$

are isomorphisms when restricted to the subcategory of π_{12}-coCartesian objects. Since π_{12} maps all morphisms in the fiber F to an identity, we have to show that the morphisms in (12) are isomorphisms when restricted to (absolutely) (co)Cartesian objects. This follows from the fact that F has an initial object (Lemma 7.21 and Corollary 7.22).

We now treat the case of q_2^*. We know by Lemma 7.15 that q_2 is a fibration with fibers of the form $(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$. Neglecting the conditions of being (co)Cartesian, we know that q_1^* has a right adjoint:

$$q_{2,*} : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_{({}^{\downarrow\uparrow}J)} {}^{\downarrow\uparrow}I)_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}J \times_J I)_{\pi_{23}^*(S^{\mathrm{op}})}$$

We will show that the unit and counit

$$\mathrm{id} \Rightarrow q_{2,*} q_2^* \qquad q_2^* q_{2,*} \Rightarrow \mathrm{id}$$

are isomorphisms *when restricted to the subcategory of π_{13}-Cartesian objects.* Since the conditions of being π_{12}-coCartesian match under q_2^* this shows the second assertion. Since q_2 is a fibration this is the same as to show that for any $\gamma \in {}^{\downarrow\uparrow\downarrow}J \times_J I$ with fiber $F = (i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$ the the unit and counit

$$\mathrm{id} \Rightarrow p_{F,*} p_F^* \qquad p_F^* p_{F,*} \Rightarrow \mathrm{id} \tag{13}$$

are isomorphisms when restricted to the subcategory of π_{13}-Cartesian objects. Since π_{13} maps all morphisms in the fiber $(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$ to an identity, this means that we have to show that (13) are isomorphisms when restricted to (absolutely) (co)Cartesian objects. This follows from the fact that $(i_2 \times_{/I_{j_2}} I_{j_2})^{\mathrm{op}}$ has a final object (Lemma 7.21 and Corollary 7.22). □

Lemma 7.24. *Let the situation be as in Theorem 7.1 and let $p' : \mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ be the morphism of 2-pre-multiderivators defined in 7.17. Let $\alpha : I \to J$ be an opfibration. Then $\alpha^* : \mathbb{E}(J)_{\alpha^* S} \to \mathbb{E}(I)_S$ has a left adjoint $\alpha_!^{(S)}$.*

Proof. We have to show that

$$({}^{\downarrow\uparrow\downarrow}\alpha)^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\text{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\text{op}})}$$

has a left adjoint. The right hand side category is by Lemma 7.23 equivalent to

$$\mathbb{D}(({}^{\downarrow\uparrow\downarrow}J) \times_J I)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^* S},$$

hence we have to show that

$$\mathrm{pr}_1^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\text{op}})} \to \mathbb{D}(({}^{\downarrow\uparrow\downarrow}J) \times_J I)^{\pi_{12}-\text{cocart},\pi_{13}-\text{cart}}_{\pi_{23}^*(S^{\text{op}})}$$

has a left adjoint. By assumption the functor

$$\mathrm{pr}_1^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}J)_{\pi_{23}^*(S^{\text{op}})} \to \mathbb{D}(({}^{\downarrow\uparrow\downarrow}J) \times_J I)_{\pi_{23}^*(S^{\text{op}})}$$

has a left adjoint $\mathrm{pr}_{2,!}$. We claim that it maps π_{12}-coCartesian objects to π_{12}-coCartesian objects and π_{13}-Cartesian objects to π_{13}-Cartesian objects. The statement then follows.
Let $\kappa : \nu \to \nu'$

$$\begin{tikzcd}
j_1 \arrow[r, "\nu_1"] \arrow[d, equal] & j_2 \arrow[r, "\nu_2"] & j_3 \arrow[d, equal] \\
j_1 \arrow[r, "\nu_1'"'] & j_2' \arrow[r, "\nu_2'"'] \arrow[u, "\kappa_2"'] & j_3
\end{tikzcd}$$

be a morphism in ${}^{\downarrow\uparrow\downarrow}J$ such that π_{13} maps it to an identity. Denote

$$f := S(\pi_{23}(\kappa)) : S(\pi_{23}(\nu)) \to S(\pi_{23}(\nu'))$$

the corresponding morphism in $\mathbb{S}(\cdot)^{\text{op}}$. Denote by (ν), resp. (ν') the inclusion of the one element category mapping to ν, resp. ν' in ${}^{\downarrow\uparrow\downarrow}I$. We have to show that the induced map

$$(\nu)^* \,\mathrm{pr}_{1,!} \to f^\bullet (\nu')^* \,\mathrm{pr}_{1,!}$$

is an isomorphism on π_{13}-Cartesian objects. Since pr_1 is an opfibration, this is the same as to show that the natural morphism

$$p_! \iota_\nu^* \to f^\bullet p_! \iota_{\nu'}^*$$

is an isomorphism on π_{13}-Cartesian objects where $p : I_{j_1} \to \cdot$ is the projection. Since $f^\bullet$ commutes with homotopy colimits by assumption 1. of Theorem 7.1, this is to say that

$$p_! \iota_\nu^* \to p_! (p^* f)^\bullet \iota_{\nu'}^*$$

is an isomorphism. However the fibers over ν and ν' in $({}^{\downarrow\uparrow\downarrow}J) \times_J I$ are both equal to I_{j_1} and the natural morphism

$$\iota_\nu^* \to (p^* f)^\bullet \iota_{\nu'}^*$$

is already an isomorphism on Cartesian objects by definition.
Let $\kappa : \nu_1 \to \nu_2$

$$\begin{tikzcd}
j_1 \arrow[r, "\nu_1"] \arrow[d, equal] & j_2 \arrow[r, "\nu_2"] \arrow[d, equal] & j_3 \arrow[d] \\
j_1 \arrow[r, "\nu_1'"'] & j_2' \arrow[r, "\nu_2'"'] & j_3'
\end{tikzcd}$$

be a morphism in ${}^{\downarrow\uparrow\downarrow}J$ such that π_{12} maps it to an identity. And denote

$$g := S(\pi_{23}(\kappa)) : S(\pi_{23}(\nu)) \to S(\pi_{23}(\nu'))$$

the corresponding morphism in $\mathbb{S}(\cdot)$. Denote by (ν), resp. (ν') the inclusion of the one element category mapping to ν, resp. ν'. We have to show that the induced map

$$g_\bullet(\nu)^* \operatorname{pr}_{1,!} \to (\nu')^* \operatorname{pr}_{1,!}$$

is an isomorphism on π_{12}-coCartesian objects. This is the same as to show that the natural morphism

$$g_\bullet p_! \iota_\nu^* \to p_! \iota_{\nu'}^*$$

is an isomorphism on π_{12}-coCartesian objects where $p : I_{j_1} \to \cdot$ is the projection. Since $g_\bullet$ commutes with homotopy colimits, this is to say that

$$p_!(p^* g)_\bullet \iota_\nu^* \to p_! \iota_{\nu'}^*$$

is an isomorphism. However the fibers over ν and ν' in $({}^{\downarrow\uparrow\downarrow}J) \times_J I$ are both equal to I_j and the natural morphism

$$(p^* g)_\bullet \iota_\nu^* \to \iota_{\nu'}^*$$

is already an isomorphism on coCartesian objects by definition of coCartesian. □

Proof of Theorem 7.1. It is clear that the 2-pre-multiderivator $\mathbb{E}$ as defined in 7.17 satisfies axioms (Der1) and (Der2) because $\mathbb{D}$ satisfies them. Axiom (FDer0 left) holds by construction of $\mathbb{E}$. Instead of Axiom (FDer3 left) it is sufficient to show Axiom (FDer3 left') which follows from Lemma 7.24. Axiom (FDer4 left') follows from the proof of Lemma 7.24. (FDer5 left) follows from the corresponding axiom for $\mathbb{D}$ and the fact that pull-back along 1-ary morphisms in $\mathbb{S}^{\mathrm{op}}$ commutes with homotopy colimits as well, by assumption. □

7.25. Let $\alpha : K \to L$ be a functor in Dia and let $\xi : (I_1, S_1), \dots, (I_n, S_n) \to (J, T)$ be a 1-morphism in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$. If we have a 1-opfibration and 2-opfibration

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{E}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$$

then the isomorphism of Lemma 5.6, 3. is transformed into an isomorphism

$$(\alpha \times \mathrm{id})^* \circ (\xi \times L)_\bullet \to (\xi \times K)_\bullet \circ ((\alpha \times \mathrm{id})^*, \dots, (\alpha \times \mathrm{id})^*)$$

which turns $K \mapsto (\xi \times K)_\bullet$ into a morphism of usual derivators

$$\xi_\bullet : \mathbb{D}_{I_1, S_1} \times \cdots \times \mathbb{D}_{I_n, S_n} \to \mathbb{D}_{J,T}. \tag{14}$$

Lemma 7.26. *The morphism of derivators (14) is left exact in each variable, i.e. the exchange*

$$(\xi \times_j L)_\bullet \circ_j (\alpha \times \mathrm{id})_! \to (\alpha \times \mathrm{id})_! \circ (\xi \times_j K)_\bullet$$

is an isomorphism for any $\alpha : K \to L$.

Proof. This follows from Lemma 5.6, 4. □

Proof of Theorem 7.2. The first assertion is a slight generalization of [3, Theorem 3.2.3 (left)]. Using Definition 4.1 of a left, resp. right fibered multiderivator over 2-pre-multiderivators we give a different slicker proof. We have to show that, under the conditions of Theorem 7.2, the constructed 1-opfibration

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{E}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$$

is a 1-fibration as well. The conditions imply:

1. $\mathrm{Dia}, \mathbb{E}$ and $\mathbb{S}^{\mathrm{cor}}$ are infinite,
2. the fibers of $\mathbb{E} \to \mathbb{S}^{\mathrm{cor}}$ (which are the same as those of $\mathbb{D} \to \mathbb{S}$) are stable and perfectly generated infinite left derivators with domain Dia, and also right derivators with domain (at least) Posf.

Any multimorphism in $(I_1, S_1), \dots, (I_n, S_n) \to (J, T)$ in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$ gives actually a morphism between fibers which are usual left and right stable derivators which are perfectly generated:

$$\mathbb{D}_{I_1,S_1} \times \cdots \times \mathbb{D}_{I_n,S_n} \to \mathbb{D}_{J,T}.$$

Lemma 7.26 shows that this morphism commutes with homotopy colimits in each variable. Thus by [3, Theorem 3.2.1 (left)] it has a right adjoint in each slot j, which, in particular, evaluated at $\cdot$ yields a right adjoint functor in the slot j:

$$\mathbb{D}(I_1)^{\mathrm{op}}_{S_1} \times \cdots \times \mathbb{D}(J)_T \times \cdots \times \mathbb{D}(I_n)^{\mathrm{op}}_{S_n} \to \mathbb{D}(I_j)_{S_j}$$

for each j. This establishes that the morphism

$$\mathrm{Dia}^{\mathrm{cor}}(\mathbb{E}) \to \mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{\mathrm{cor}})$$

is 1-fibered as well.

The lax extension of this 1-fibration is given as follows. For each diagram I we again specify a 1-fibered, and 2-opfibered multicategory with 1-categorical fibers $\mathbb{E}'(I) \to \mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}$.

The category

$$\mathbb{E}'(I)$$

has the same objects as $\mathbb{E}(I)$, i.e. pairs $(S, \mathcal{E})$ consisting of an admissible diagram $S : {}^{\downarrow\uparrow}I \to \mathcal{S}$ and an object

$$\mathcal{E} \in \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{12}-\mathrm{cocart},\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}.$$

The 1-Morphisms are the morphisms in $\mathbb{S}^{\mathrm{cor},G,\mathrm{lax}}(I)$, i.e. lax morphisms, which can be given by a multicorrespondence

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (S_1, \dots, S_n) & & T \end{tikzcd}$$

in which f is type 2 admissible, and g is arbitrary, together with a morphism

$$\rho \in \mathrm{Hom}_{\mathbb{E}(I)}\left((\mathcal{E}_1, S_1), \dots, (\mathcal{E}_n, S_n), (\mathcal{F}, T)\right) = \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(T^{\mathrm{op}})}}\left((\pi_{23}^* f)^{\bullet}(\pi_{23}^* g)_{\bullet}(\mathcal{E}_1, \dots, \mathcal{E}_n), \mathcal{F}\right).$$

Note that the multivalued functor $(\pi_{23}^* g)_{\bullet}$ does not necessarily have values in the subcategory of π_{13}-Cartesian objects.

A 2-morphism $(f, g, \rho) \Rightarrow (f', g', \rho')$ is given by a morphism of multicorrespondences

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\
X_1, \dots, X_n & & Y \\
 & A' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

where h is an arbitrary morphism (which is automatically type 2 admissible, cf. Lemma 7.10) such that the diagram

$$\begin{tikzcd}
(\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet (\mathcal{E}_1, \dots, \mathcal{E}_n) \arrow[drr, "\rho"] & & \\
(\pi_{23}^* f')^\bullet (\pi_{23}^* h)^\bullet (\pi_{23}^* h)_\bullet (\pi_{23}^* g')_\bullet (\mathcal{E}_1, \dots, \mathcal{E}_n) \arrow[u, "\sim"'] & & \mathcal{F} \\
(\pi_{23}^* f')^\bullet (\pi_{23}^* g')_\bullet (\mathcal{E}_1, \dots, \mathcal{E}_n) \arrow[u] \arrow[urr, "\rho'"'] & &
\end{tikzcd}$$

commutes, where the lower left vertical morphism is the unit.
A functor $\alpha : I \to J$ is mapped to the functor $({}^{\downarrow\uparrow\downarrow}\alpha)^*$ which obviously preserves the (co)Cartesianity conditions. Natural morphisms are treated in the same way as in the plain case because no lax morphisms are involved.
We will now discuss the axioms:
(FDer0 right): It is clear from the definition that

$$\mathbb{E}'(I) \to \mathbb{S}^{\mathrm{cor}, G, \mathrm{lax}}(I)$$

is 2-opfibered and has 1-categorical fibers. It is also 1-fibered because we have

$$\begin{aligned}
& \mathrm{Hom}_{\mathbb{E}(I)}((\mathcal{E}_1, S_1), \dots, (\mathcal{E}_n, S_n), (\mathcal{F}, T)) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^* T^{\mathrm{op}}}}((\pi_{23}^* f)^\bullet (\pi_{23}^* g)_\bullet (\mathcal{E}_1, \dots, \mathcal{E}_n), \mathcal{F}) \\
\cong\ & \mathrm{Hom}_{\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^* S_j^{\mathrm{op}}}}(\mathcal{E}_j, \Box_* (\pi_{23}^* g)^{\bullet, j} (\mathcal{E}_1, \overset{\widehat{j}}{\dots}, \mathcal{E}_n; (\pi_{23}^* f)_? \mathcal{F})).
\end{aligned}$$

Here $\Box_*$ is the right coCartesian projection defined and discussed in Section 8 and $(\pi_{23}^* f)_?$ is a right adjoint of $(\pi_{23}^* f)^\bullet$, which exists by the reasoning in the first part of the proof. (Note that $(\pi_{23}^* f)_?$ would be denoted $f^!$, i.e. exceptional pull-back, in the usual language of six-functor-formalisms. Our notation, unfortunately, has reached its limit here.) Therefore Cartesian morphisms exist w.r.t. to any slot j with pull-back functor explicitly given by

$$\Box_* (\pi_{23}^* g)^{\bullet, j} (-, \overset{\widehat{j}}{\dots}, -; (\pi_{23}^* f)_? -).$$

The second part of (FDer0 right) follows from the corresponding statement for $\mathbb{D}$ and the fact that $\Box_*$ is "point-wise the identity" (cf. Proposition 8.5). The axioms (FDer3–4 right) do not involve lax morphisms. (FDer5 right) follows because the corresponding axiom holds for $\mathbb{D}$, because $(\pi_{23}^* f)_?$, as right adjoint, commutes with homotopy limits, and because $\Box_*$ is "point-wise the identity" (cf. Proposition 8.5). $\square$

8 Cocartesian projectors

8.1. We will show in this section that the fully-faithful inclusion

$$\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart},\pi_{12}-\mathrm{cocart}}_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$$

(cf. Definitions 7.16, 7.17) has a right adjoint $\Box_*$ which we will call a **right coCartesian projector** (cf. also [3, Section 2.4]).
A right coCartesian projector (or rather its composition with the fully-faithful inclusion) can be specified by an endofunctor $\Box_*$ of $\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart}}_{\pi_{23}^*(S^{\mathrm{op}})}$ together with a natural transformation

$$\nu : \Box_* \Rightarrow \mathrm{id}$$

such that

1. $\Box_*\mathcal{E}$ is π_{12}-coCartesian for all objects $\mathcal{E}$,
2. $\nu_{\mathcal{E}}$ is an isomorphism on π_{12}-coCartesian objects $\mathcal{E}$,
3. $\nu_{\Box_*\mathcal{E}} = \Box_*\nu_{\mathcal{E}}$ holds true.

This, in particular, gives a pullback functor

$$\Box_* f^\bullet : \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart},\pi_{12}-\mathrm{cocart}}_{\pi_{23}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)^{\pi_{13}-\mathrm{cart},\pi_{12}-\mathrm{cocart}}_{\pi_{23}^*(T^{\mathrm{op}})}$$

for *any* morphism (not necessarily type 2 admissible)

$$f : S \to T$$

of admissible diagrams in $\mathcal{S}({}^{\downarrow\uparrow}I)$.
Note that, of course, $f^\bullet$ preserves automatically the condition of being π_{13}-Cartesian. Proposition 8.5 below shows that this is still computed point-wise, i.e. that we have for any $\alpha : I \to J$

$$\alpha^* \Box_* f^\bullet \cong \Box_*(\alpha^* f)^\bullet.$$

8.2. We need some technical preparation. Consider the projections:

$$\pi_{123}, \pi_{125}, \pi_{145}\pi_{345} : {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \to {}^{\downarrow\uparrow\downarrow}I$$

We have obvious natural transformations

$$\pi_{123} \Rightarrow \pi_{125} \Leftarrow \pi_{145} \Rightarrow \pi_{345}$$

and therefore

$$\pi_{123}^* \Rightarrow \pi_{125}^* \Leftarrow \pi_{145}^* \Rightarrow \pi_{345}^*$$

If we plug in $\pi_{23}^*(S^{\mathrm{op}})$ for an admissible diagram $S \in \mathbb{S}({}^{\downarrow\uparrow}I)$, we get morphisms of diagrams in $\mathbb{S}^{\mathrm{op}}$:

$$\pi_{23}^*(S^{\mathrm{op}}) \xrightarrow{g} \pi_{25}^*(S^{\mathrm{op}}) \xleftarrow{f} \pi_{45}^*(S^{\mathrm{op}}) = \pi_{45}^*(S^{\mathrm{op}})$$

and therefore natural transformations

$$\begin{array}{ccc} g_\bullet \pi_{123}^* & \Rightarrow & \pi_{125}^* \\ f^\bullet \pi_{125}^* & \Leftarrow & \pi_{145}^* \end{array}$$

of functors between fibers.

Lemma 8.3. *π_{123} and π_{345} are opfibrations.*

Proof. This was explained in 7.5. $\square$

Lemma 8.4. *The natural transformation*

$$\pi_{345,!}^{(\pi_{45}^* S)} \pi_{145}^* \Rightarrow \mathrm{id}$$

induced by the natural transformation

$$\pi_{145}^* \Rightarrow \pi_{345}^*$$

of functors

$$\pi_{145}^*, \pi_{345}^* : \mathbb{D}({}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I)_{\pi_{45}^*(S^{\mathrm{op}})} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(S^{\mathrm{op}})}$$

is an isomorphism.

Proof. Since π_{345} is an opfibration, we have for any object $\alpha = \{i \to j \to k\}$ in ${}^{\downarrow\uparrow\downarrow}I$:

$$\alpha^* \pi_{345,!} \pi_{145}^* = p_! \pi_{145}^*$$

where $p : {}^{\downarrow\uparrow}(I \times_{/I} i) \to \{\cdot\}$. We can factor p in the following way:

$${}^{\downarrow\uparrow}(I \times_{/I} i) \xrightarrow{\pi_1} I \times_{/I} i \xrightarrow{P} \{\cdot\}$$

The functor π_1 is an opfibration with fibers of the form $\beta \times_{/(I \times_{/I} i)} (I \times_{/I} i)$. Since these fibers have an initial object, and the objects in the image of π_{145}^* are constant along it, the homotopy colimit over objects in the image of π_{145}^* along it are equal to this constant value by Corollary 7.22. Furthermore, the homotopy colimit over $I \times_{/I} i$ is the same as evaluation at id_i because id_i is the final object. $\square$

If $\mathcal{E}$ is an object in $\mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^*(S^{\mathrm{op}})}^{\pi_{13}-\mathrm{cart}}$ we have that the morphism

$$f^\bullet \pi_{125}^* \mathcal{E} \leftarrow \pi_{145}^* \mathcal{E}$$

is an isomorphism.

Proposition 8.5. *Using the notation of 8.2, denote $\square_* := \pi_{345;!} f^\bullet g_\bullet \pi_{123}^*$. This functor, together with the composition*

$$\begin{tikzcd} \mathcal{E} & \pi_{345;!}\pi_{145}^*\mathcal{E} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \pi_{345;!} f^\bullet \pi_{125}^*\mathcal{E} & \pi_{345;!} f^\bullet g_\bullet \pi_{123}^*\mathcal{E} = \square_*\mathcal{E} \arrow[l] \arrow[lll, bend left, "\nu_{\mathcal{E}}"] \end{tikzcd},$$

defines a right coCartesian projector:

$$\square_* : \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^* S}^{\pi_{13}-\mathrm{cart}} \to \mathbb{D}({}^{\downarrow\uparrow\downarrow}I)_{\pi_{23}^* S}^{\pi_{13}-\mathrm{cart}, \pi_{12}-\mathrm{cocart}}.$$

This projector has the following property:

- *For each $i \in I$ the natural transformation*

$$({}^{\downarrow\uparrow\downarrow}i)^* \square_* \to ({}^{\downarrow\uparrow\downarrow}i)^*$$

is an isomorphism. (Here i denotes, by abuse of notation, the subcategory of I consisting of i and id_i. Hence ${}^{\downarrow\uparrow\downarrow}i$ is the subcategory of ${}^{\downarrow\uparrow\downarrow}I$ consisting of $i = i = i$ and its identity.)

Proof. If suffices to show that

1. $\Box_* \mathcal{E}$ is π_{12}-coCartesian for all objects $\mathcal{E}$,
2. $\nu_{\mathcal{E}}$ is an isomorphism on π_{12}-coCartesian objects $\mathcal{E}$,
3. the equation $\Box_* \nu_{\mathcal{E}} = \nu_{\Box_* \mathcal{E}}$ holds true.

1. Since π_{345} is an opfibration, the evaluation of $\Box_* \mathcal{E}$ at an object $i \to j \to k$ of ${}^{\downarrow\uparrow\downarrow}I$ is equal to

$$p_{{}^{\downarrow\uparrow}(I \times_{/I} i),!} \iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$$

where $\iota_{i,j,k} : {}^{\downarrow\uparrow}(I \times_{/I} i) \hookrightarrow {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I$ is the inclusion

$$(l \to m \to i) \mapsto (l \to m \to i \to j \to k).$$

For any morphism μ in ${}^{\downarrow\uparrow\downarrow}I$ such that π_{12} (resp. π_{13}) maps it to an an identity we have to see that the morphism

$$\mu^*(\Box_* \mathcal{E})$$

in $\mathbb{D}(\cdot)$ is coCartesian (resp. Cartesian). In the first case, such a morphism μ is of the form

$$\begin{tikzcd} i \arrow[r] \arrow[d, equal] & j \arrow[r] \arrow[d, equal] & k \arrow[d] \\ i \arrow[r] & i \arrow[r] & k' \end{tikzcd}$$

and since homotopy colimits commute with push-forward it suffices to show that all morphisms

$$(S^{\mathrm{op}}(\mathrm{pr}_{23}(\mu)))_\bullet \iota_{i,j,k'}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E} \to \iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$$

are isomorphisms. This follows immediately from the base-change formula 1.5.
In the second case, such a morphism μ is of the form

$$\begin{tikzcd} i \arrow[r] \arrow[d, equal] & j' \arrow[r] & k \arrow[d, equal] \\ i \arrow[r] & j \arrow[r] \arrow[u] & k \end{tikzcd}$$

and since by assumption homotopy colimits commute with pull-backs as well, it suffices to show that all morphisms

$$(S^{\mathrm{op}}(\mathrm{pr}_{23}(\mu)))^\bullet \iota_{i,j',k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E} \to \iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$$

are isomorphisms which is obvious.
Assertion 2. follows from the fact that for a π_{13}-coCartesian and π_{12}-Cartesian diagram $\mathcal{E}$ the diagram

$$\iota_{i,j,k}^* f^\bullet g_\bullet \pi_{123}^* \mathcal{E}$$

is (co)Cartesian over the constant diagram $S(j \to k)$. Therefore (as in the proof of Lemma 7.23) its homotopy colimit is the same as evaluation at $(i = i) \in {}^{\downarrow\uparrow}(I \times_{/I} i)$ which is not affected by $f^\bullet g_\bullet$ and $(i = i)$ is mapped by $\pi_{123} \circ \iota_{i,j,k}$ to $i \to j \to k$.

We give a sketch of proof of assertion 3. for which we need a bit of preparation. Note that the following diagrams are Cartesian:

$$\begin{tikzcd} {}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{12345}"] \arrow[d, "\pi_{34567}"'] & {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[d, "\pi_{345}"] \\ {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{123}"] & {}^{\downarrow\uparrow\downarrow}I \end{tikzcd} \qquad \begin{tikzcd} {}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{12347}"] \arrow[d, "\pi_{34567}"'] & {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[d, "\pi_{345}"] \\ {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{125}"] & {}^{\downarrow\uparrow\downarrow}I \end{tikzcd} \qquad \begin{tikzcd} {}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{12367}"] \arrow[d, "\pi_{34567}"'] & {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[d, "\pi_{345}"] \\ {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I \arrow[r, "\pi_{145}"] & {}^{\downarrow\uparrow\downarrow}I \end{tikzcd}$$

Consider the following commutative diagram in $\mathbb{S}({}^{\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow}I)$ in which the square is Cartesian:

$$\begin{tikzcd} & & \pi_{67}^*(S^{\mathrm{op}}) \arrow[d, "f_2"] \arrow[dd, bend left=60, "f_3"] \\ & \pi_{45}^*(S^{\mathrm{op}}) \arrow[r, "g_4"] \arrow[d, "f_5"] & \pi_{47}^*(S^{\mathrm{op}}) \arrow[d, "f_1"] \\ \pi_{23}^*(S^{\mathrm{op}}) \arrow[r, "g_3"'] \arrow[rr, bend right=40, "g_2"'] & \pi_{25}^*(S^{\mathrm{op}}) \arrow[r, "g_5"'] & \pi_{27}^*(S^{\mathrm{op}}) \end{tikzcd}$$

We have an isomorphism

$$(\Box_*)^2 = \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \ \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \xleftarrow[\Theta]{\sim} \pi_{567,!}f_3^\bullet g_{2,\bullet}\pi_{123}^*$$

given as the following composition

$$\begin{aligned} \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \ \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \quad &\xleftarrow{\sim} \quad \pi_{345,!}f^\bullet g_\bullet \pi_{34567,!}\pi_{12345}^* f^\bullet g_\bullet \pi_{123}^* \\ &\xleftarrow{\sim} \quad \pi_{345,!}\pi_{34567,!}f_2^\bullet g_{4,\bullet} f_5^\bullet g_{3,\bullet}\pi_{12345}^*\pi_{123}^* \\ &\xleftarrow{\sim} \quad \pi_{567,!}f_3^\bullet g_{2,\bullet}\pi_{123}^* \end{aligned}$$

involving that $()_\bullet$ and $()^\bullet$ commute with both α^* and $\alpha_!$ for any functor α in Dia, and the base-change formula for f_1, f_5, g_4, g_5. For the first isomorphism note that π_{345} is an opfibration.
Similarly, we construct the other horizontal isomorphisms in the following diagram

$$\begin{tikzcd} \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \arrow[rr, equal] & & \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \\ \pi_{345;!}\pi_{145}^* \ \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \arrow[u, "\sim"'] \arrow[d, "\sim"'] & & \pi_{567,!}f_3^\bullet g_{2,\bullet}\pi_{123}^* \arrow[ll, "\sim"'] \arrow[u, "①"'] \arrow[d, equal] \\ \pi_{345;!}f^\bullet \pi_{125}^* \ \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* & & \pi_{567,!}f_3^\bullet g_{2,\bullet}\pi_{123}^* \arrow[ll, "\sim"'] \arrow[d, equal] \\ \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \ \pi_{345,!}f^\bullet g_\bullet \pi_{123}^* \arrow[u, "\sim"'] \arrow[uuu, bend left=60, "\nu * \Box_!"] & & \pi_{567,!}f_3^\bullet g_{2,\bullet}\pi_{123}^* \arrow[ll, "\sim", "\Theta"'] \end{tikzcd} \tag{15}$$

The morphism 1 is the following composition, in which the first morphism is induced by the counit of the pair of adjoint functors $\pi_{12567,!}$, π^*_{12567}:

$$\begin{aligned}
\pi_{345,!} f^\bullet g_\bullet \pi^*_{123} &\longleftarrow \pi_{345,!} f^\bullet g_\bullet \pi_{12567,!} \pi^*_{12567} \pi^*_{123} \\
&\xleftarrow{\sim} \pi_{567,!} \pi_{12567,!} f_3^\bullet g_{5,\bullet} \pi^*_{125} \\
&\longleftarrow \pi_{567,!} f_3^\bullet g_{5,\bullet} g_{3,\bullet} \pi^*_{123} \\
&\xleftarrow{\sim} \pi_{567,!} f_3^\bullet g_{2,\bullet} \pi^*_{123}.
\end{aligned}$$

One checks that the diagram (15) commutes.
There is an analogous commutative diagram

$$\begin{tikzcd}
\pi_{345,!} f^\bullet g_\bullet \pi^*_{123} \arrow[r, equal] & \pi_{345,!} f^\bullet g_\bullet \pi^*_{123} \\
\pi_{345,!} f^\bullet g_\bullet \pi^*_{123}\ \pi_{345;!} \pi^*_{145} \arrow[u, "\sim"'] \arrow[d, "\sim"] & \pi_{567,!} f_2^\bullet g_{4,\bullet} \pi^*_{145} \arrow[l, "\sim"'] \arrow[u, "2"] \arrow[d, "3"', "\sim"] \\
\pi_{345,!} f^\bullet g_\bullet \pi^*_{123}\ \pi_{345;!} f^\bullet \pi^*_{125} & \pi_{567,!} f_3^\bullet g_{5,\bullet} \pi^*_{125} \arrow[l, "\sim"'] \\
\pi_{345,!} f^\bullet g_\bullet \pi^*_{123}\ \pi_{345,!} f^\bullet g_\bullet \pi^*_{123} \arrow[u, "\sim"'] \arrow[uuu, bend left=60, "\square_! * \nu"] & \pi_{567,!} f_3^\bullet g_{2,\bullet} \pi^*_{123} \arrow[l, "\sim"', "\Theta"] \arrow[u, "4"', "\sim"]
\end{tikzcd} \tag{16}$$

in which the morphism 2 is constructed similarly using the counit of the pair of adjoint functors $\pi_{14567,!}$, π^*_{14567}, and 3 and 4 are constructed as in 8.2.
Furthermore, there is a morphism 5 constructed similarly using the counit of the pair of adjoint functors $\pi_{12567,!}$, π^*_{12567} again, making the diagram

$$\begin{tikzcd}
& \pi_{567,!} f_3^\bullet g_{5,\bullet} \pi^*_{125} \arrow[dd, "5"] & \\
\pi_{567,!} f_3^\bullet g_{2,\bullet} \pi^*_{123} \arrow[ur, "4"] \arrow[dr, "1"'] & & \pi_{567,!} f_2^\bullet g_{4,\bullet} \pi^*_{145} \arrow[ul, "3"'] \arrow[dl, "2"] \\
& \pi_{345,!} f^\bullet g_\bullet \pi^*_{123} &
\end{tikzcd} \tag{17}$$

commutative. This proves assertion 3.
For the additional property given in the statement of the Proposition observe that $\pi_{345,!}\mathcal{E}$ at an arrow $i \to i \to i$ is the homotopy colimit over the diagram $\iota^*\mathcal{E}$ for $\iota : {}^{\downarrow\uparrow}(I \times_{/I} i) \hookrightarrow {}^{\downarrow\uparrow\downarrow\uparrow\downarrow}I$ pulled back to $S(i = i)$. The projection $\mathrm{pr}_1 : {}^{\downarrow\uparrow}(I \times_{/I} i) \to (I \times_{/I} i)$ is an opfibration with fibers of the form $\beta \times_{/(I \times_{/I} i)} (I \times_{/I} i)$. These categories have an initial object and the restriction of the diagram $\pi^*_{123}\mathcal{E}$ is constant on it, because of the assumption that $\mathcal{E}$ is π_{13}-Cartesian already. Hence the homotopy colimit over the restriction of $\pi^*_{123}\mathcal{E}$ to these fibers is the corresponding constant value by Lemma 7.21. The colimit over $(I \times_{/I} i)$, furthermore, is evaluation at id_i because it is a final object. In total, the natural morphism

$$({}^{\downarrow\uparrow\downarrow}i)^* \,\square_*\, \mathcal{E} \to ({}^{\downarrow\uparrow\downarrow}i)^*\mathcal{E}$$

is an isomophism. $\square$

9 The (co)localization property and n-angels in the fibers of a stable proper or etale derivator six-functor-formalism

9.1. Let $\mathcal{S}$ be a category and $\mathcal{S}_0$ a class of "proper" morphisms. Let

$$\mathbb{D}' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}} \quad \text{resp.} \quad \mathbb{D}'' \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$$

be a proper derivator six-functor-formalism (cf. Definition 6.1) with stable fibers (cf. Definition 4.4). The multi-aspect will not play any role in this section. The reasoning in this section has an "etale" analogue that we leave to the reader to state.

9.2. If $\mathcal{S}$ is a category of some kind of spaces, we are often given a class of elementary squares as follows. Assume that in $\mathcal{S}$ there are certain distinguished morphisms called "closed immersions" or "open immersions" respectively, with an operation of taking complements. For a morphism $f : X \to Y$ in $\mathcal{S}$ we denote by f, resp. f^{op} the correspondences

$$f: \quad \begin{tikzcd} & S \arrow[dl, equal] \arrow[dr, "f"] & \\ S & & T \end{tikzcd} \qquad f^{\mathrm{op}}: \quad \begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, equal] & \\ T & & S \end{tikzcd}$$

in $\mathcal{S}^{\mathrm{cor}}$. Let

$$\begin{tikzcd} U \arrow[r, hook, "i"] & V \arrow[r, hook, "j"] & X \end{tikzcd}$$

be a sequence of "open embeddings". And let $\bar{i} : V \smallsetminus U \hookrightarrow V$, resp. $\overline{j \circ i} : X \smallsetminus U \hookrightarrow X$ be "closed embeddings of the complements". For now these morphisms can be arbitrary, but to make sense of these definitions in applications they should satisfy the properties of 9.3 below.
We then have the following square in $\Xi_{U,V,X} \in \mathbb{S}^{\mathrm{cor}}(\square)$:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "\bar{i}^{\mathrm{op}}"] & X \arrow[d, "\overline{j \circ i}^{\mathrm{op}}"] \\ V \smallsetminus U \arrow[r, "j"] & X \smallsetminus U \end{tikzcd}$$

Assume that the "closed embeddings" lie in the class $\mathcal{S}_0$ which was fixed to define the notion of proper derivator six-functor-formalism. Then the above square comes equipped with a morphism $\xi : \Xi_{U,V,X} \to p^* X$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}(\square)$ represented by the cube (as a morphism from the front face to the back face):

$$\begin{tikzcd} & X \arrow[rr, equal] \arrow[dd, equal] & & X \arrow[dd, equal] \\ V \arrow[ur, "j"] \arrow[rr, "j"] \arrow[dd, "\bar{i}^{\mathrm{op}}"'] & & X \arrow[ur, equal] \arrow[dd, "\overline{j \circ i}^{\mathrm{op}}"] & \\ & X \arrow[rr, equal] & & X \\ V \smallsetminus U \arrow[ur, "j \circ \bar{i}"] \arrow[rr, "j"] & & X \smallsetminus U \arrow[ur, "\overline{j \circ i}"'] & \end{tikzcd}$$

The top and bottom squares are 2-commutative, whereas the left and right squares are only oplax 2-commutative, e.g. there is a 2-morphism making the diagram

$$\begin{tikzcd} X \arrow[r, "\overline{j \circ i}^{\mathrm{op}}"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & X \smallsetminus U \arrow[d, "\overline{j \circ i}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutative, which is given by the morphism of correspondences

$$\begin{tikzcd} & X \smallsetminus U \arrow[dl, "j \circ i"'] \arrow[dr, "j \circ i"] \arrow[dd, "j \circ i"] & \\ X \arrow[dr, equal] & & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$

From now on, we forget about the provenance of these squares and just consider a proper derivator six-functor-formalism (more precisely, its oplax left fibered derivator)

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$$

with a class of distinguised squares $\Xi \in \mathbb{S}^{\mathrm{cor}}(\Box)$ with given morphisms $\xi : \Xi \to p^* X$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}(\Box)$.

Definition 9.3. *Let $\mathbb{S}$ be a 2-pre-derivator with all 2-morphisms invertible. We call a square $\Xi \in \mathbb{S}(\Box)$* **Cartesian***, if the natural functor*

$$\mathrm{Hom}(X, (0,0)^* \Xi) \to \mathrm{Hom}(p^* X, i_{\lrcorner}^* \Xi)$$

is an equivalence of groupoids for all $X \in \mathbb{S}(\cdot)$, and **coCartesian** *if the natural functor*

$$\mathrm{Hom}((1,1)^* \Xi, X) \to \mathrm{Hom}(i_{\ulcorner}^* \Xi, p^* X)$$

is an equivalence of groupoids for all $X \in \mathbb{S}(\cdot)$. We call a square $\Xi \in \mathbb{S}(\Box)$ **biCartesian** *if it is Cartesian and coCartesian.*

Remark 9.4. *If $\mathbb{S}$ is a usual derivator then this notion coincides with the usual notion [1].*

9.5. One can show that the squares $\Xi_{U,V,X} \in \mathbb{S}^{\mathrm{cor}}(\Box)$ constructed in the last paragraph are actually Cartesian in $\mathbb{S}^{\mathrm{cor}}$ provided that for all pairs $U, X \smallsetminus U$ of "open and closed embeddings" used above we have

$$\mathrm{Hom}_{\mathcal{S}}(A, U) = \{\alpha \in \mathrm{Hom}_{\mathcal{S}}(A, X) \mid A \times_{\alpha, X} (X \smallsetminus U) = \varnothing\}$$

and coCartesian provided that we have

$$\mathrm{Hom}_{\mathcal{S}}(A, X \smallsetminus U) = \{\alpha \in \mathrm{Hom}_{\mathcal{S}}(A, X) \mid A \times_{\alpha, X} U = \varnothing\}$$

where $\varnothing$ is the initial object.

9.6. There is a dual variant of the previous construction (not to be confused with the transition to an etale six-functor-formalism). We consider instead the square $\Xi'_{U,V,X}$ with morphism

$$\begin{tikzcd}
& X \arrow[rr, equal] \arrow[dl, "\overline{j\circ i}^{\mathrm{op}}"'] \arrow[dd, equal] & & X \arrow[dl, equal] \arrow[dd, equal] \\
X \setminus U \arrow[rr, "\overline{j\circ i}"] \arrow[dd, "i^{\mathrm{op}}"'] & & X \arrow[dd, "j^{\mathrm{op}}"] & \\
& X \arrow[rr, equal] \arrow[dl, "(j\circ \bar{i})^{\mathrm{op}}"] & & X \arrow[dl, "j^{\mathrm{op}}"] \\
V \setminus U \arrow[rr, "\bar{i}"'] & & V &
\end{tikzcd}$$

In this case the top and bottom squares are only *lax* 2-commutative, e.g. there is a 2-morphism making the diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\overline{j\circ i}^{\mathrm{op}}"'] \arrow[dr, phantom, "\Nearrow"] & X \arrow[d, equal] \\
X \setminus U \arrow[r, "\overline{j\circ i}"] & X
\end{tikzcd}$$

2-commutative. This means that for a stable proper derivator six-functor-formalism it is also reasonable to consider a class of distinguished squares with given morphisms $\xi' : p^*X \to \Xi$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}(\Box)$. The morphism $p^*X \to \Xi'_{U,V,X}$ is just the *dual* of the morphism $\Xi_{U,V,X} \to p^*X$ for the absolute duality on $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{S}^{cor})$ (cf. 6.2).

Let $i_\ulcorner : \ulcorner \hookrightarrow \Box$ and $i_\lrcorner : \lrcorner \hookrightarrow \Box$ be the inclusions. Analogously to the situation for stable derivators [1, 4.1] we define:

Definition 9.7. *A square $\mathcal{E} \in \mathbb{D}(\Box)$ over $\Xi \in \mathbb{S}(\Box)$ is called* **relatively coCartesian**, *if for the inclusion $i_\ulcorner : (\ulcorner, i_\ulcorner^*\Xi) \to (\Box, \Xi)$ the unit $\mathcal{E} \to i_{\ulcorner,*} i_\ulcorner^* \mathcal{E}$ is an isomorphism, and it is called* **relatively Cartesian** *if for the inclusion $i_\lrcorner : (\lrcorner, i_\lrcorner^*\Xi) \to (\Box, \Xi)$ the counit $i_{\lrcorner,!} i_\lrcorner^* \mathcal{E} \to \mathcal{E}$ is an isomorphism*[13]*. $\mathcal{E}$ is called* **relatively biCartesian** *if it is relatively Cartesian and relatively coCartesian.*

If Ξ is itself (co)Cartesian in the sense of Definition 9.3 then $\mathcal{E}$ relatively (co)Cartesian implies (co)Cartesian in the sense of Definition 9.3.

Lemma 9.8. *Assume $V = U$ and let $\mathbb{D}(\Box)^{\mathrm{bicart}}_{\Xi_{U,U,X}}$ be the full subcategory of relatively biCartesian squares. Let $(1,0) : (\cdot, X) \to (\Box, \Xi_{U,U,X})$ be the inclusion. Then the functor*

$$(1,0)^* : \mathbb{D}(\Box)^{\mathrm{bicart}}_{\Xi_{U,U,X}} \to \mathbb{D}(\cdot)_X$$

and the composition

$$\mathbb{D}(\cdot)_X \xrightarrow{1_*} \mathbb{D}(\to)_{U \to X} \xrightarrow{0_!} \mathbb{D}(\Box)^{\mathrm{bicart}}_{\Xi_{U,U,X}}$$

define an equivalence of categories.
Also if $V \neq U$ the functor $1_* 0_!$ takes values in relatively biCartesian squares.

[13] The functors $i_{\lrcorner,!}$ and $i_{\ulcorner,*}$ are in both cases considered w.r.t. the base Ξ.

Recall that the functor

$$0^* : \mathbb{D}(\square)^{\mathrm{bicart},0}_{p^*X} \to \mathbb{D}(\to)_{p^*X}$$

is an equivalence, where $\mathbb{D}(\square)^{\mathrm{bicart},0}_{p^*X}$ is the full subcategory of (relatively) biCartesian objects whose $(1,0)$-entry is zero. (This is a statement about usual derivators.)

Definition 9.9. *We say that a distinguished square Ξ together with $\xi : \Xi \to p^*X$ is a* **localizing square** *if the push-forward $\xi_\bullet$ maps relatively biCartesian squares to relatively biCartesian squares. We say that a distinguished square Ξ together with $\xi : p^*X \to \Xi$ is a* **colocalizing square** *if the pull-back $\xi^\bullet$ maps relatively biCartesian squares to relatively biCartesian squares.*

If every object in $\mathbb{D}(\cdot)$ is dualizable w.r.t. the absolute monoidal product in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ then $\xi : \Xi \to p^*X$ is localizing if and only if $\xi^\vee : p^*X \to \Xi^\vee$ is colocalizing.

Remark 9.10. *If the proper derivator six-functor-formalism with its oplax extension*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$$

has stable *fibers, and the square $\Xi_{U,V,X}$ constructed above is distinguished, then the property of being a localizing square implies that for $\mathcal{E} \in \mathbb{D}(\cdot)_X$ the triangle*

$$j_! j^! \mathcal{E} \longrightarrow (j \circ \overline{i})_! \overline{i}^* j^! \mathcal{E} \oplus \mathcal{E} \longrightarrow \overline{j \circ i}_* \overline{j \circ i}^* \mathcal{E} \xrightarrow{[1]}$$

is distinguished. If $U = V$ this is just the sequence

$$j_! j^! \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \overline{j}_* \overline{j}^* \mathcal{E} \xrightarrow{[1]}$$

Remark 9.11. *If the proper derivator six-functor-formalism with its lax extension*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$$

has stable *fibers, and the square $\Xi_{U,V,X}$ constructed above is distinguished, then the property of being a colocalizing square implies that for $\mathcal{E} \in \mathbb{D}(\cdot)_X$ the triangle*

$$\overline{j \circ i}_! \overline{j \circ i}^! \mathcal{E} \longrightarrow (j \circ \overline{i})_* i^* (\overline{j \circ i})^! \mathcal{E} \oplus \mathcal{E} \longrightarrow j_* j^* \mathcal{E} \xrightarrow{[1]}$$

is distinguished. If $U = V$ this is just the sequence:

$$\overline{j}_! \overline{j}^! \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow j_* j^* \mathcal{E} \xrightarrow{[1]}$$

Definition 9.12. *We say that the proper derivator six-functor-formalism with its extension as oplax left fibered derivator*

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$$

satisfies **the localization property** *w.r.t. a class of distinguished squares $\xi : \Xi \to p^*X$ if these are localizing squares.*
We say that the proper derivator six-functor-formalism with its extension as lax right fibered derivator

$$\mathbb{D} \to \mathbb{S}^{\mathrm{cor},0,\mathrm{lax}}$$

satisfies **the colocalization property** *w.r.t. a class of distinguished squares $\xi : p^*X \to \Xi$ if these are colocalizing squares.*

There is an analogous notion in which an *etale* derivator-six-functor-formalism w.r.t. a class of "etale morphisms" $\mathcal{S}_0$ in $\mathcal{S}$ satisfies the (co)localization property

9.13. Consider again the situation in 9.2. More generally we may consider a sequence

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n$$

of open embeddings. They lead to a diagram Ξ

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[d] \\
\varnothing \arrow[r] & X_2 \setminus X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \setminus X_1 \arrow[d] \\
 & \varnothing \arrow[r] & \ddots \arrow[d] & \vdots \arrow[d] \\
 & & \varnothing \arrow[r] & X_n \setminus X_{n-1}
\end{tikzcd}$$

in which all squares are biCartesian in $\mathbb{S}^{\mathrm{cor}}$. Starting from an object $\mathcal{E} \in \mathbb{D}(\cdot)_{X_n}$ we may form again

$$0_*(n)_!\mathcal{E}$$

where $(n) : \cdot \to [n]$ is the inclusion of the last object and $0 : [n] \to \Xi$ is the inclusion of the first line. It is easy to see that in the object $0_*(n)_!\mathcal{E}$ all squares are biCartesian. There is furthermore again a morphism $\xi : \Xi \to p^*X_n$ in $\mathbb{S}^{\mathrm{cor},0,\mathrm{oplax}}$ such that all squares in $\xi_\bullet 0_*(n)_!\mathcal{E}$ are biCartesian with zero's along the diagonal. This category is equivalent to $\mathbb{D}([n])_{p^*X_n}$ by the embedding of the first line. It can be seen as a category of n-angels in the stable derivator $\mathbb{D}_{X_n}$ (the fiber of $\mathbb{D}$ over X).
Hence for an oplax derivator six-functor-formalism with localization property, and for any filtration of a space X by n open subspaces, and for any object $\mathcal{E} \in \mathbb{D}(\cdot)_X$ we get a corresponding $(n+1)$-angle in the derivator $\mathbb{D}_X$ in the sense of [2, §13].

A Representable 2-pre-multiderivators

Proof (sketch) of Proposition 2.6: We show exemplarily 1. for 1-fibrations of 2-categories and 2. for 1-fibrations of 2-categories. The same proof works for 1-opfibrations (even of 2-multicategories). If we have a 1-bifibration of 2-multicategories *with 1-categorical fibers* then a slight extension of the proof of [3, Proposition 4.1.6] shows 3. For 1-opfibrations and 2-fibrations of 2-multi-categories with 1-categorical fibers this is actually easier to prove using the encoding by a pseudo-functor as follows: The 1-opfibration $\mathcal{D} \to \mathcal{S}$ with 1-categorical fibers is encoded in a pseudo-functor

$$\mathcal{S} \to \mathcal{MCAT}$$

The category $\mathbb{D}(I) \to \mathbb{S}(I)$ is encoded in the pseudo-functor

$$\mathbb{S}(I) \to \mathcal{MCAT}$$

which maps a pseudo-functor $F : I \to \mathcal{S}$ to the multicategory of natural transformations and modifications

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{MCAT})}(\cdot, F),$$

where $\cdot$ is the constant functor with value the 1-point category.
This construction may be adapted to 2-categorical fibers by using "pseudo-functors" of 3-categories. The problem with *1-fibrations* of (1- or 2-)*multi*categories comes from the fact that the internal Hom cannot be computed point-wise but involves a limit construction (cf. [3, Proposition 4.1.6]). The difference between external and internal monoidal product in $\mathrm{Dia}^{\mathrm{cor}}(\mathbb{D})$ gives a theoretical explanation of this phenomenon (cf. [4, Example 7.5]).
1-fibration of 2-categories $\mathcal{D} \to \mathcal{S}$ implies 1-fibration of 2-categories $\mathbb{D}(I) \to \mathbb{S}(I)$:
Let I be a diagram in Dia. Let $Y, Z : I \to \mathcal{S}$ be pseudo-functors, $f : Y \Rightarrow Z$ be a pseudo-natural transformation and $\mathcal{E} : I \to \mathcal{D}$ be a pseudo-functor over Z. For each morphism $\alpha : i \to i'$ we are given a 2-commutative diagram

$$\begin{tikzcd} Y(i) \arrow[r, "f(i)"] \arrow[d, "Y(\alpha)"'] \arrow[dr, phantom, "\nearrow f_\alpha"] & Z(i) \arrow[d, "Z(\alpha)"] \\ Y(i') \arrow[r, "f(i')"] & Z(i') \end{tikzcd}$$

Since f is assumed to be a pseudo-functor, the morphism f_α is invertible. We will construct a pseudo-functor $\mathcal{G} : I \to \mathcal{D}$ over Y and a 1-coCartesian morphism $\xi : \mathcal{G} \to \mathcal{E}$ over f. For each i, we choose a 1-coCartesian morphism

$$\xi(i) : \mathcal{G}(i) \to \mathcal{E}(i)$$

over $f(i) : Y(i) \to Z(i)$. For each α, we look at the 2-Cartesian diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i')) \arrow[r, "\xi(i')\circ"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i')) \arrow[d] \\ \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i')) \arrow[r, "f(i')\circ"] & \mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i')) \end{tikzcd}$$

The triple $(\mathcal{E}_\alpha \circ \xi(i), f_\alpha, Y_\alpha)$ is an object in the category

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i')) \times^{\sim}_{/\mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i'))} \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i'))$$

Define $\mathcal{G}(\alpha)$ to be an object in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i'))$ such that there exists a 2-isomorphism

$$\Xi_\alpha : (\xi(i') \circ \mathcal{G}_\alpha, \mathrm{id}, p(\mathcal{G}_\alpha)) \Rightarrow (\mathcal{E}_\alpha \circ \xi(i), f_\alpha^{-1}, Y_\alpha).$$

Such an object exists because the above square is 2-Cartesian.
We get a 2-commutative square

$$\begin{tikzcd} \mathcal{G}(i) \arrow[r, "\xi(i)"] \arrow[d, "\mathcal{G}_\alpha"'] \arrow[dr, phantom, "\nearrow \xi_\alpha"] & \mathcal{E}(i) \arrow[d, "\mathcal{E}_\alpha"] \\ \mathcal{G}(i') \arrow[r, "\xi(i')"] & \mathcal{E}(i') \end{tikzcd}$$

Here ξ_α is the first component of Ξ_α.
This defines a pseudo-functor $\mathcal{G} : I \to \mathcal{D}$ as follows. Let $\alpha : i \to i'$ and $\beta : i' \to i''$ be two morphisms in I. We need to define a 2-isomorphism $G_{\beta\alpha} \Rightarrow G_\beta \circ G_\alpha$. It suffices to define the 2-isomorphism after applying the embedding

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i'')) \hookrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i'')) \times^{\sim}_{/\mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i''))} \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i''))$$

which maps $G_\beta \circ G_\alpha$ to

$$(\xi(i'') \circ \mathcal{G}_\beta \circ \mathcal{G}_\alpha, \mathrm{id}, p(\mathcal{G}_\beta) \circ p(\mathcal{G}_\alpha))$$

and $G_{\beta\alpha}$ to

$$(\xi(i'') \circ \mathcal{G}_{\beta\alpha}, \mathrm{id}, p(\mathcal{G}_{\beta\alpha})).$$

We have the chains of 2-isomorphisms

$$\begin{tikzcd}
\xi(i'') \circ \mathcal{G}_\beta \circ \mathcal{G}_\alpha \arrow[d, "\xi_\beta * \mathcal{G}_\alpha"] & p(\mathcal{G}_\beta) \circ p(\mathcal{G}_\alpha) \arrow[d, "\Xi_{\beta,2} * p(\mathcal{G}_\alpha)"] \\
\mathcal{E}_\beta \circ \xi(i') \circ \mathcal{G}_\alpha \arrow[d, "\mathcal{E}_\beta * \xi_\alpha"] & Y_\beta \circ p(\mathcal{G}_\alpha) \arrow[d, "Y_\beta * \Xi_{\alpha,2}"] \\
\mathcal{E}_\beta \circ \mathcal{E}_\alpha \circ \xi(i) \arrow[d] & Y_\beta \circ Y_\alpha \arrow[d] \\
\mathcal{E}_{\beta\alpha} \circ \xi(i) & Y_{\beta\alpha} \\
\xi(i'') \circ \mathcal{G}_{\beta\alpha} \arrow[u] & p(\mathcal{G}_{\beta\alpha}) \arrow[u]
\end{tikzcd}$$

Applying p to the first chain and $f(i'')\circ$ to the second chain, we get the commutative diagram

$$\begin{tikzcd}
 & f(i'') \circ p(\mathcal{G}_\beta) \circ p(\mathcal{G}_\alpha) \arrow[r, equal] \arrow[d] & f(i'') \circ p(\mathcal{G}_\beta) \circ p(\mathcal{G}_\alpha) \arrow[d] \\
 & Z_\beta \circ f(i') \circ p(\mathcal{G}_\alpha) \arrow[r] \arrow[d] \arrow[dl] & f(i'') \circ Y_\beta \circ p(\mathcal{G}_\alpha) \arrow[d] \\
Z_\beta \circ Z_\alpha \circ f(i) \arrow[r] \arrow[d] & Z_\beta \circ f(i') \circ Y_\alpha \arrow[r] & f(i'') \circ Y_\beta \circ Y_\alpha \arrow[d] \\
Z_{\beta\alpha} \circ f(i) \arrow[rr, "f_{\beta\alpha}"] & & f(i'') \circ Y_{\beta\alpha} \\
f(i'') \circ p(\mathcal{G}_{\beta\alpha}) \arrow[u] \arrow[rr, equal] & & f(i'') \circ p(\mathcal{G}_{\beta\alpha}) \arrow[u]
\end{tikzcd}$$

hence a valid isomorphism in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i'')) \tilde{\times}_{/\mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i''))} \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i''))$. One checks that this satisfies the axioms of a pseudo-functor [4, Definition 1.3] and that ξ is indeed a pseudo-natural transformation ([4, Definition 1.4]).

Now assume that we have a lax natural transfomation, i.e. the f_α go into the opposite direction and are no longer invertible. We assume that we have a 2-fibration as well. Then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i')) \arrow[r, "\xi(i')\circ"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i')) \arrow[d] \\
\mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i')) \arrow[r, "f(i')\circ"] & \mathrm{Hom}_{\mathcal{S}}(Y(i), Z(i'))
\end{tikzcd}$$

is Cartesian as well. Moreover we have an adjunction with the full comma category

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{G}(i')) \underset{\mathrm{can}}{\overset{\rho_{\xi(i'),\mathcal{G}(i)}}{\leftrightarrows}} \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(i), \mathcal{E}(i')) \times_{/\,\mathrm{Hom}_{\mathcal{S}}(Y(i),Z(i'))} \mathrm{Hom}_{\mathcal{S}}(Y(i), Y(i'))$$

with $\rho \circ \mathrm{can} = \mathrm{id}$, in particular with the morphism 'can' fully faithful. See below for the precise definition of ρ. Hence we define

$$\mathcal{G}(\alpha) := \rho_{\xi(i'),\mathcal{G}(i)}(\mathcal{E}_\alpha \circ \xi(i), f_\alpha, Y_\alpha)$$

and get at least a morphism (coming from the unit of the adjunction):

$$\Xi_\alpha : (\xi(i') \circ \mathcal{G}(\alpha), \mathrm{id}, p(\mathcal{G}(\alpha))) \Rightarrow (\mathcal{E}_\alpha \circ \xi(i), f_\alpha, Y_\alpha).$$

The first component of Ξ_α this time (potentially) define a lax-natural transformation $\xi : \mathcal{G} \to \mathcal{E}$ only. To turn $\mathcal{G}$ into a pseudo-functor, we have to see that ρ is functorial.
For a Cartesian arrow $\xi : \mathcal{E} \to \mathcal{F}$, we define

$$\rho_{\xi,\mathcal{G}} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}, \mathcal{F}) \times_{/\,\mathrm{Hom}_{\mathcal{S}}(U,T)} \mathrm{Hom}_{\mathcal{S}}(U, S) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}, \mathcal{E})$$

as follows: Let (τ, μ, g) be a tuple with $g \in \mathrm{Hom}_{\mathcal{S}}(U, S)$, $\tau \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}, \mathcal{F})$ and

$$\mu : f \circ g \Rightarrow p(\tau)$$

a 2-morphism. We may choose a coCartesian 2-morphism

$$\widetilde{\mu} : X \Rightarrow \tau$$

above μ. We set $\rho_\xi(\tau, \mu, g)$ equal to an object with an isomorphism

$$(\xi \circ \rho_\xi(\tau, \mu, g), \mathrm{id}, p(\rho_\xi(\tau, \mu, g))) \xrightarrow{\sim} (X, \mathrm{id}, g).$$

Together with the morphism

$$\widetilde{\mu} : (X, \mathrm{id}, g) \longrightarrow (\tau, \mu, g)$$

we get the counit

$$\mathrm{can} \circ \rho_\xi \Rightarrow \mathrm{id}\,.$$

We need to define a 2-isomorphism $G_{\beta\alpha} \Rightarrow G_\beta \circ G_\alpha$. i.e.

$$\rho_{\xi(i''),\mathcal{G}(i)}(\mathcal{E}_{\beta\alpha} \circ \xi(i), f_{\beta\alpha}, Y_{\beta\alpha}) \to$$
$$\rho_{\xi(i''),\mathcal{G}(i')}(\mathcal{E}_\beta \circ \xi(i'), f_\beta, Y_\beta) \circ \rho_{\xi(i'),\mathcal{G}(i)}(\mathcal{E}_\alpha \circ \xi(i), f_\alpha, Y_\alpha)$$

First of all, we get three Cartesian 2-morphisms

$$\begin{array}{rlcll} \widetilde{f_{\beta\alpha}} : & X_{\beta\alpha} & \Rightarrow & \mathcal{E}_{\beta\alpha} \circ \xi(i) & \text{over } f_{\beta\alpha} \\ \widetilde{f_\alpha} : & X_\alpha & \Rightarrow & \mathcal{E}_\alpha \circ \xi(i) & \text{over } f_\alpha \\ \widetilde{f_\beta} : & X_\beta & \Rightarrow & \mathcal{E}_\beta \circ \xi(i') & \text{over } f_\beta \end{array}$$

and have to define an isomorphism (after applying can)

$$(X_{\beta\alpha}, \mathrm{id}, Y(\beta\alpha)) \xrightarrow{\sim} (X_\beta, \mathrm{id}, Y(\beta)) \circ (X_\alpha, \mathrm{id}, Y(\alpha)).$$

We have the diagram

$$
\begin{tikzcd}
\mathcal{G}(i) \arrow[r, "f(i)"] \arrow[d, "\mathcal{G}(\alpha)"'] \arrow[dr, "X_\alpha"'] \arrow[r, phantom, shift right=4ex, "\Nearrow \mu_\alpha"'] \arrow[d, phantom, shift left=4ex, "\Nearrow \sim"] & \mathcal{E}(i) \arrow[d, "\mathcal{E}(\alpha)"] \arrow[dd, bend left=60, "\mathcal{E}(\beta\alpha)"] \\
\mathcal{G}(i') \arrow[r, "f(i')"] \arrow[d, "\mathcal{G}(\beta)"'] \arrow[dr, "X_\beta"'] \arrow[r, phantom, shift right=4ex, "\Nearrow \mu_\beta"'] \arrow[d, phantom, shift left=4ex, "\Nearrow \sim"] & \mathcal{E}(i') \arrow[d, "\mathcal{E}(\beta)"] \arrow[r, phantom, "\Nearrow \sim"] & {} \\
\mathcal{G}(i'') \arrow[r, "f(i'')"] & \mathcal{E}(i'')
\end{tikzcd}
\tag{18}
$$

and the diagram

$$
\begin{tikzcd}
\mathcal{G}(i) \arrow[r, "f(i)"] \arrow[d, "\mathcal{G}(\beta\alpha)"'] \arrow[dr, "X_{\beta\alpha}"'] \arrow[r, phantom, shift right=4ex, "\Nearrow \mu_{\beta\alpha}"'] \arrow[d, phantom, shift left=4ex, "\Nearrow \sim"] & \mathcal{E}(i) \arrow[d, "\mathcal{E}(\beta\alpha)"] \\
\mathcal{G}(i'') \arrow[r, "f(i'')"] & \mathcal{E}(i'')
\end{tikzcd}
\tag{19}
$$

The two pastings are both Cartesian (using Lemma A.1 below) over the pastings in the diagram

$$
\begin{tikzcd}
Y(i) \arrow[r, "f(i)"] \arrow[d, "Y(\alpha)"'] \arrow[dr, phantom, "\Nearrow f_\alpha"] & Z(i) \arrow[d, "Z(\alpha)"] \arrow[dd, bend left=60, "Z(\beta\alpha)"] \\
Y(i') \arrow[r, "f(i')"] \arrow[d, "Y(\beta)"'] \arrow[dr, phantom, "\Nearrow f_\beta"] & Z(i') \arrow[r, phantom, "\Nearrow Z_{\beta,\alpha}"] & {} \\
Y(i'') \arrow[r, "f(i'')"] & Z(i'')
\end{tikzcd}
\quad \text{resp.} \quad
\begin{tikzcd}
& Y(i) \arrow[r, "f(i)"] \arrow[d, "Y(\alpha)"] \arrow[dd, bend right=60, "Y(\beta\alpha)"'] \arrow[dr, phantom, "\Nearrow f_\alpha"] & Z(i) \arrow[d, "Z(\alpha)"] \arrow[dd, bend left=60, "Z(\beta\alpha)"] \\
{} \arrow[r, phantom, "\Nearrow Y_{\beta,\alpha}"] & Y(i') \arrow[r, "f(i')"] \arrow[d, "Y(\beta)"] \arrow[dr, phantom, "\Nearrow f_\beta"] & Z(i') \arrow[r, phantom, "\Nearrow Z_{\beta,\alpha}"] & {} \\
& Y(i'') \arrow[r, "f(i'')"] & Z(i'')
\end{tikzcd}
$$

The pasting in the second diagram is just $f_{\beta,\alpha}$ by definition of lax natural transformation for f. This yields an isomorphism between the pastings in diagram (18) and (19) *over* $Y_{\beta,\alpha}$ which we define to be $\mathcal{G}_{\beta,\alpha}$. One checks that this defines indeed a pseudo-functor $\mathcal{G}$ such that $\xi : \mathcal{G} \to \mathcal{E}$ is a lax natural transformation which is 1-Cartesian. □

Lemma A.1. *Let $\mathcal{D} \to \mathcal{S}$ be a 2-(op)fibration of 2-categories. Let $\mu : \alpha \Rightarrow \beta$ be a 2-(co)Cartesian morphism, where $\alpha, \beta : \mathcal{E} \to \mathcal{F}$ are 1-morphisms. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a 1-morphism then $\gamma * \mu$ is 2-(co)Cartesian. Similarly, if $\gamma' : \mathcal{G} \to \mathcal{E}$ is a 1-morphism then $\mu * \gamma'$ is 2-(co)Cartesian.*

Proof. This follows immediately from the axiom that composition is a morphism of (op-)fibrations. □

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