Galois groups of first order theories.

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18 October 2000*

Abstract

We study the groups $Gal_L(T)$ and $Gal_{KP}(T)$, and the associated equivalence relations E_L and E_{KP}, attached to a first order theory T. An example is given where $E_L \neq E_{KP}$ (a non G-compact theory). It is proved that E_{KP} is the composition of E_L and the closure of E_L. Other examples are given showing this is best possible.

1 Introduction and preliminaries.

Let T be a complete first order theory, possibly many-sorted. We will be studying a certain group, $Gal_L(T)$, the Lascar group of T. If T is the theory of algebraically closed fields of characteristic 0, this will be the absolute Galois group of $\mathbf{Q}$ (a profinite group). For "G-compact" theories, $Gal_L(T)$

*revised 8-8-2001

[1]The first author was partially supported by a grant from the Spanish Ministry of Education. The third author was partially supported by NSF grants DMS-9696268 and DMS-0070179. Some of the work reported here was carried out during visits of the second author to Urbana and of the third author to Paris which were supported by a CNRS-UIUC collaboration fund.

has naturally the structure of a compact (Hausdorff) topological group. In general $Gal_L(T)$ will be more of a "descriptive set-theoretic" invariant of T.

Let us begin with an informal description of the relevant groups and equivalence relations. Let $\bar{M}$ be a very saturated model of T. "Small" or "bounded" means of strictly smaller cardinality than that of $\bar{M}$. An equivalence relation will be called bounded (finite) if it has a bounded (finite) number of classes. Type-definable over $\emptyset$ (or $\emptyset$-type-definable) means defined by a possibly infinite set (conjunction) of L-formulas. Let S be any sort. E_L^S is the finest bounded invariant (under $Aut(\bar{M})$) equivalence relation on S. E_{KP}^S is the finest bounded type definable over $\emptyset$ equivalence relation on S. E_{Sh}^S is the intersection of all finite $\emptyset$-definable equivalence relations on S. E_L^S refines E_{KP}^S which in turn refines E_{Sh}^S. These equivalence relations have explicit syntactic descriptions which do not depend on the choice of $\bar{M}$. For each of these equivalence relations E, S/E denotes the quotient space, on which clearly $Aut(\bar{M})$ acts. We obtain the corresponding "Galois groups", Gal_L^S, Gal_{KP}^S and Gal_{Sh}^S. All this can be done with some $\emptyset$-type-definable set X of possibly infinite tuples in place of the sort S. Roughly speaking, taking the projective limit of these groups as X varies, yields groups $Gal_L(T)$, $Gal_{KP}(T)$ and Gal_{Sh}^T which are invariants of the bi-interpretability type of T. Precise statements and definitions will be given below, but for now let us say that these Galois groups come equipped with "additional structure": $Gal_{Sh}(T)$ is a profinite group and $Gal_{KP}(T)$ is a compact group (and in fact $Gal_{Sh}(T)$ is the maximal profinite quotient of $Gal_{KP}(T)$). $Gal_L(T)$ can be described as a "quasicompact" group (that is compact but not necessarily Hausdorff). However possibly more interesting is that $Gal_L(T)$ arises naturally as the quotient of a certain "space of types" by a certain equivalence relation which is a countable union of closed sets. As such $Gal_L(T)$ is a kind of "descriptive set-theoretic" invariant of T. In many cases (such as when T is stable) all these equivalence relations and Galois groups coincide.

$Gal_L(T)$, the Lascar group, was introduced by the second author in [6]. He also introduced the notion of a G-compact theory and remarked that all known theories were G-compact. Essentially G-compactness of T means that $Gal_L(T) = Gal_{KP}(T)$.

Additional interest was generated by the work on simple theories [5], where Lascar strong types (E_L-classes) took the place of strong types. Kim [4] subsequently showed that simple theories are G-compact.

The second author, in [6], defined a topology on $Gal_L(T)$ in the case where T is G-compact, making $Gal_L(T)$ into a compact (Hausdorff) topological group. In [2], Hrushovski gave another account of the topology, working directly with Gal_{KP} (whether T is G-compact or not). In fact in that paper the E_{KP} notation was introduced. Similar things were done in [7]. The main point was that the spaces S/E_{KP} or even X/E_{KP} are naturally equipped with compact Hausdorff topologies (the closed sets being precisely the type-definable sets). There has been considerable attention paid to the issue of proving that $E_{KP} = E_{Sh}$ in certain situations. For example in [1] this is proved for supersimple theories. The simple case is still open although Hrushovski [2] found a counterexample in the more general (non first order) context of Robinson theories.

The current paper is concerned with the issue of when and how E_L differs from E_{KP}, in particular the existence of non G-compact theories. The starting point for our work was the discovery by the fourth author of such a theory (non G-compact). This example together with two other related examples, appears in section 4. In section 2 we prove that E_{KP} is the composition of E_L and $\bar{E}_L$. (Here, working on a sort S say, $\bar{E}_L$ denotes the closure of E_L in the Stone space of complete types in $S \times S$.) This is done by characterizing closure in the quasicompact group $Gal_L(T)$. In section 3, we look at *products* and *co-products* of structures and study the resulting Galois groups. Informed by this analysis, we present the examples in section 4. The "basic example" (a product of circles with specified structures) has E_{KP} different from E_L. A modification gives an example where E_{KP} is different from $\bar{E}_L$ (showing that the results in section 2 are best possible). In a third example we show that on E_L, closure (in the Stone space on $S \times S$) need not commute with restriction to a complete type $p(x)$.
We also ask some questions, which possibly need new kinds of examples to settle.

In the rest of this section we repeat some definitions, fix notation, summarise earlier relevant results, and give some additional information. As above S denotes a sort in $\bar{M}$.

Definition 1.1 *(i) E_L^S is the finest bounded invariant equivalence relation on S.*
(ii) E_{KP}^S is the finest bounded $\emptyset$-type-definable equivalence relation on S.
(iii) E_{Sh}^S is the intersection of all finite $\emptyset$-definable equivalence relations on S.
(iv) Let X be a $\emptyset$-type-definable set of possibly infinite tuples (for example X could be a product of infinitely many sorts). E_L^X, E_{KP}^X, E_{Sh}^X are defined as

above. For example E^X_{KP} is the finest bounded type-definable over $\emptyset$ equivalence relation on X.

Remark 1.2 *(i) E^S_L refines E^S_{KP} which in turn refines E^S_{Sh}. In stable theories they are all equal. We will not say much about E_{Sh}.*
(ii) Define a subset of X/E^X_{KP} to be closed if its preimage in X is type-definable (maybe with parameters). Then X/E^X_{KP} is a compact Hausdorff topological space. Similarly for X/E^X_{Sh} except that this space is now profinite.

Definition 1.3 *(i) $Autf_L(\bar{M})$, the group of Lascar strong automorphisms, is the subgroup of $Aut(\bar{M})$ consisting of f which fix each class of each E^X_L (X $\emptyset$-type-definable). Similarly for $Autf_{KP}(\bar{M})$ and $Autf_{Sh}(\bar{M})$.*
(iii) $Gal_L(T)$, the Lascar group of T is the quotient of $Aut(\bar{M})$ by the (normal) subgroup $Autf_L(\bar{M})$. By definition it acts faithfully on $\{a/E^X_L : X\emptyset$-type-definable, $a \in X\}$. Similarly for $Gal_{KP}(T)$ and $Gal_{Sh}(T)$. The latter two groups have the structure of compact Hausdorff topological groups (via the Tychonoff topology for example). These (topological) groups are invariants of (the bi-interpretability type of) T (and so do not depend on the choice of $\bar{M}$).

Fact 1.4 *(i) Let $X \subseteq Y$ be $\emptyset$-type-definable (or even invariant) sets of possibly infinite tuples. Then $E^Y_L|X$ coincides with E^X_L and it is precisely the relation (on X) of being in the same orbit under $Autf_L(\bar{M})$.*
(ii) Similarly for E_{KP} and $Autf_{KP}(\bar{M})$. In particular, given a sort S and a complete type $p(x)$ of that sort, $E^S_{KP}|p = E^p_{KP}$.

Proof. (i) is immediate.
(ii) is contained in Lemma 4.18 of [7]. But we will give another proof. Work for simplicity in a sort S. It suffices to show that the equivalence relation on S of being in the same orbit under $Autf_{KP}(\bar{M})$ is type-definable over $\emptyset$ (and thus has to be E^S_{KP}). a and b are in the same orbit under $Autf_{KP}(\bar{M})$ iff $tp(a/e) = tp(b/e)$ whenever e is a bounded hyperimaginary. The latter is seen easily to be type-definable.

Corollary 1.5 *Let S be a sort and X a $\emptyset$-type-definable subset of S. Let E be any bounded $\emptyset$-type-definable equivalence relation on X. Then there is a bounded $\emptyset$-type-definable equivalence relation E' on S such that E is the restriction of E' to X.*

Proof By (ii) of the previous lemma, $E^S_{KP}|X$ refines E. The disjunction $E^S_{KP} \vee E$ is type-definable and as required.

Remark 1.6 *By Fact 1.4, we can speak unambiguously of E_L and E_{KP}. We will say that (possibly infinite) tuples a, b from $\bar{M}$ have the same Lascar strong type if $E_L(a, b)$ and the same KP-strong type if $E_{KP}(a, b)$.*

Fact 1.7 *$Autf_{KP}(\bar{M})$ is closed in $Aut(\bar{M})$ (where $Aut(\bar{M})$ is equipped with the pointwise convergence topology: the basic open sets are the stabilizers of finite tuples).*

Proof. This is in 4.18 of [7].

Fact 1.8 *The following are equivalent:*
(i) $Autf_L(\bar{M}) = Autf_{KP}(\bar{M})$ (and so canonically $Gal_L(T) = Gal_{KP}(T)$).
(ii) E_L coincides with E_{KP} (even on infinite tuples)
(iii) E_L coincides with E_{KP} on finite tuples and $Autf_L(\bar{M})$ is closed in $Aut(\bar{M})$.

Proof. See [4] and 4.20 of [7].

Fact 1.9 *(i) $E_L(a, b)$ if and only if there is some $n < \omega$ and there are models $M_1, .., M_n$ and $a_0, .., a_n$ such that $a_0 = a$, $a_n = b$ and $tp(a_i/M_{i+1}) = tp(a_{i+1}/M_{i+1})$ for each $i = 0, .., n-1$.*
(ii) $Autf_L(\bar{M})$ is the subgroup of $Aut(\bar{M})$ generated by the subgroups $Fix(m)$ where m ranges over enumerations of small submodels of $\bar{M}$.

Proof. (i) is well-known, see for example [5]. (ii) follows from (i) (and is actually Lascar's original definition of $Autf_L(\bar{M})$).

Suppose that $E_L(a, b)$. We define $d(a, b)$ to be the smallest n as in Fact 1.9(i).

We now discuss thick formulas, although these will play a minor role in our proofs.

Definition 1.10 *Fix a sort S. An L-formula $\theta(x, y)$, x, y of sort S, is said to be thick, if for some n, for any $a_1, ..., a_n \in S$, $\theta(a_i, a_j)$ for some $i \neq j$. Similarly working inside a type-definable set of possibly infinite tuples.*

Note that thickness is a symmetric notion: if $\theta(x,y)$ is thick, then so is $\theta(y,x)$. In particular
(*) any thick formula is implied by a symmetric thick formula.

Fact 1.11 *Suppose $a \neq b$. The following are equivalent:*
(i) $\theta(a,b)$ for all thick θ.
(ii) a and b belong to some infinite indiscernible sequence.
(iii) $\theta(a,b)$ holds for all symmetric thick formulas.

Proof. (i) implies (ii) is by compactness. (ii) implies (iii): if $(a_i : i < \omega)$ is infinite indiscernible and $\models \neg\theta(a_0, a_1)$, then $\models \neg\theta(a_i, a_j)$ for all $i < j < \omega$, so θ is not thick. (iii) implies (i) holds by (*) above.

Let Θ be the set of thick formulas.

Fact 1.12 *(i) $\Theta(a,b)$ implies that $tp(a/M) = tp(b/M)$ for some model M.*
(ii) If $tp(a/M) = tp(b/M)$ then there is c such that $\Theta(a,c)$ and $\Theta(c,b)$.

Proof. (i) is well-known. (ii) This actually lies behind Fact 1.9. The usual proof is by choosing a coheir of $tp(a/M)$ and constructing a sequence I such that both aI and bI are (infinite) indiscernible. Alternatively one could argue directly with thick formulas: As M is a model, for each thick symmetric $\theta(x,y)$ we can find $a_1, .., a_n \in M$ (suitable n) such that for all a' there is $i \leq n$ such that $\theta(a', a_n)$. As $tp(a/M) = tp(b/M)$ we have $\theta(a, a_i) \wedge \theta(b, a_i)$ for some i. By compactness, we find c as required.

Note that if (a,b) begins an infinite indiscernible sequence then $f(a) \neq f(b)$ for any $\emptyset$-definable finite-to-one function (in $\bar{M}^{eq}$). So we easily get examples of $a \neq b$ with $tp(a/M) = tp(b/M)$ for some model, but where a, b does not begin an infinite indiscernible sequence.

Fact 1.13 *$E_L(a,b)$ iff there is $n < \omega$ and there are $a_0, a_1, .., a_n$ with $a_0 = a$, $a_n = b$ and $\Theta(a_i, a_{i+1})$ for all $i < n$.*

Proof. By 1.9 and 1.12.

Finally we mention a certain topology on $Gal_L(T)$ which was given in [7]. Let $\mu : Aut(\bar{M}) \to Gal_L(T)$ be the canonical surjective homomorphism. Fix a small elementary substructure M_0 of $\bar{M}$, enumerated by m_0. In Definition 4.9

of [7], a subset C of $Gal_L(T)$ was defined to be *closed* if whenever $g_i \in \mu^{-1}(C)$ for $i \in I$, g' is some ultraproduct of the g_i, and $g \in Aut(\bar{M})$ is such that $tp(g(m_0)/m_0) = tp(g'(m_0)/m_0)$ then $g \in \mu^{-1}(C)$. This defines a "topology" on $Gal_L(T)$, independent of the choice of M_0. When T is G-compact (so $Gal_L = Gal_{KP}$) this agrees with the topology on Gal_{KP} referred to earlier.

2 $Gal_L(T)$

We try to get a better understanding of the object $Gal_L(T)$ and see how explicitly E_{KP} can be obtained from E_L. By studying closure in $Gal_L(T)$ we will see that E_{KP} is obtained by first taking the closure of E_L (in the Stone space sense) and then closing under E_L.

Let us fix a small submodel M_0 of $\bar{M}$, enumerated by m_0. If T happens to be countable, we may assume M_0 is too. Let $S_{m_0}(m_0)$ be the space of extensions of $tp(m_0/\emptyset)$ to complete types over m_0, namely $\{tp(g(m_0)/m_0) : g \in Aut(\bar{M})\}$ equipped with the usual Stone space topology. Note that if M_0 is countable, $S_{m_0}(m_0)$ is a Polish space. Let $\mu : Aut(\bar{M}) \to Gal_L(T)$ be the canonical homomorphism. Let $\mu_1 : Aut(\bar{M}) \to S_{m_0}(m_0)$ be the surjective map taking $g \in Aut(\bar{M})$ to $tp(g(m_0)/m_0)$. We begin by making explicit some observations from [7].

Fact 2.1 *μ factors through μ_1. That is, for $g \in Aut(\bar{M})$, $\mu(g)$ depends only on $tp(g(m_0)/m_0)$.*

Proof. If $tp(g(m_0)/m_0) = tp(h(m_0)/m_0)$, let $\gamma \in Fix(m_0)$ be such that $\gamma(g(m_0)) = h(m_0)$. Then, by Fact 1.9 (ii), both γ and $h^{-1}.\gamma.g$ are in $Autf_L(\bar{M})$. As $Autf_L(\bar{M})$ is normal in $Aut(\bar{M})$, $\gamma.g = g.\gamma'$ for some $\gamma' \in Autf_L(\bar{M})$. Thus $h^{-1}.g.\gamma' \in Autf_L(\bar{M})$, whereby $h^{-1}.g \in Autf_L(\bar{M}) = Ker(\mu)$.

So we obtain a canonical surjective map $\nu : S_{m_0}(m_0) \to Gal_L(T)$ such that $\nu.\mu_1 = \mu$.

Remark 2.2 *Let $p, q \in S_{m_0}(m_0)$. Then*
(i) $\nu(p) = \nu(q)$ if and only if for any (some) m realizing p and any (some) n realizing q, $E_L(m, n)$ (equivalently m and n are in the same $Autf_L(\bar{M})$-orbit).
(ii) The equivalence relation $\nu(p) = \nu(q)$ is (by 1.13 and (i)) a countable

union of closed subsets of $S_{m_0}(m_0) \times S_{m_0}(m_0)$. In particular if T (and M_0) are countable, this is a Borel equivalence relation.
(iii) The set of Lascar strong types extending $tp(m_0/\emptyset)$ is (naturally) a principal homogeneous space for $Gal_L(T)$, so can be identified with $Gal_L(T)$ after fixing a point (such as the class of m_0).
(iv) (T countable.) The set of Lascar strong types on any sort S can be considered as the quotient of a Boolean space by a Borel equivalence relation: Let m_0 be a countable model, and let X be the space of complete types over m_0 in sort S. Define $E_L(p,q)$ if for some (any) realizations a of p and b of q, $E_L(a,b)$.

Fact 2.3 *(i) The topology on $Gal_L(T)$ referred to in the last paragraph of section 1 is precisely the quotient topology under the map ν. That is $C \subseteq Gal_L(T)$ is closed if and only if $\nu^{-1}(C)$ is closed in the space $S_{m_0}(m_0)$.*
(ii) $Gal_L(T)$ is a topological (not necessarily Hausdorff) group.
(iii) $Autf_{KP}(\bar{M})$ is precisely $\mu^{-1}(\overline{\{id\}})$ where id is the identity element of $Gal_L(T)$.

Proof. (i) Lemma 4.10 of [7] says that C is closed in $Gal_L(T)$ just if $\{g(m_0) : g \in \mu^{-1}(C)\}$ is type-definable over m_0, namely if $\{tp(g(m_0)/m_0) : g \in \mu^{-1}(C)\}$ (which is precisely $\nu^{-1}(C)$) is closed in $S_{m_0}(m_0)$.
(ii) Let $C \subset Gal_L(T)$ be closed. We have to show that
(a) $X = \{(g,h) \in Gal_L(T) \times Gal_L(T) : g.h \in C\}$ is closed and
(b) $Y = \{g \in Gal_L(T) : g^{-1} \in C\}$ is closed.
We will just deal with (a). Let $\Phi(x, m_0)$ define the closed set $\nu^{-1}(C)$. Let $p, q \in S_{m_0}(m_0)$, and let m, n be any realizations of p, q respectively. We claim that $(\nu(p), \nu(q)) \in X$ if and only if there is x such that $\Phi(x, m_0)$ and $tp(x, n/\emptyset) = tp(m, m_0/\emptyset)$. The verification is left to the reader: among the points is that $Autf_L(\bar{M})$ is a normal subgroup of $Aut(\bar{M})$.
(iii) Let H be the closure of the identity in $Gal_L(T)$, which is, by (ii) a (normal) subgroup. Let $H_1 = \mu(Autf_{KP}(\bar{M}))$. So $\nu^{-1}(H_1)$ is closed in $S_{m_0}(m_0)$, being defined by $E_{KP}(x, m_0)$. Thus $H \subseteq H_1$. Let $\Phi(x, m_0)$ define $\nu^{-1}(H)$. So $\Phi(x,y) \rightarrow E_{KP}(x,y)$, and $E_L(x,y) \rightarrow \Phi(x,y)$ (on realizations of $tp(m_0)$). As H is a subgroup of $Gal_L(T)$ we see that $\Phi(x,y)$ is an equivalence relation, so has to be E_{KP}. Thus $H = H_1$.

Proposition 2.4 *Let $C \subseteq Gal_L(T)$. Suppose that $q \in \overline{\nu^{-1}(C)}$ and $\nu(p) = \nu(q)$. Then for any realization n of p there is n' such that*

(i) (n, n') *begins an indiscernible sequence (namely* $\Theta(n, n')$ *holds) and*
(ii) $tp(n'/m_0) \in \overline{\nu^{-1}(C)}$.

Proof. Let m realize q. By the assumption on q there is a set I, an ultrafilter U on I and $q^i \in S_{m_0}(m_0)$ for $i \in I$, such that $\nu(q^i) \in C$ for all $i \in I$, and q is the ultraproduct via U of the q^i. Let m^i realize q^i such that $tp(m^i/m, m_0)$ is finitely satisfiable in M_0. As $\nu(p) = \nu(q)$ there is, by Remark 2.2(i), some $g \in Autf_L(M)$ such that $g(m) = n$. Let $n^i = g(m^i)$ $(i \in I)$. The ultraproduct with respect to U of $\{tp(m, m^i, n, n^i, m_0) : i \in I\}$ can be realized by (m, m', n, n', m_0) for some m', n' in $\bar{M}$. We have, by the assumptions:
(a) $tp(m'/m_0) = q$.
(b) $tp(m'/m, m_0)$ is finitely satisfiable in M_0.
(a) and (b) imply
(c) (m, m') begins an indiscernible sequence.
As $tp(n, n^i) = tp(m, m^i)$ for all i, we see that $tp(n, n') = tp(m, m')$ and so by (c),
(d) (n, n') begins an indiscernible sequence.
By Remark 2.2, $\nu(tp(n^i/m_0)) \in C$ for all i, whereby
(e) $tp(n'/m_0) \in \overline{\nu^{-1}(C)}$.
(d) and (e) give the desired conclusion.

Corollary 2.5 *Let* $C \subseteq Gal_L(T)$*. Then* $\bar{C} = \nu(\overline{\nu^{-1}(C)})$.

Proof. Clearly $\bar{C}$ contains $\nu(\overline{\nu^{-1}(C)})$, so it suffices to show that the latter is closed, that is, its preimage, X say, is closed in $S_{m_0}(m_0)$. Let $\Psi(x, m_0)$ be the partial type defining $\overline{\nu^{-1}(C)}$. By Proposition 2.4, X is the closed subset of $S_{m_0}(m_0)$ defined by:
$\exists y(\Theta(x, y) \wedge \Psi(y, m_0))$.

Let X be a $\emptyset$-type-definable set (of possibly infinite tuples). $\overline{E_L^X}$ denotes the closure of E_L^X in the space of complete types $p(x, y)$ over $\emptyset$ extending $x \in X \wedge y \in X$. If X is the set of realizations of a complete type p, we notationally replace X by p.

Corollary 2.6 *For any complete type* $p(x)$ *over* $\emptyset$*, and realizations* a, b *of* p*.* $E_{KP}(a, b)$ *iff there is some* c *such that* $\Theta(a, c)$ *and* $\overline{E_L^p}(c, b)$*. In particular, on realizations of* p*,* E_{KP} *is the composition of* E_L *with* $\overline{E_L^p}$*.*

Proof. It is enough to prove this where $p = tp(m_0)$. Suppose $E_{KP}(a, b)$ where a, b are realizations of p. We may assume that $b = m_0$. By 2.3(iii), $\nu(tp(a/m_0))$ is in the closure of the identity in $Gal_L(T)$. By Proposition 2.4, there is c such that $\Theta(a, c)$ holds and $tp(c/m_0)$ is in the closure of the set of $tp(c'/m_0)$ where c' realises p and $E_L(c', m_0)$. So $\overline{E^p_L}(c, m_0)$.

Various examples will be given in section 4 which answer some obvious "qualitative" questions: such as can $E_{KP} \neq E_L$, can $E_{KP} \neq \overline{E_L}$? etc. But many questions remain (regarding $Gal_L(T)$ and E_L). Let us suppose T to be countable. What are the possible cardinalities of $Gal_L(T)$, of the kernel $K(T)$ of $Gal_L(T) \to Gal_{KP}(T)$ and of the set of E_L-classes in a given E_{KP}-class? As, by Remark 2.2, $Gal_L(T)$ is the quotient of the Polish space $S_{m_0}(m_0)$ by the Borel equivalence relation $E : \nu(p) = \nu(q)$, by well-known results, CH holds for the cardinality of $Gal_L(T)$. Similarly for the other questions. For example, let X be the subspace of $S_{m_0}(m_0)$ consisting of those $tp(m/m_0)$ such that $E_{KP}(m, m_0)$. Then X is a Polish space and $K(T)$ is the quotient of X by $E|X$. Maybe the right question to ask is what can be the "Borel cardinality" (in the sense of [3]) of this $E|X$. We would like to conjecture that it is very complicated (if nontrivial). A possibly related question is: Suppose Y is a KP-class and E_L is trivial on Y (so Y is also an E_L-class). Does it follow that there is a finite bound on the $d(a, b)$ for $a, b \in X$?

3 Products and Galois groups.

The examples in section 4 will involve (elaborations of) infinite products of structures. We find it worthwhile to give some generalities about products of structures and the effect on the Galois groups Gal_{KP} and Gal_L.

Let $(M_i : i \in I)$ be a family of structures in disjoint languages L_i. We assume for simplicity that these are 1-sorted structures. By the *disjoint sum* or *coproduct* $\coprod_i M_i$ of this family we mean the family $(M_i : i \in I)$ considered as a many-sorted structure. That is, the sorts are labelled by the elements of I, the *ith* sort is M_i equipped with all its L_i-structure, and there are no additional relations or functions. By the product $\prod_i M_i$ we mean the structure M whose universe is the set of sequences $(a_i)_{i \in I}$ where $a_i \in M_i$ for each i, equipped with, for each $i \in I$, the equivalence relation E_i of having the same *ith* coordinate, and also equipped with all the L_i structure on $M/E_i = M_i$ (all i). We emphasize that M denotes both the structure

$\prod_i M_i$ as well as its underlying set.

If the M_i are all saturated, then so is $\coprod_i M_i$. If I is finite, then $\coprod_i M_i$ is bi-interpretable with $\prod_i M_i$. In fact it is convenient in this case (I finite) to identify both structures with the structure $(M, M_i, f_i)_i$ where f_i is the projection map from M to $M/E_i = M_i$.

If I is infinite $\prod_i M_i$ is still bi-interpretable with $(M, M_i, f_i)_i$. However, even if the M_i are all saturated, $\prod_i M_i$ will not be saturated, unless it is finite. Let E be the intersection of the E_i (so type-definable). In $\prod_i M_i$ each equivalence class of E is a singleton. Adjoining a suitably large number of elements to each E-class yields a saturated elementary extension of $\prod_i M_i$, which we call $(\prod_i M_i)^*$ or M^*. If I or M is finite we set $M^* = M$. Note that in passing to M^* no new elements were added to any of the sorts M_i, and that $\coprod_i M_i$ is interpretable in M^*.

From now on we assume that each M_i is saturated and we stick with notation above.

Remark 3.1 *(i) $Aut(\coprod_i M_i)$ is precisely the product of the $Aut(M_i)$.*
(ii) The canonical map from $Aut(M^*)$ to $Aut(\coprod_i (M_i))$ is surjective.
(iii) Let $X \subset M_{i_1} \times .. \times M_{i_n}$ be definable in the structure M^* with parameters from some M_j's where $j \neq i_1, .., i_n$. Then X is already $\emptyset$-definable in the coproduct (or product) of M_{i_1},..,M_{i_n}.

Corollary 3.2 *(i) E_L on (even infinite tuples from) $\coprod_i M_i$ is the same in the structure M^* as in the structure $\coprod_i M_i$.*
(ii) Similarly for E_{KP}.
(iii) Fix $i_0 \in I$, and let c be a finite tuple from some other M_j's ($j \neq i_0$). Then E_L and E_{KP} on M_{i_0} in the sense of the structure $(\coprod_i M_i, c)$ is the same as in the sense of the structure M_{i_0}.

Proof. Immediate.

Lemma 3.3 *Suppose I is finite.*
(i) For $a, b \in M$ (the universe of $\prod_i M_i$). $E_L(a, b)$ iff $E_L(f_i(a), f_i(b))$ (in M_i for each i).
(ii) Similarly for E_{KP}.

Proof. By induction it is enough to look at the case where $|I| = 2$.
We give a proof for E_{KP} which also works for E_L (although the E_L case is immediate from Fact 1.13).

So now $M = M^*$ is the product of M_1 and M_2. Suppose $a, b \in M$. Let $f_i(a) = a_i \in M_i$ for $i = 1, 2$ and likewise $f_i(b) = b_i$. Suppose that $E_{KP}(a_i, b_i)$ in M_i for $i = 1, 2$. We must show that $E_{KP}(a, b)$ in M. By (iii) of Corollary 3.2, $E_{KP}(a_1, b_1)$ in the sense of M. So by Fact 1.4 there is $f \in Autf_{KP}(M)$ such that $f(a_1) = b_1$. Let $a_2' = f(a_2)$. So $E_{KP}(a_2, a_2')$ in the sense of M and so also of M_2. Thus $E_{KP}(a_2', b_2)$ in the structure M_2. By (iii) of Corollary 3.2 again there is $g \in Autf_{KP}(M, b_1)$ such that $g(a_2') = b_2$. So $g \cdot f \in Autf_{KP}(M)$ and takes a to b, thus $E_{KP}(a, b)$ in M.

Remark 3.4 *So for I finite one sees (generalizing the above to finite tuples) that $Autf_{KP}(M)$ is the product of the $Autf_{KP}(M_i)$ and even $Gal_{KP}(M)$ is isomorphic to the product of the $Gal_{KP}(M_i)$ as topological groups.*

We now consider the case where I is possibly infinite. Let us call a subset X of the underlying set M of $\prod_i M_i$ *dense* if for any $i_1, .., i_n \in I$ and $a_{i_j} \in M_{i_j}$ for $j = 1, .., n$ there is $x \in X$ such that $f_{i_j}(x) = a_{i_j}$ for $j = 1, .., n$. The following is easy.

Lemma 3.5 *Any dense subset of M is the underlying set of an elementary substructure of M (and also of M^*).*

Lemma 3.6 *Suppose that $a, b \in M^*$ and $E(a, b)$. Then $E_L(a, b)$, in fact a and b have the same type over some model.*

Proof. We may assume I to be infinite. Choose an automorphism α of M^* which maps a to b and fixes all other E-classes pointwise. The set of elements fixed by α is an elementary submodel by the last lemma.

Lemma 3.7 *(I arbitrary.) (i) Let $a_i, b_i \in M_i$ for all i. Then $E_L((a_i)_{i \in I}, (b_i)_{i \in I})$ in M^* iff there is $n < \omega$ such that $d(a_i, b_i) \leq n$ for all $i \in I$.*
(ii) Let $a, b \in M^*$. Then $E_L(a, b)$ iff there is $n < \omega$ such that $d(f_i(a), f_i(b)) \leq n$ for all $i \in I$.
(iii) For $a, b \in M^*$, $E_{KP}(a, b)$ (in M^*) iff $E_{KP}(f_i(a), f_i(b))$ in M_i for all $i \in I$.

Proof. (i) By Corollary 3.2, we make work in the structure $\prod_i M_i$. The assertion then follows immediately from Fact 1.13.
(ii) Note that $E(a, (f_i(a))_{i \in I})$ and similarly for b. So, by Lemma 3.6, (ii) is an immediate consequence of (i).

(iii) Left implies right is clear. Now suppose the right-hand side holds. Let $a_i = f_i(a)$. By Lemma 3.3, for each finite $J \subseteq I$,
(*) $E_{KP}((a_i)_{i\in J}, (b_i)_{i\in J})$
in the sense of structure $\coprod_{i\in J} M_i$. By Corollary 3.2 (iii) for each finite J, (*) holds in the sense of the structure $\coprod_{i\in I} M_i$. By Fact 1.7,
(**) $E_{KP}((a_i)_{i\in I}, (b_i)_{i\in I})$
holds in the sense of $\coprod_{i\in I} M_i$. By Corollary 3.2 (ii), (**) holds in the structure M^*. As in the proof of (ii), we deduce, that $E_{KP}(a, b)$ in M^*.

Corollary 3.8 *Suppose that $I = \omega$. Suppose moreover that there are $i_1 < i_2 < ....$ and $a_{i_N}, b_{i_N} \in M_{i_N}$ for each N such that $E_L(a_{i_N}, b_{i_N})$ but $d(a_{i_N}, b_{i_N}) > N$ in M_{i_N}. Then*
(i) On (the underlying set of) M^*, $E_L \neq E_{KP}$. Moreover some E_{KP} class splits into continuum many E_L classes.
(ii) In $\coprod_i M_i$, $Autf_L$ is not closed in the full automorphism group. There is an E_{KP} class of infinite tuples $(a_i)_i$ ($a_i \in M_i$) which splits into continuum many E_L-classes.

Proof.(i) We can find some $X \subset 2^\omega$ of size continuum such that if $\eta \neq \nu \in X$ then for arbitrarily large N, $\eta(N) \neq \nu(N)$. Define elements $c_\eta \in M^*$ or even in M for $\eta \in X$ as follows: if $\eta(N) = 0$ then $f_{i_N}(c_\eta) = a_{i_N}$, if $\eta(N) = 1$, $f_{i_N}(c_\eta) = b_{i_N}$ and for j different from all i_N, $f_j(c_\eta)$ is some fixed element of M_j. By the above lemma, we have in M^* that $E_{KP}(c_\eta, c_\nu)$ for all $\eta, \nu \in X$, but for distinct η, ν, c_η and c_ν are in different E_L-classes.
(ii) Similar.

4 Examples

By Corollary 3.8, to find an example where $E_{KP} \neq E_L$ and where $Autf_L$ is not closed, we only have to find structures M_i for $i < \omega$ as in the hypothesis of 3.8. We proceed to do this now.

We will define structures M_n for $n = 1, 2, ...$ Let $n \geq 1$. M_n will be the structure whose universe is the circle of centre the origin, and radius 1 say (in a saturated real-closed field), equipped with a ternary relation S_n and a unary function g_n. S_n is the "circular order": $S_n(a, b, c)$ holds just if a, b, c are distinct and b comes before c going around the circle clockwise starting at a. g_n is rotation (clockwise) by $2\pi/n$ radians. Note that g_n is a bijection, $(g_n)^n$ is the identity, and for any a, $S_n(a, x, y)$ is a dense linear ordering on

$M_n \setminus \{a\}$. We will call a subset X of M_n *dense* if for all $a \neq c \in X$ there is $b \in X$ with $S_n(a, b, c)$. (Hopefully there will be no confusion with denseness as defined before 3.5.) Note also that any finitely generated substructure is finite. In fact the substructure generated by $\{a_1, .., a_k\}$ is precisely $\{g_n^j(a_i) : i = 1, .., k, j = 0, .., n-1\}$. It will be convenient later to let R_n denote the binary (symmetric) relation : $x = y \vee S_n(x, y, g_n(x)) \vee S_n(y, x, g_n(y))$ (so meaning that the shortest arc joining x and y has length $< 2\pi/n$).

Lemma 4.1 *(i) $Th(M_n)$ has a unique* 1*-type over* $\emptyset$.
(ii) Let $a \in M_n$, and let $I_a = \{x \in M_n : S_n(a, x, g_n(a))\}$. For $x, y \in I_a$, write $x < y$ iff $S_n(a, x, y)$. (So $(I_a, <)$ is a dense linear ordering with no first or last element.) Then there is a natural $1-1$ correspondence between partial isomorphisms between finite tuples in $(I_a, <)$ and partial isomorphisms between finite substructures in (M_n, a): if $\bar{b}$, $\bar{c}$ are finite tuples in I_a with the same quantifier-free type in $(I_a, <)$ then $(\bar{b}, g_n(\bar{b}), .., (g_n)^{n-1}(\bar{b}))$ has the same quantifier-free type as $(\bar{c}, g_n(\bar{c}), .., (g_n)^{n-1}(\bar{c}))$ in (M_n, a).

Proof. (i) In the elementary substructure $M_n(\mathbf{R})$ of M_n, rotation by r degrees (any r) is an automorphism. So there is a unique 1-type realised in $M_n(\mathbf{R})$, so $Th(M_n)$ has a unique 1-type.
(ii) Easy.

Proposition 4.2 *(i) $Th(M_n)$ has quantifier-elimination.*
(ii) $X \subseteq M_n$ is an elementary substructure of M_n just if X is a substructure (i.e. closed under g_n) and X is dense.

Proof. (i) We do back-and-forth in M_n. Let $(a_1, ...a_k)$, $(b_1, .., b_k)$ have the same quantifier-free type. Given c we want d such that $(a_1, .., a_k, c)$ and $b_1, .., b_k, d)$ have the same quantifier-free type. By Lemma 4.1(i), we may assume that $a_1 = b_1 = a$. We may assume that the tuple of a_i's enumerates a substructure. By relabelling we may assume that this substructure is generated by $(a, a_2, .., a_r)$ where $a_i \in I_a$ for $i = 2, .., r$. Now $(a_2, .., a_r)$ has the same quantifier-free type as $(b_2, .., b_r)$ in I_a. Replacing c by some $(g_n)^j(c)$ we may assume $c \in I_a$. So we find $d \in I_a$ such that $(a_2, .., a_r, c)$ has the same quantifier-free type as $(b_2, .., b_r, d)$ in I_a. By Lemma 4.1(ii), $(a_2, ., a_r, c)$ and $(b_2, .., b_r, d)$ have the same quantifier-free type in (M, a) which is enough.
(ii) Left implies right is clear. Right implies left: Working over a given finite tuple from X the back-and-forth argument above still works even if we also require that the first player always chooses in X. So X is an elementary substructure.

Corollary 4.3 *Let $a, b \in M_n$. Then a and b have the same type over some elementary substructure of M_n if and only if $R_n(a, b)$.*

Proof. Suppose first that $R_n(a, b)$, and we may assume that $S_n(a, b, g_n(a))$. Let X be the substructure of M_n generated by $\{c \in I_a : b < c\}$. By Proposition 4.2(ii), X is an elementary substructure of M_n and $tp(a/X) = tp(b/X)$. Conversely, suppose $\neg R_n(a, b)$. Let N be any elementary substructure of M_n. Let $c \in N$. Then for some $0 \leq i, j < n$ we have $S_n(a, (g_n)^i(c), b)$ and $S_n(b, (g_n)^j(c), a)$, whereby $S_n((g_n)^i(c), b, (g_n)^j(c))$ but $\neg S_n((g_n)^i(c), a, (g_n)^j(c))$ so a and b have different types over N.

We immediately obtain:

Corollary 4.4 *E_L (and so E_{KP}) is trivial on (the universe of) M_n. On the other hand there are $a, b \in M_n$ such that $d(a, b) > n/2$.*

Let M^* be the saturated structure built from the M_n as in the previous section. From Corollary 4.4, Lemma 3.7, and Corollary 3.8 we obtain:

Proposition 4.5 *Neither $Th(M^*)$ nor $Th(\coprod_n M_n)$ are G-compact. E_{KP} is trivial on the underlying set of M^* but there are continuum many E_L-classes on this set. In $\coprod_n M_n$, $Autf_L$ is not closed in the full automorphism group.*

The "space" of Lascar strong 1-types in M^* has a rather easy representation. It is the space $(S_1)^\omega$ quotiented by the Borel equivalence relation F, where $F((a_i)_i, (b_i)_i)$ holds iff there is m such that for each n the length of the shortest arc between a_n and b_n is less than m/n. F appears to be substantially more complicated, in the sense of Borel cardinalities, than the Borel equivalence relation E_1, eventual agreement on countable sequences of reals. Again we would suppose this to be the case for *any* example produced using Corollary 3.8.

We now modify the above example to give an example where E_{KP} is not $\overline{E_L}$, showing that Corollary 2.6 is best possible.

We consider again the circles M_n mentioned at the end of section 2, but we now take M to be $\prod(M_n : n \text{ even})$. Clearly the previous analysis still goes through. Recall that f_n is the projection map from M to M_n. Now define a new binary function h on M: for $a \in M$, $h(a)$ is the unique element $b \in M$ such that $f_n(b) = (g_n)^{n/2}(f_n(a))$ for all n (that is, $f_n(a)$ and $f_n(b)$

are "antipodal" in M_n for all n). Note that no new structure is added to any M_n. Let N be the structure (M, h). Note that h is an involution, and is an automorphism of M, $\emptyset$-definable on each M_n. Let N^* be the saturated elementary extension M^* of M described in section 3, equipped with an extension of h which establishes a bijection between the E-class of a and the E-class of b (whenever $h(a) = b$ in N). By a back and forth argument N^* is a saturated elementary extension of N. As before the many-sorted saturated structure $\coprod_n M_n$ is interpretable in N^*.

Lemma 4.6 *The canonical map from $Aut(N^*)$ to $Aut(\coprod_n M_n)$ is surjective.*

Proof. Clear.

We conclude:

Lemma 4.7 *E_L in the sense of N^* on infinite tuples $(a_n)_n$ (where $a_n \in M_n$) is the same as in the sense of $\coprod_n M_n$. Likewise for E_{KP}. In particular, E_{KP} is trivial on such infinite tuples in the structure N^*.*

Lemma 4.8 *Let $a, b \in N^*$ (namely in the home sort) be such that $E(a, b)$. Then there is an elementary substructure of N^* (even of N) over which a and b have the same type.*

Proof. Choose a small dense subset X of N which is closed under h and does not meet the E-class of a. Then X is an elementary submodel and $tp(a/X) = tp(b/X)$.

As in Lemma 3.7, we conclude

Lemma 4.9 *Both E_L and E_{KP} on the home sort in N^* agree with the corresponding relations in the reduct M^*. In particular E_{KP} is trivial.*

Lemma 4.10 *The formula $h(x) \neq y$ is in $\overline{E_L}$ in N^*.*

Proof. If $E_L(a, b)$ then for some n, $(g_n)^{n/2}(f_n(a)) \neq f_n(b)$ in the circle M_n. So $h(a) \neq b$.

We conclude

Proposition 4.11 *In N^*, $\overline{E_L}$ is properly contained in E_{KP} (on the home sort)*

The final example is one of a complete type $p(x)$ in a sort S such that $\overline{E_L^p} \neq \overline{E_L^S}|p$. Infinite products enter the picture again, but only "at infinity". We will describe a many-sorted structure W. It will be convenient to put in $\coprod(M_n : n \text{ even})$ at the beginning (M_n the circles as above). So the sorts will be the M_n for n even together with another sort S say. S will be a disjoint union of predicates P_n (n even), where P_n is (in bijection with the underlying set of) $M_2 \times M_4 \times ... \times M_n$. We also give ourselves projection functions f_i, i even (as part of the structure). f_i is defined only on those P_n for $n \geq i$, and takes $(a_2, a_4, .., a_n) \in P_n$ to $a_i \in M_i$. (Formally we could instead add the graph of f_i as a relation.) So far we have defined a structure W_0 say. Let us say a few words about this structure before continuing. Let $p(x)$ be the type in sort S which says $\{\neg P_n(x) : n < \omega\}$. p is a complete type. f_i is defined on all of p. Let E be the (type-definable) equivalence relation on p: $f_i(x) = f_i(y)$ for all i. A saturated elementary extension $(W_0)^*$ can be obtained by adding realizations of p, a suitable number in each E-class. The set of realizations of p in $(W_0)^*$ is essentially the structure M^* from section 2. On elements (or finite tuples) from the base model W_0, E_L is trivial. On realizations of p, E_L is as in M^*. E_{KP} is trivial on W_0^*.

We will add a function h to W_0 to obtain W. h is a function from S to S: Suppose $a \in P_n$. Then $h(a)$ is the unique $b \in P_n$ such that for each even $i \leq n$, $f_i(b) = (g_i)^{i/2}(f_i(a)$. Extend h to W_0^* by making it a bijection between E-classes X and Y whenever for some (any)$x \in X, y \in Y$, $f_i(y) = g_i^{i/2}(f_i(x))$ for all (even) i. We obtain a structure W^*. We leave it to the reader to check that W^* is a saturated elementary extension of W, and that $p(x)$ remains a complete type in $Th(W^*)$.

As in the previous examples we have:

Fact 4.12 *(i) $\coprod_n M_n$ has no more induced structure in W^*.*
(ii) For a, b realizing $p(x)$, $E(a, b)$ implies a and b have the same type over some model.

From this fact, we see as before that on $p(x)$, $E_L(a, b)$ holds if and only if for some n, $d(f_i(a), f_i(b)) \leq n$ for all i. It follows that the formula $h(x) \neq y$ is in $\overline{E_L^p}$. (Note also that E_{KP} is trivial on $p(x)$.) On the other hand on each predicate P_n, E_L is trivial. So if $a^n, b^n \in P_n$ with $h(a^n) = b^n$ for each n, then any limit of $tp(a^n, b^n)_n$ will be in $\overline{E_L}|p$ and will contain the formula $h(x) = y$. Thus

Proposition 4.13 *In the structure W^*, $\bar{E}_L^p \neq \bar{E}_L|p$.*

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