NOETHERIAN THEORIES

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Abstract. A first-order theory is Noetherian with respect to the collection of formulae $\mathcal{F}$ if every definable set is a Boolean combination of instances of formulae in $\mathcal{F}$ and the topology whose subbasis of closed sets is the collection of instances of arbitrary formulae in $\mathcal{F}$ is Noetherian. We show the Noetherianity of the theory of proper pairs of algebraically closed fields in any characteristic with respect to the family of tame formulae as introduced in [14], thus answering a question which was left open there.

1. Introduction

Consider a first-order theory T and a collection of formulae $\mathcal{F}$ closed under finite conjunctions. The family $\mathcal{F}$ is *Noetherian* if in every model M of T the family of instances of arbitrary formulae in $\mathcal{F}$ has the descending chain condition. The theory T is *Noetherian* with respect to the Noetherian collection of formulae $\mathcal{F}$ if every formula $\varphi(x; y)$ (in a fixed partition of the variables into tuples x and y) is equivalent modulo T to a Boolean combination of formulae $\psi(x; y)$ in $\mathcal{F}$ (in the same partition).

After having submitted a first draft of this article, we were made aware that Noetherian theories had already been defined by Hoffmann and Kowalski [8], albeit in a slightly different formulation. We will discuss the equivalent formulations in Remark 2.19.

Quantifier elimination implies that the theory of algebraically closed fields as well as the theory of differentially closed fields of characteristic 0 are Noetherian, since both the Zariski and the Kolchin topology are Noetherian. Every differentially closed field (in characteristic 0) expands the structure of a proper pair of algebraically closed fields, where the distinguished algebraically closed subfield is given by the *constant* elements whose derivative is 0.

In [6, 14] it was shown that proper pairs of algebraically closed fields of characteristic 0 are Noetherian. This follows from the fact that definable sets in the pair are Boolean combination of certain definable sets which happen to be Kolchin-closed in the corresponding expansion as a differentially closed field. However, this approach cannot be carried over to the case of positive characteristic since the Kolchin topology for differentially closed fields of positive characteristic is not Noetherian.

A weakening of Noetherianity is equationality, in which we only require that each partitioned formula is a boolean combination of *equations* (in the same partition). A

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partitioned formula $\varphi(x; y)$ is an equation if in every model of T the family of finite intersections of instances $\varphi(x, a)$ has the descending chain condition. The authors showed in [14] that the theory of pairs of algebraically closed fields is equational.

In this paper, we will prove that the family of equations exhibited in [14] for the theory of proper pairs of algebraically closed fields is in fact Noetherian, so every proper pair of algebraically closed fields is Noetherian, regardless of the characteristic (cf. Section 9 of the extended version of [14]).

Main Theorem. (*Corollary 4.2*) *The theory of proper pairs of algebraically closed fields is Noetherian.*

The structure of the papers is as follows: In Section 2, we explore the notion of Noetherianity. It will follow from Corollary 2.14 that Noetherianity is equivalent to the fact that every type contains a minimal instance of a formula in $\mathcal{F}$. Moreover, we relate Morley rank to the foundational rank relative to $\mathcal{F}$ and show that equality holds under a mild condition, called *Noetherian isolation*. Section 3 contains a short overview of the main properties of the theory of proper pairs of algebraically closed fields, which will be used in Section 4 in order to give a proof of the Noetherianity of this theory. In Section 5 we show that the theory of proper pairs of algebraically closed fields has Noetherian isolation using Poizat's description of Morley and Lascar ranks. Finally, in Section 6, we use the techniques of Hilbert polynomials and schemes (in a self-contained presentation) in order to explicitly exhibit the minimal tame formula of a type.

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2. Noetherianity and chain conditions

Definition 2.1. A collection $\mathcal{C}$ of subsets of a set X is *Noetherian* if it satisfies the following two conditions:

- The collection $\mathcal{C}$ is closed under finite intersections and contains the set X itself.
- The collection $\mathcal{C}$ has the descending chain condition (DCC): every descending chain
$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots,$$
with C_n in $\mathcal{C}$ for n in $\mathbb{N}$, eventually stabilises, that is, there is some n_0 such that $C_n = C_{n+1}$ for all $n \geq n_0$.

It is easy to see that $\mathcal{C}$ is has DCC if and only if every non-empty subset of $\mathcal{C}$ has a minimal element (with respect to set-theoretic inclusion).

Lemma 2.2. *Consider a collection $\mathcal{C}$ of subsets of a set X such that $\mathcal{C}$ contains X and is closed under finite intersections. Then the following are equivalent;*

(a) The collection $\mathcal{C}$ is Noetherian.
(b) For every ultrafilter $\mathcal{U}$ on X, the intersection $\mathcal{U} \cap \mathcal{C}$ has a minimal element D.
(c) Every ultrafilter $\mathcal{U}$ on X contains a set Y (possibly not in $\mathcal{C}$) which is contained in every element of $\mathcal{U} \cap \mathcal{C}$.

Since $\mathcal{U} \cap \mathcal{C}$ is closed under finite intersections, the subset D in (b) is uniquely determined. We refer to D as *the minimal element of $\mathcal{U}$ with respect to $\mathcal{C}$*.

Proof. The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are immediate. For the implication $(c) \Rightarrow (a)$, consider a strictly decreasing chain

$$C_0 \supsetneq C_1 \supsetneq \cdots \supsetneq C_n \supsetneq \cdots,$$

of elements of $\mathcal{C}$. Set $Z = \bigcap_{n\in\mathbb{N}} C_n$ and notice that the collection $\{C_n \setminus Z\}_{n\in\mathbb{N}}$ has the finite intersection property. Thus, there is some ultrafilter $\mathcal{U}$ on X containing every $C_n \setminus Z$. Assume that $\mathcal{U}$ contains an element Y as in (c). Since $\mathcal{U} \cap \mathcal{C}$ has empty intersection, we deduce that $Y = \emptyset$ which gives the desired contradiction. □

Definition 2.3. An element C of $\mathcal{C}$ is *irreducible* if it is non-empty and cannot be written as a finite union $C = C_1 \cup \cdots \cup C_n$, with each $C_k \subsetneq C$ in $\mathcal{C}$. Equivalently, whenever C is contained in some finite union $\bigcup_{i=1}^m D_i$, with D_i in $\mathcal{C}$, then $C \subset D_i$ for some $1 \leq i \leq m$.

Remark 2.4. If $\mathcal{C}$ is Noetherian, it follows immediately from König's Lemma that every C in $\mathcal{C}$ can be written as an irredundant union of finitely many irreducible subsets $C_1, \ldots, C_n$. The decomposition $C = C_1 \cup \cdots \cup C_n$ is *irredundant* if $C_i \not\subset C_j$ for $i \neq j$. The irreducible subsets appearing in an irredundant expression of C are unique up to permutation. We refer to them as the *irreducible components* of C.

A straightforward application of Lemma 2.2 yields the following result, which justifies our choice of terminology.

Fact 2.5. ([17, Lemma 2.7]) If the collection $\mathcal{C}$ of subsets of X is Noetherian, then so is the family $\mathcal{C}'$ of finite unions of elements of $\mathcal{C}$.

A topology is *Noetherian* if the family of closed sets is Noetherian. If $\mathcal{C}$ is Noetherian, it follows that $\mathcal{C}'$ consists of the closed sets of a Noetherian topology $\mathcal{T}_\mathcal{C}$ on X.

Definition 2.6. Given a Noetherian collection $\mathcal{C}$ of subsets of X, we assign an ordinal rank $\mathrm{Rk}_\mathcal{C}(Y)$ to every subset Y of X as follows. For irreducible sets C it is the foundational rank, that is,

- $\mathrm{Rk}_\mathcal{C}(C) \geq 0$ always holds;
- $\mathrm{Rk}_\mathcal{C}(C) \geq \alpha + 1$ if and only if $\mathrm{Rk}_\mathcal{C}(D) \geq \alpha$ for some irreducible $D \subsetneq C$;
- $\mathrm{Rk}_\mathcal{C}(C) \geq \alpha$ with α limit if and only if $\mathrm{Rk}_\mathcal{C}(C) \geq \beta$ for every $\beta < \alpha$.

We set $\mathrm{Rk}_\mathcal{C}(C)$ the largest α with $\mathrm{Rk}_\mathcal{C}(C) \geq \alpha$. (Such an ordinal always exist by Noetherianity of the family). The rank of a closed set is the largest rank of its irreducible components, whilst $\mathrm{Rk}_\mathcal{C}(\emptyset) = -\infty$. Finally the rank of an arbitrary subset Y is the rank of its closure $\overline{Y}$ with respect to the topology $\mathcal{T}_\mathcal{C}$.

Remark 2.7. If A is closed in X with respect to $\mathcal{T}_\mathcal{C}$, we have that

$$\mathrm{Rk}_\mathcal{C}(A) = \max\{\mathrm{Rk}_\mathcal{C}(C) \mid C \subset A \text{ irreducible }\}.$$

This follows from the fact that every irreducible subset of A is contained in an irreducible component of A. Using the above equality, we deduce that

$$\mathrm{Rk}_\mathcal{C}(Y_1 \cup Y_2) = \max\{\mathrm{Rk}_\mathcal{C}(Y_1), \mathrm{Rk}_\mathcal{C}(Y_2)\}$$

for any subsets Y_1 and Y_2 of X. Now, if the subset Y of X is *constructible*, that is, it is a Boolean combination of closed sets, write $Y = \bigcup_{1\leq i\leq n} C_i \cap O_i$ for some

irreducible closed subsets C_i and open subsets O_i with each $C_i \cap O_i \neq \emptyset$. It follows that that
$$\mathrm{Rk}_{\mathcal{C}}(Y) = \max_{1 \le i \le n} \mathrm{Rk}_{\mathcal{C}}(C_i),$$
since the closure of each $C_i \cap O_i$ is C_i.

The following lemma will be used in the proof of 2.17.

Lemma 2.8. *If Y is constructible and non-empty, then $\mathrm{Rk}_{\mathcal{C}}(\overline{Y} \setminus Y) < \mathrm{Rk}_{\mathcal{C}}(Y)$.*

Proof. Write $Y = \bigcup_{1 \le i \le n} C_i \cap O_i$ as in Remark 2.7 and notice that
$$\overline{Y} \setminus Y \subset \bigcup_{1 \le i \le n} C_i \setminus O_i.$$
Since $C_i \setminus O_i$ is a proper closed subset of C_i, we have that $\mathrm{Rk}_{\mathcal{C}}(C_i \setminus O_i) < \mathrm{Rk}_{\mathcal{C}}(C_i)$, which gives the desired inequality. □

Definition 2.9. The *degree* of a closed subset A of X is the number $\deg_{\mathcal{C}}(A)$ of irreducible subsets of A of rank the rank of of A. The degree of an arbitrary subset of X is the degree of its closure.

The following observation follows from the fact that an irreducible subset of $\overline{Y^1} \cup \overline{Y^2}$ is contained in $\overline{Y^1}$ or in $\overline{Y^2}$.

Lemma 2.10. *If $\mathrm{Rk}_{\mathcal{C}}(Y^1) > \mathrm{Rk}_{\mathcal{C}}(Y^2)$, then $\deg_{\mathcal{C}}(Y^1 \cup Y^2) = \deg_{\mathcal{C}}(Y^1)$.* □

Note that the degree of a constructible set $Y = \bigcup_{1 \le i \le n} C_i \cap O_i$ equals the number of different C_i's of maximal rank. This yields the following result.

Lemma 2.11. *Given two disjoint constructible subsets Y^1 and Y^2 of X of the same rank, we have $\deg_{\mathcal{C}}(Y_1 \cup Y_2) = \deg_{\mathcal{C}}(Y^1) + \deg_{\mathcal{C}}(Y^2)$.*

Proof. For j in $\{1, 2\}$, write $Y^j = \bigcup_{1 \le i \le n_i} C_i^j \cap O_i^j$ for some irreducible closed subsets C_i^j and open subsets O_i^j with $C_i^j \cap O_i^j \neq \emptyset$. We need only show that $C_i^1 \neq C_k^2$ for all i, k. Assume otherwise. Since $C = C_i^1 = C_k^2$ is not the union of the two closed proper subsets $C \setminus O_i^1$ and $C \setminus O_k^2$, it follows that $C \cap O_i^1$ and $C \cap O_k^2$ cannot be disjoint, which gives the desired contradiction. □

Fix now a first-order theory T in a language $\mathcal{L}$.

Notation. Consider a collection of partitioned formulae $\mathcal{F}$ closed under renaming of variables, and finite conjunctions. For simplicity, we will always assume that the tautologically true *sentence* $\top$ belongs to $\mathcal{F}$. We allow dummy free variables, so $\top$ may be considered as a formula in any partitioned set of variables.

Definition 2.12. The collection $\mathcal{F}$ is *Noetherian* if in every model M of T and for every length $n = |x|$, the family of instances
$$\mathcal{C} = \{\varphi(M, a) \mid \varphi(x, y) \in \mathcal{F} \ \&\ a \in M\}$$
is Noetherian. We call a definable set $\theta(x, b)$ *closed* if $\theta(M, b)$ belongs to $\mathcal{C}$, that is, if $\theta(M, b)$ equals $\varphi(M, a)$ for some $\varphi(x, y)$ in $\mathcal{F}$ and a in M.

If the theory T is complete, it suffices to check that the family of instances with parameters in some $\aleph_0$-saturated model has the descending chain condition.

Remark 2.13. Recall that a formula $\varphi(x,y)$ is an *equation* if the collection of finite intersections of instances of $\varphi(x,y)$ has the DCC, or equivalently, if the collection of all conjunctions $\bigwedge_{i=1}^{n}\varphi(x,y_i)$ is Noetherian. Every formula in a Noetherian family $\mathcal{F}$ is an equation.

If M is a model of T, every ultrafilter on $M^{|x|}$ determines a type $p(x)$ over M, and thus over any subset A of M. Hence, we deduce from Lemma 2.2 and the observation after Definition 2.12 the following result.

Corollary 2.14. *The following conditions are equivalent:*

(a) The collection $\mathcal{F}$ is Noetherian.

(b) Every type $p(x)$ over a model M of T contains a minimal formula $\varphi(x,a)$ *with respect to $\mathcal{F}$, that is, the formula $\varphi(x,y)$ belongs to $\mathcal{F}$ and*

$$\psi(x,b) \text{ belongs to } p \text{ if and only if } \varphi(M,a)\subset\psi(M,b)$$

for every $\psi(x,z)$ in $\mathcal{F}$ and every tuple b in M.

(c) Every type $p(x)$ over a model M of T contains an $\mathcal{L}_M$-formula $\theta(x,a)$ such that

$$\psi(x,b) \text{ belongs to } p \text{ if and only if } \theta(M,a)\subset\psi(M,b)$$

for every formula $\psi(x,y)$ in $\mathcal{F}$ and every tuple b in M.

□

Since every two minimal formulae in the type p over M are equivalent, we will say that $\varphi(x,a)$ is *the* minimal formula of p (with respect to the Noetherian family $\mathcal{F}$). In condition (c), we do not require that $\theta(x,y)$ belongs to $\mathcal{F}$, so a type may admit two non-equivalent formulae $\theta(x,a)$ and $\theta'(x,a')$ as in (c).

Remark 2.15. If $\mathcal{F}$ is Noetherian, it is easy to see that every type $p(x)$ over a subset A of a model M contains a closed formula $\psi(x,a)$, which is minimal among all closed formulae in p. We will refer to $\psi(x,a)$ as the *minimal formula* of p.

If A is an arbitrary subset of parameters and not necessarily an elementary substructure of M, it need not be the case that the minimal formula of p is of the form $\varphi(x,a)$ for some $\varphi(x,y)\in\mathcal{F}$ and a in A. The easiest example is the theory of a 2-element set with $\mathcal{F}$ the family generated by $(x \doteq y)$. For those readers who do not feel at ease with finite models (which is the case of the the first author), we provide a more *standard* example: Consider the theory of a structure with two infinite equivalence classes modulo a definable equivalence relation $E(x,y)$ and set $\mathcal{F}$ the family generated by $\{(x\doteq y), E(x,y)\}$. This family is clearly Noetherian by Corollary 2.14, since there are only finitely many atomic formulae over any subset of parameters. If a is any element, the closed formula $\neg E(x,a)$ is clearly invariant over $A=\{a\}$, yet it is not an instance over A of an $\mathcal{F}$-formula.

We will see in Proposition 3.11 that minimal closed formulae for the theory of proper pairs of algebraically closed fields are indeed equivalent to instances of tame formulae with the same parameters.

Using Definition 2.3, we can define whether a closed formula $\psi(x,a)$ in the model M is irreducible. More generally, given a subset A of some model M of T, we say that a closed formula with parameters in A is *irreducible over A* if it cannot be written as a proper union of a finite number of closed formulas with parameters in A. If $A=M$, we will simply say that $\psi(x,a)$ is irreducible.

Remark 2.16. Let A be a subset of a model M of T. The minimal formula of a type over A is irreducible over A.

A closed formula with parameters in M is irreducible over M if and only it is irreducible over any elementary extension of M. Moreover, if $\theta(x,a)$ is any formula with parameters in M, then a closed formula $\varphi(x,b)$ equals the topological closure $\overline{\theta(x,a)}$ of $\theta(x,a)$ in M if and only $\varphi(x,b)$ is the closure of $\theta(x,a)$ in any elementary extension of M. It follows that $\varphi(x,b)$ can actually be defined using the same tuple a of parameters.

Notation. Using Definition 2.6, given a formula $\theta(x,a)$ with parameters in a model M of T, we denote by $\mathrm{R}_{\mathcal{F}}\,\theta(x,a)$ the $\mathrm{Rk}_{\mathcal{C}}$-rank of the set $\theta(N,a)$ with respect to the Noetherian family $\mathcal{C} = \{\varphi(N,b) \mid \varphi(x,y) \in \mathcal{F} \ \&\ b \in N\}$, where N is some $\aleph_0$-saturated elementary extension of M. We define the degree $\deg_{\mathcal{F}}\theta(x,a)$ similarly. The rank $\mathrm{R}_{\mathcal{F}}(p)$ of a type is the smallest rank of a formula in p. The degree $\deg_{\mathcal{F}}(p)$ is the smallest degree of a formula in p of rank $\mathrm{R}_{\mathcal{F}}(p)$.

Since the closure of a formula has the same rank and degree, it is easy to see that the rank and the degree of a type are exactly the rank and the degree of its minimal formula. Whence, the degree of a a type p over a model is always 1, since its minimal formula is irreducible.

Lemma 2.17. *Given a Noetherian family $\mathcal{F}$, let $\theta(x,a)$ be a formula with parameters in a subset A of a model M of T. Then*

$$\mathrm{R}_{\mathcal{F}}\,\theta(x,a) = \max\{\mathrm{R}_{\mathcal{F}}(p) \mid \textit{the type } p \textit{ over } A \textit{ contains } \theta(x,a)\},$$

where $\max\emptyset = -\infty$.

Proof. This follows easily from Remark 2.7: If α is the rank of $\theta(x,a)$ (so $\theta(x,a)$ is in particular consistent), then the set $\Sigma(x)$ of all negations of formulae of rank $< \alpha$ together with $\theta(x,a)$ is consistent. Any type over A which extends $\Sigma(x)$ has rank α. □

Definition 2.18. A first-order theory T is *Noetherian* with respect to the Noetherian family of formulae $\mathcal{F}$ if every partitioned formula $\psi(x,y)$ is equivalent modulo T to a Boolean combination of formulae $\varphi(x,y)$ in $\mathcal{F}$.

Remark 2.19. In [8] Hoffmann and Kowalski defined T to be Noetherian with respect to a set Σ of formulae if the following two conditions hold in every model M:

- The collection of definable sets $\sigma(M,a)$, with $\sigma(x,y)$ in Σ and a in M, form the class of closed sets of a Noetherian topology on $M^{|x|}$.
- For every subset A of parameters, every definable subset of $M^{|x|}$ over A is a Boolean combination of instances $\sigma(M,a)$, with $\sigma(x,y)$ in Σ and a in A.

It is easy to see that both notions are equivalent: Given $\mathcal{F}$, set Σ to be all finite disjunctions of formulae in $\mathcal{F}$. Conversely, given Σ as above, let $\mathcal{F}$ be the collection of formulae of the form $\sigma(x,y) \wedge \psi(y)$, where σ is in Σ and ψ is arbitrary.

Remark 2.20. Recall that a theory T is *equational* if every partitioned formula is equivalent to a boolean combination of equations. We conclude from Remark 2.13 that Noetherian theories are equational.

Question. As pointed out in [14, p. 830], a theory is equational if and only if every completion is. We do not know whether the same holds for Noetherianity.

We will now explore some of the model-theoretic properties of theories which are Noetherian and determine their stability spectrum. For that, we will adapt the previous notions of rank and degree to formulae in terms of their underlying definable set. It is easy to see that these definitions do not depend on the choice of the model. Even if no parameters occur in $\theta(x)$, we still need to choose an ambient model M.

Lemma 2.21. *Noetherian theories are totally transcendental.*

Proof. Assume for a contradiction that in some model M of T there is a binary tree of consistent formulae $\theta_s(x, a_s)$, with a_s in M for s in ${}^{<\omega}2$. Since T is Noetherian, each definable set $\theta_s(x, a_s)$ is constructible. Choose thus an instance $\theta_s(x, a_s)$ in the tree whose $\mathrm{R}_{\mathcal{F}}$-rank and degree are least possible in the lexicographic order. By minimality of the rank, both $\theta_{s\frown 0}(x, a_{s\frown 0})$ and $\theta_{s\frown 1}(x, a_{s\frown 1})$ must have the same $\mathrm{R}_{\mathcal{F}}$-rank as $\theta_s(x, a_s)$. Now, the instances $\theta_{s\frown 0}(x, a_{s\frown 0})$ and $\theta_{s\frown 1}(x, a_{s\frown 1})$ are disjoint, so we deduce from Lemma 2.11 that the degree of $\theta_s(x, a_s)$ is strictly larger than $\deg_{\mathcal{C}}(\theta_{s\frown 0}(x, a_{s\frown 0}))$, which gives the desired contradiction. □

It follows that Noetherian theories are κ-stable for every $\kappa \geq |\mathcal{L}|$. We will provide a direct proof of this in terms of the natural correspondence between types and their minimal formulae.

Proposition 2.22. *Suppose that the first-order theory T is Noetherian with respect to $\mathcal{F}$. For every subset A of a model M of T, there is bijection between types over A and (equivalence classes with respect to logical equivalence of) irreducible formulas over A. In particular, every Noetherian theory is κ-stable whenever $\kappa \geq |\mathcal{L}|$.*

Proof. By Remark 2.15, given a type $p(x)$ over A, we denote its minimal formula by $\varphi_p(x, a)$, which is unique up to equivalence and irreducible over A.

Given now a closed formula $\varphi(x, a)$ with parameters from A, the collection

$$\Sigma_\varphi = \{\varphi(x, a)\} \cup \{\neg\psi(x, a') \mid \psi(x, a') \text{ closed, } a' \text{ in } A \text{ and } \varphi(M, a) \not\subset \psi(M, a')\}$$

is consistent exactly if $\varphi(x, a)$ is irreducible over A. In that case, it admits a unique completion $p_\varphi(x)$, since A-definable subsets are Boolean combinations of A–definable closed subsets.

To conclude, we need only observe that $\varphi(x, a)$ is the minimal formula of a type p over A if and only if $\Sigma_\varphi \subset p$. □

It is not hard to see that the type $p(x)$ can be recovered from its minimal formula $\varphi(x, a)$ as the set of all formula $\theta(x, a')$ over A such $\varphi(x, a)$ is the topological closure of $\varphi(x, a) \wedge \theta(x, a')$.

Remark 2.23. In contrast to Remark 2.20 and Lemma 2.21, totally transcendental equational complete theories need not be Noetherian as the following example shows: Bonnet and Si-Kaddour [2] constructed a superatomic Boolean algebra B of Cantor-Bendixson rank 3 which is not generated by any well-founded subset closed under intersections. Such a Boolean algebra must necessarily be uncountable and have rank at least 3.

We will now consider the language $\mathcal{L}$ consisting of a unary predicate P_b for each element b of B as well as the complete theory T of the structure whose universe consists of the atoms of B such that the each predicate P_b is interpreted as the collection of atoms in B contained in the element b of B. It is easy to see that T

eliminates quantifier (for T is a complete relational monadic theory) and furthermore that $P_b = P_{b_1}$ if and only if $b = b_1$ (for B is atomic). An easy induction over the complexity of the formula yields that every definable subset of a model m of T is a disjunction of subsets of the form

$$\{(a_1, \ldots, a_n) \in M^n \mid \varphi(a_1, \ldots, a_n) \wedge \bigwedge_{i=1}^{n} P_{b_i}(a_i)\},$$

for some $b_1, \ldots, b_n$ in B and φ a quantifier-free formula in the empty language. In particular, the theory T is equational and has Morley rank 3, yet it is not Noetherian.

Whilst it is conceivable that a totally transcendental equational complete theory in a countable language or of Morley rank ≤ 2 has to be Noetherian, we have not further pursued this direction and leave the question open.

Corollary 2.24. *If M is a model, then $\deg_{\mathcal{F}} \theta(x, a)$ is the number of types p over M containing $\theta(x, a)$ with $\mathrm{R}_{\mathcal{F}}(p) = \mathrm{R}_{\mathcal{F}}\, \theta(x, a)$.*

Proof. Let α be the $\mathrm{R}_{\mathcal{F}}$-rank of $\theta(x, a)$. By Lemma 2.8, a type over M of rank α contains $\theta(x, y)$ if and only it contains the topological closure $\overline{\theta(x, a)}$. Now, the types over M of rank α containing $\overline{\theta(x, a)}$ correspond exactly to the irreducible components of $\overline{\theta(x, a)}$ of rank α. □

Lemma 2.25. *Let M be a model of the Noetherian theory T and p a type over a subset A of M. The minimal formula of p isolates it among all types over A of rank at least $\mathrm{R}_{\mathcal{F}}(p)$.*

Proof. Let $\varphi(x, a)$ be the minimal formula of p. Choose another type $q \neq p$ over A containing $\varphi(x, a)$. There is a formula $\theta(x, b)$ in p which implies $\varphi(x, a)$ and does not belong to q. Now, both $\varphi(x, a)$ and $\theta(x, b)$ have the same rank and degree as p, so by Lemma 2.11 the rank of $\varphi(x, a) \wedge \neg\theta(x, b)$, and therefore also the rank of q, is strictly smaller than $\mathrm{R}_{\mathcal{F}}(p)$, as desired. □

Total transcendence means that Morley rank is ordinal-valued. For the rest of the section, we will compare Morley rank to the foundational rank $\mathrm{R}_{\mathcal{F}}$ and show equality of these ranks under some mild assumption (see Definition 2.28) on the Noetherian theory T.

Corollary 2.26. *Assume that T is Noetherian with respect to $\mathcal{F}$. Then, for every formula $\theta(x, a)$ with parameters in a model of T, we have that $\mathrm{RM}\, \theta(x, a) \leq \mathrm{R}_{\mathcal{F}}\, \theta(x, a)$.*

Note that both ranks are computed in reference to the ambient model under consideration.

Proof. By Lemma 2.17, it is enough to show that $\mathrm{RM}(p) \leq \mathrm{R}_{\mathcal{F}}(p)$ for every type over an $\aleph_0$-saturated model M. Let $\alpha = \mathrm{R}_{\mathcal{F}}(p)$ and $\varphi(x, a)$ the minimal formula of p. Then $\mathrm{R}_{\mathcal{F}}(q) < \alpha$ for all types $q \neq p$ containing $\varphi(x, a)$. By induction on α, we deduce that $\mathrm{RM}(q) < \alpha$ for all such q. Since M is $\aleph_0$-saturated, it follows that $\mathrm{RM}(p) \leq \alpha$. □

Remark 2.27. Even for Noetherian theories of finite Morley rank, we need not always have equality between Morley rank and the foundational rank $\mathrm{R}_{\mathcal{F}}$. Indeed, consider the language $\mathcal{L}$ consisting of a single unary predicate P and the theory T

whose models are exactly the $\mathcal{L}$-structures where P denotes an infinite co-infinite subset. The theory T is Noetherian with respect to the class $\mathcal{F}$ consisting of finite conjunctions of atomic formulas. However, the irreducible $x \doteq x$ has Morley rank 1 (and Morley degree 2), yet $\mathrm{R}_{\mathcal{F}}$-rank 2.

One of the reasons why equality of both ranks does not hold in the above example is the fact that the unique non-algebraic 1-type over a model M determined by the formula $\neg P(x)$ contains no irreducible formula isolating it among all types over M of Morley rank at least 1. We will therefore introduce the notion of Noetherian isolation to ensure equality in Corollary 2.26.

Definition 2.28. The Noetherian theory T with respect to $\mathcal{F}$ admits *Noetherian isolation* if every type p over a subset A of a model of T contains a closed formula $\varphi(x,a)$ which isolates p among all types over A of Morley rank at least $\mathrm{RM}(p)$.

Clearly T admits Noetherian isolation if and only if the minimal formula of p isolates it among all types of Morley rank at least $\mathrm{RM}(p)$.

Theorem 2.29. *The following are equivalent for a Noetherian theory T:*
(a) The theory T has Noetherian isolation.
(b) For every formula $\theta(x,a)$ with parameters in a model of T, we have that $\mathrm{RM}\,\theta(x,a) = \mathrm{R}_{\mathcal{F}}\,\theta(x,a)$.
(c) For every consistent formula $\theta(x,a)$ with parameters in some model T, we have that $\mathrm{RM}\big(\overline{\theta(x,a)} \wedge \neg\theta(x,a)\big) < \mathrm{RM}\,\theta(x,a)$

Proof. For (a)⇒(b): By Lemma 2.17 and Corollary 2.26, it is enough to show that $\mathrm{RM}(p) \geq \mathrm{R}_{\mathcal{F}}(p)$ for all types over an $\aleph_0$-saturated model M. We proceed by induction on $\alpha = \mathrm{RM}(p)$. By assumption, the minimal formula $\varphi(x,a)$ of p isolates p among all types over M of Morley rank at least α, so $\mathrm{RM}(q) < \alpha$ for all types $q \neq p$ containing $\varphi(x,a)$. By induction, we have that $\mathrm{R}_{\mathcal{F}}(q) < \alpha$, so it follows that $\mathrm{R}_{\mathcal{F}}\,\psi(x,b) < \alpha$ for all irreducible proper subformulas of $\varphi(x,a)$, and thus $\mathrm{R}_{\mathcal{F}}\,\varphi(x,a) \leq \alpha$, as desired.

The implication (b)⇒(c) follows from Lemma 2.8, so we need only show (c)⇒(a). Consider a type $p(x)$ over A and let $\theta(x,a)$ in p isolate it among types over A of Morley rank at least $\mathrm{RM}(p)$. Since $\mathrm{RM}\big(\overline{\theta(x,a)} \wedge \neg\theta(x,a)\big) < \mathrm{RM}\,\theta(x,a)$, we have that $\overline{\theta(x,a)}$ is a closed formula which also isolates p among types over A of Morley rank at least $\mathrm{RM}(p)$. Hence, the theory T has Noetherian isolation, as desired. □

Notice that the above proof yields immediately the following corollary.

Corollary 2.30. *Suppose that every type over a model M of T is isolated by its minimal formula among all types over M of Morley rank at least* $\mathrm{RM}(p)$*. Then T has Noetherian isolation.* □

Remark 2.31. It is easy to see that a theory has Noetherian isolation if Morley rank and the foundational rank agree on closed formulas.

Corollary 2.32. *If the Noetherian theory T has Noetherian isolation, then Morley degree of a formula $\theta(x,a)$ equals* $\deg_{\mathcal{F}}\theta(x,a)$*.*

Proof. By Lemma 2.8 and Theorem 2.29 part (c), we may assume that $\theta(x,a)$ is a closed formula. Furthermore, we may also assume that our ambient model M is $\aleph_0$-saturated. Now, Morley degree of $\theta(x,a)$ is the number of types over M containing

$\theta(x,a)$ of Morley rank $\mathrm{RM}\,\theta(x,a)$. Since Morley rank and the foundational rank are the same, we have that Morley degree is exactly $\deg_{\mathcal{F}}\theta(x,a)$, by Corollary 2.24. $\square$

Corollary 2.33. *If the Noetherian theory T has Noetherian isolation, then for every type p over a set A with minimal formula $\varphi(x,a)$ we have that* $\mathrm{RM}(p) = \mathrm{R}_{\mathcal{F}}\,\varphi(x,a)$ *and its Morley degree is* $\deg_{\mathcal{F}}\varphi(x,a)$. $\square$

We will conclude this section with an easy observation regarding imaginaries in Noetherian theories (cf. [16, Corollary 2.8]).

Remark 2.34. Given a complete Noetherian theory T with respect to the Noetherian family $\mathcal{F}$ of formulae, the theory T has weak elimination of imaginaries after adding sorts for the canonical parameter of instances of formulae in $\mathcal{F}$.

Moreover, if $\mathcal{F}$ is closed under finite disjunctions, then every imaginary is interdefinable with the canonical parameter of some instance of a formula in $\mathcal{F}$.

Proof. Consider an $\emptyset$-definable equivalence relation E and an equivalence class $E(x,a)$. By Remark 2.4, the closure of the constructible subset $E(x,a)$ can be written as an irredundant union of its irreducible components $C_1,\ldots,C_n$. Write each C_i as $C_i = \varphi_i(x,b_i)$ for some formula φ_i in $\mathcal{F}$. Clearly, the tuple of canonical parameters $\bar\eta = (\ulcorner\varphi_1(x,b_1)\urcorner,\ldots,\ulcorner\varphi_n(x,b_n)\urcorner)$ is algebraic over the canonical parameter of the closure, and thus algebraic over $\ulcorner E(x,a)\urcorner$.

Now, the canonical parameter of the closure is clearly definable over the tuple $\bar\eta$, so we need only show that $\ulcorner E(x,a)\urcorner$ is definable over the canonical parameter of its closure. Otherwise, there would be an automorphism σ such that constructible set $E(x,\sigma(a))$ differs (and thus is disjoint) from $E(x,a)$, but they have the same closure, which is not possible since $E(x,a)$ and $E(x,\sigma(a))$ are constructible.

If $\mathcal{F}$ is now closed under finite disjunctions, then the closure of $E(x,a)$ is given by a single instance $\varphi(x,c)$ of a formula $\varphi(x,y)$ in $\mathcal{F}$, so $\ulcorner E(x,a)\urcorner$ and $\ulcorner\varphi(x,c)\urcorner$ are interdefinable, as desired. $\square$

3. Proper pairs of algebraically closed fields

As mentioned in the introduction, we will show in Sections 4 and 6 that the theory of proper pairs of algebraically closed fields is Noetherian. Most of the results mentioned here already appear in [10, 18], unless explicitly stated.

A *pair* (K,E) *of algebraically closed fields* is an extension $E \subset K$ of algebraically closed fields. Every pair is a structure in the language $\mathcal{L}_P = \mathcal{L}_{ring} \cup \{P\}$ with $E = P(K)$. If $E = K$, the pair is bi-interdefinable with the theory ACF of algebraically closed fields, which is clearly Noetherian, since definable sets are Zariski constructible.

Henceforth, we will restrict from now on our attention to proper pairs (K,E) of algebraically closed fields, so $E \subsetneq K$.

We denote by ACFP the $\mathcal{L}_P$-theory of proper pairs of algebraically closed fields, which expands the incomplete $\mathcal{L}_{ring}$-theory ACF of algebraically closed fields. We will use the index P to refer to the theory ACFP. In particular, given a subset of parameters A of K, by $\mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ we mean its definable and algebraic closures in the pure field language, whereas $\mathrm{dcl}_P(A)$ or $\mathrm{acl}_P(A)$ mean the corresponding objects in the structure of the pair (K,E). In particular, the independence symbol $\mathop{\smile\kern-0.9em\raise0.9ex\hbox{$\scriptstyle|$}}$ refers to the algebraic independence in the reduct ACF.

As shown by Robinson [20], every completion of the theory ACFP is obtained by fixing the characteristic. Each of these completions is ω-stable of Morley rank ω [19, p. 1659]. The induced structure on the proper subfield E agrees with its structure as a pure field, so E has Morley rank 1. Over any subset of parameters A of K, there is a unique type of Morley rank ω over A given by an element (inside a sufficiently saturated pair) which is transcendental over the compositum field $E \cdot \mathrm{Quot}(A)$, where $\mathrm{Quot}(A)$ denotes the subfield generated by A.

Delon [3] provided a definable expansion of the language $\mathcal{L}_P$ for ACFP to have quantifier elimination. The suggested language is $\mathcal{L}_D = \mathcal{L}_P \cup \{\mathrm{Ind}_n, \lambda_n^i\}_{1\leq i\leq n\in\mathbb{N}}$, where $K \models \mathrm{Ind}_n(a_1, \ldots, a_n)$ if and only if $a_1, \ldots, a_n$ are E-linearly independent. Each function λ_n^i takes values in E. If $a_1, \ldots, a_n$ are E-linearly independent, but $a_0, a_1, \ldots, a_n$ are not, its values are determined by $a_0 = \sum\limits_{i=1}^{n} \lambda_n^i(a_0; a_1 \ldots, a_n)\, a_i$. Otherwise, the value $\lambda_n^i(a_0; a_1 \ldots, a_n)$ is zero.

Notation. From now on, given a set A of parameters, we will denote by $\lambda(A)$ the subfield of E generated by the values of the λ-functions applied to tuples of A.

Note that the subfield $E \cap k$ is always contained in $\lambda(k)$, since an element a belongs to E if and only if $a = \lambda(a; 1)$.

Remark 3.1. For every subfield k of K, we have that $\lambda(k)$ is the smallest subfield F of E such that $F \cdot k$ and E are linearly disjoint over F. This implies $\lambda(k(e)) = \lambda(k)(e)$ for every tuple e in E. In particular,

$$\lambda(k \cdot \lambda(k)) = \lambda(k).$$

Lemma 3.2. *Let $k \subset L$ be subfields of K. Then,*

$$L \underset{\lambda(k)\cdot k}{\downarrow} E \;\Leftrightarrow\; \lambda(L) \subset \mathrm{acl}\big(\lambda(k)\big).$$

Proof. By Remark 3.1, we have $\lambda(k) \cdot k \downarrow_{\lambda(k)} E$. Now, transitivity and monotonicity of non-forking yield that

$$L \underset{\lambda(k)\cdot k}{\downarrow} E \;\Leftrightarrow\; L \underset{\lambda(k)}{\downarrow} E.$$

The latter condition is equivalent to $\mathrm{acl}(\lambda(k)) \cdot L \downarrow^{\mathrm{ld}}_{\mathrm{acl}(\lambda(k))} E$, which again by Remark 3.1 is equivalent to $\lambda(L) \subset \mathrm{acl}(\lambda(k))$. □

Definition 3.3. A subfield k of K is λ-*closed* if $\lambda(k)$ is a subfield of k, or equivalently, if k is linearly disjoint from E over the subfield $E \cap k$, which again is equivalent to $E \cap k = \lambda(k)$.

A straightforward application of Remark 3.1 and Lemma 3.2 to the subfields $k(e) \subset k(a, e)$ yields the following result:

Corollary 3.4. *If k is λ-closed, then for all tuples a in K and e in E, we have*

$$a \underset{k(e)}{\downarrow} E \;\Leftrightarrow\; \lambda(k(a)) \subset \mathrm{acl}\big(\lambda(k)(e)\big)$$

□

Fact 3.5. ([3, Théorème 1]) The fraction field of an $\mathcal{L}_D$-substructure is λ-closed. The $\mathcal{L}_P$-type of a λ-closed field (seen as a long tuple with respect to some fixed enumeration) is uniquely determined by its quantifier-free $\mathcal{L}_P$-type, so the theory ACFP has quantifier elimination in the language $\mathcal{L}_D$.

Remark 3.6. Every $\mathcal{L}_P$-definably closed subset of a model (K, E) of ACFP is λ-closed as a subfield of K. Moreover, if a subfield k is λ-closed, then its $\mathcal{L}_P$-definable closure is its inseparable closure k^{ins} and its $\mathcal{L}_P$-algebraic closure is the field algebraic closure k^{alg}.

We now introduce (cf. [14, Definition 6.3]) the collection of *tame formulae*, which will be shown in Sections 4 and 6 to be Noetherian.

Definition 3.7. A *tame formula* on the tuple x of variables is an $\mathcal{L}_P$-formula of the form

$$\exists \zeta \in P^r \Big(\neg \zeta \doteq 0 \ \wedge \ \bigwedge_{j=1}^{m} q_j(x, \zeta) \doteq 0 \Big)$$

for some polynomials $q_1, \ldots, q_m$ in $\mathbb{Z}[X, Z]$, homogeneous in the variables Z.

Fact 3.8. ([14, Lemma 6.4, Corollaries 6.5 and 6.8 & Proposition 7.3])

- Given polynomials $q_1, \ldots, q_m$ in $\mathbb{Z}[X, Y, Z]$ homogeneous in the variables Y and Z separately, the $\mathcal{L}_P$-formula

$$\exists \xi \in P^r \ \exists \zeta \in P^s \Big(\neg \xi \doteq 0 \ \wedge \ \neg \zeta \doteq 0 \ \wedge \ \bigwedge_{j=1}^{m} q_j(x, \xi, \zeta) \doteq 0 \Big)$$

  is equivalent in ACFP to a tame formula.
- The collection of tame formulae is, up to equivalence, closed under finite conjunctions and disjunctions.
- Every tame formula, in any partition of the variables, is an equation (cf. Remark 2.13).
- Every $\mathcal{L}_P$-formula is equivalent modulo ACFP to a Boolean combination of tame formulae, so ACFP is equational.

The fundamental reason why tame formulae are equations is due to the following observation, which will be again relevant in order to show that ACFP is Noetherian:

Remark 3.9. Projective varieties are *complete*: Given a projective variety Z and an algebraic variety X, the projection map $X \times Z \to X$ is closed with respect to the Zariski topology.

By Fact 3.8, in order to show that ACFP is Noetherian, we need only show in Sections 4 and 6 that the family of instances of tame formulae, which is already closed under finite intersections, has the DCC. For this, we need a couple of auxiliary lemmata. The next result already appeared implicitly in [14, Lemma 7.2] (or Lemma 9.1 in the ArXiv version), so we will not provide a proof thereof.

Lemma 3.10. *Let k be a λ-closed subfield of K. For every instance $\varphi(x, a)$ of a tame formula φ with parameters in k, there exists a Zariski closed subset V of $E^{|x|}$ defined over $\lambda(k)$ such that for every e in E,*

$$(K, E) \models \varphi(e, a) \iff e \in V.$$

If the polynomials in φ are homogeneous in X, then so are the polynomials defining V. □

We will finish this section by showing (cf. Remark 2.15) that every *closed* formula (as in Definition 2.12 with $\mathcal{F}$ the family of tame formulae) over a subset A of parameters is indeed equivalent to an instance of a tame formula with parameters over A. This result resonates with [9, Proposition 2.9].

Proposition 3.11. *An instance of a tame formula which is equivalent to an $\mathcal{L}_P$-formula with parameters in A is equivalent to an instance of a tame formula with parameters over A.*

Proof. To render the presentation of the proof more structured, we will first prove some intermediate claims.

Claim 1. Every instance of a tame formula with parameters in $\mathrm{dcl}_P(A)$ is equivalent to an instance of a tame formula over A.

Proof of Claim 1. By Remark 3.6, the definable closure $\mathrm{dcl}_P(A)$ is the inseparable closure of the smallest λ-closed subfield containing A. Thus, the parameters from $\mathrm{dcl}_P(A)$ are obtained from A using the ring operations, as well as inversion, extraction of p^{th}-roots (if the characteristic of K is $p > 0$) and applying the λ-functions. The cases of the ring operations, inversion and extraction of p^{th}-roots are easy (for the distinguished algebraically closed subfield E is perfect), so we will solely focus on the application of the λ-functions. For the sake of the presentation, assume that $a_0, \dots, a_n$ are elements of A with $e_1 = \lambda^1_n(a_0; a_1, \dots, a_n) \neq 0$ (so $a_1, \dots, a_n$ are linearly independent over E). Consider now the instance

$$\varphi(x, a', e_1) = \exists \zeta \in P^r \Big(\neg\, \zeta \doteq 0 \ \wedge \ \bigwedge_{j=1}^{m} q_j(x, a', e_1, \zeta) \doteq 0 \Big)$$

of a tame formula, where a' is a tuple in A containing $a_0, \dots, a_n$. Let N be the largest integer such that e_1^N occurs non-trivially in some q_j. Set now

$$\psi(x, a') = \exists \zeta \in P^r \ \exists \xi \in P^{n+1} \Big(\neg\, \zeta \doteq 0 \ \wedge \ \neg\, \xi \doteq 0 \ \wedge \\ \xi_0 a_0 = \sum_{i=1}^{n} \xi_i a_i \ \wedge \ \bigwedge_{j=1}^{m} \xi_0^N q_j(x, a', \frac{\xi_1}{\xi_0}, \zeta) \doteq 0 \Big),$$

which is an instance of a tame formula over A by Fact 3.8. Observe that the element ξ_0 in $\psi(x, a')$ is never 0, for $a_1, \dots, a_n$ are linearly independent over E, so $e_1 = \frac{\xi_1}{\xi_0}$. Thus, the two instances are equivalent, as desired. $\square_{\text{Claim 1}}$

Claim 2. An instance of a tame formula with parameters in $\mathrm{acl}_P(A)$ which is equivalent to an $\mathcal{L}_P$-formula with parameters in A is equivalent to an instance of a tame formula over A.

Proof of Claim 2. Suppose the element b is algebraic over A and consider the instance

$$\varphi(x, b) = \exists \zeta \in P^r \Big(\neg\, \zeta \doteq 0 \ \wedge \ \bigwedge_{j=1}^{m} q_j(x, b, \zeta) \doteq 0 \Big),$$

or equivalently using a different notation

$$\varphi(x, b) = \exists \zeta \in P^r \Big(\neg\, \zeta \doteq 0 \ \wedge \ (x, \zeta) \in V\big(I(b)\big) \Big),$$

where $I(b)$ is the ideal of $K[X,Z]$, homogeneous in Z, generated by $q_1(X,b,Z)$,..., $q_n(X,b,Z)$.

Let now $b=b_1,\dots,b_n$ be the $\mathcal{L}_P$-conjugates of b over A. Since $\varphi(x,b)$ is equivalent to an $\mathcal{L}_P$-formula with parameters in A, we have that $\varphi(x,b)$ is equivalent to the disjunction $\bigvee_{i=1}^n \varphi(x,b_i)$, which is again an instance of a tame formula, namely

$$\exists\,\zeta\in P^r\big(\neg\,\zeta\doteq 0\ \wedge\ (x,\zeta)\in \mathrm{V}(J)\big),$$

where J is the product ideal $I(b_1)\cdots I(b_n)$. The ideal J is invariant under all automorphisms of (K,E) fixing A pointwise, so by Weil's theorem its field of definition is contained in $\mathrm{dcl}_P(A)$. Hence, the ideal J can be generated by polynomials over $\mathrm{dcl}_P(A)$ which are homogeneous in Z. Therefore, the instance $\varphi(x,b)$ is equivalent to an instance of a tame formula with parameters in $\mathrm{dcl}_P(A)$. By Claim 1, we conclude that $\varphi(x,b)$ is equivalent to an instance of a tame formula with parameters over A, as desired. $\square_{\text{Claim 2}}$

We now have all the ingredients to prove the statement of the proposition. Consider thus an instance $\varphi(x,b)$ of a tame formula and assume that $\varphi(x,b)$ is equivalent to an $\mathcal{L}_p$-formula $\theta(x,a)$ with parameters over A. Consider first the case that A is not fully contained in E, so by Fact 3.5 and Remark 3.6, the subset $\mathrm{acl}_P(A)$ is the universe of an elementary substructure k of (K,E). Hence, there is some b' in $\mathrm{acl}_P(A)$ such that $\varphi(x,b')$ is equivalent to $\theta(x,a)$ (and thus to $\varphi(x,b)$). We deduce from Claim 2 that $\varphi(x,b)$ is equivalent to an instance of a tame formula over A, as desired.

Thus, we need only consider the case that the parameter set A is a subset of E. Choose a small elementary substructure k of (K,E) containing A. By saturation, there is some element a' in K which is transcendental over the subfield $E\cdot k$. Set now $A'=A\cup\{a'\}$ and deduce from the above paragraph as well as from Claim 2 that $\varphi(x,b)$ is equivalent to an instance

$$\varphi_1(x,a,a')=\exists\,\zeta\in P^r\left(\neg\,\zeta\doteq 0\ \wedge\ \bigwedge_{j=1}^m q_j(x,a,a',\zeta)\doteq 0\right),$$

where a is a tuple of elements in A. Let c in k be a realisation of $\varphi_1(x,a,a')$ and e some tuple in E . Since a' is transcendental over $E\cdot k$, we have that $q_j(c,a,a',e)=0$ if and only if the polynomial $q_j(c,a,Y,e)$ is the trivial polynomial (which is equivalent to a finite system of polynomial equations). Set now

$$\varphi_2(x,a)=\exists\,\zeta\in P^r\left(\neg\,\zeta\doteq 0\ \wedge\ \bigwedge_{j=1}^m q_j(x,a,Y',\zeta)\doteq 0\right),$$

which is again an instance of a tame formula with parameters in A. It is now clear that

$$\theta(k,a)=\varphi(k,b)=\varphi_1(k,a,a')=\varphi_2(k,a).$$

Since $\theta(x,a)$ and $\varphi_1(x,a)$ have parameters in $A\subset k$, we conclude that $\varphi_2(x,a)$ is equivalent to $\theta(x,a)$, and thus to $\varphi(x,b)$, as desired. $\square$

4. An indirect proof of the Noetherianity of ACFP

In this section we will give a simple proof that the instances of tame formulae have the DCC in the theory ACFP of proper pairs of algebraically closed fields. Whilst the methods we will use for the proof are elementary, using the strength of

Corollary 2.14, they do not explicitly allow to produce the minimal tame formula in a given type. The subsequent Section 6 will provide an explicit description of the minimal tame formula of a given type using results on the Hilbert polynomials of saturated ideals.

Proposition 4.1. *Given a λ-closed subfield k and a finite tuple a of K, there is some $\mathcal{L}_P$-formula $\theta(x)$ in $\mathrm{tp}_P(a/k)$ which implies every instance of a tame formula in $\mathrm{tp}_P(a/k)$.*

Proof. With respect to a fixed compatible total order of the collection of monomials on X, choose a Gröbner basis $r_1, \ldots, r_m$ of the vanishing ideal $\mathrm{I}(a/E \cdot k)$, that is, a collection of polynomials $r_1, \ldots, r_m$ in the ideal $\mathrm{I}(a/E \cdot k)$ such that the leading (or *initial*) monomial (with respect to the fixed total order) of every polynomial in the ideal is divisible by the leading monomial of one of the r_j's. Such a set is always a generating set for the ideal thanks to the corresponding division algorithm of Gröbner [12, Proposition 5.4.2 & Corollary 5.4.5].

Clearing denominators, we may assume that each r_i has coefficients in the ring $k[E]$ generated by $k \cup E$. Since every element of the ring $k[E]$ is a sum of products of the form $b \cdot e'$, where b is in k and e' is an element of E, we may write (after possibly adding additional variables) each r_i as $r_i = r_i(X, e)$, where $r_i(X, Z)$ is a polynomial in $k[X, Z]$, linear in Z and e is a tuple from E. Let $\sigma_i(e)$ be the leading coefficient of $r_i(x, e)$ with respect to our compatible total order and denote by $\sigma(e)$ the product of all the $\sigma_i(e)$'s. Choosing a system of generators of the vanishing ideal $I(e/k)$, denote by $\gamma(Z)$ the locus of e over $k \cap E = \lambda(k)$.

We will show that the $\mathcal{L}_P$-formula

$$\theta(x) = \exists\, \zeta \in P \left(\gamma(\zeta) \wedge \neg\, \sigma(\zeta) \doteq 0 \wedge \bigwedge_{i=1}^{l} r_i(x, \zeta) \doteq 0 \right)$$

has the desired property as in the statement. Notice first of all that the above formula belongs to $\mathrm{tp}_P(a/k)$, setting $\zeta = e$.

Consider now an instance of a tame formula

$$\varphi(x) = \exists\, \zeta' \in P \left(\neg\, \zeta' \doteq 0 \ \wedge \ \bigwedge_{j=1}^{m} q_j(x, \zeta') \doteq 0 \right)$$

in $\mathrm{tp}_P(a/k)$, where the polynomials q_j in $k[X, Z']$ are homogeneous in the tuple of variables Z'. Since a realises φ, there exists a non-trivial tuple e' in E with $q_j(a, e') = 0$ for every $1 \leq j \leq m$. Now, each polynomial $q_j(X, e')$ belongs to $\mathrm{I}(a/E \cdot k)$, so for large enough N and each j Gröbner's division algorithm allows us to write

$$\sigma(e)^N q_j(X, e') = \sum_{i=1}^{l} h_{j,i}(X, e, e') r_i(X, e)$$

for some polynomials $h_{j,i}(X, Z, Z')$ over k, homogeneous in the tuple of variables Z' of the same degree as $q_j(X, Z')$.

The formula

$$\rho(\zeta) = \exists\, \zeta' \in P \left(\neg\, \zeta' \doteq 0 \ \wedge \ \bigwedge_{j=1}^{m} \left(\sigma(\zeta)^N q_j(X, \zeta') \ - \sum_{i=1}^{l} h_{l,i}(X, \zeta, \zeta') r_i(X, \zeta) \doteq 0 \right) \right)$$

is an instance of a tame formula in the type $\mathrm{tp}_P(e/k)$. Since k is λ-closed, Lemma 3.10 yields that $\rho(E)$ is equivalent to the E-rational points of a Zariski closed set defined over $\lambda(k) = k \cap E$. In particular, every solution of the locus $\gamma(\zeta)$ of e over k must satisfy $\rho(\zeta)$.

Choose now a realisation b of the above formula $\theta(x)$ and let f be the corresponding tuple from E. Since f is a solution of $\gamma(\zeta)$, it realises $\rho(\zeta)$, so there exists a non-trivial tuple f' in E such that $\sigma(f)^N q_j(X, f') = \sum_{i=1}^{l} h_{j,i}(X, f, f') r_i(X, f)$ for every $1 \leq j \leq m$. Since $r_i(b, f) = 0$ for all i yet $\sigma(f) \neq 0$, it follows that $q_j(b, f') = 0$ for all $1 \leq j \leq m$. We conclude that every realisation b of θ realises φ, as desired. □

Corollary 2.14, Fact 3.8 and Proposition 4.1 immediately yield the following.

Corollary 4.2. *The theory* ACFP *of proper pairs of algebraically closed fields is Noetherian with respect to the collection of tame formulae.* □

By Remark 2.15 and Proposition 3.11, we deduce the following result.

Corollary 4.3. *Every type over a subset A of K contains an instance $\varphi(x, a)$ of a tame formula which implies every formula in p which is equivalent to an instance of a tame formula.* □

5. Morley, Lascar and Poizat

In [19, Subsection 2.2, p. 1660], Poizat stated (without proof) that the following rank equality holds for every type $p = \mathrm{tp}(a/k)$ over an elementary substructure k of a sufficiently saturated proper pair (K, E) of algebraically closed fields:

$$\mathrm{U}(p) = \mathrm{RM}(p) = \omega \cdot \mathrm{tr}(a/E \cdot k) + \mathrm{tr}(\alpha/E \cap k),$$

where α is the canonical base in the reduct ACF of $k(a)$ over E. (Note that there is a misprint in [19]). He deduced from the above identity that the dimension associated to the unique generic type of K is additive. Though Poizat's formula (and its proof) is probably well-known, we will nonetheless take the opportunity to give a detailed proof in this section. Our proof yields in particular that the Morley rank of a type over a λ-closed subfield can be isolated by a tame formula, and thus the theory ACFP admits Noetherian isolation, by Corollary 2.30. In order to prove Poizat's formula, we will need some auxiliary results regarding the behaviour of non-forking independence in ACFP.

The theory ACFP of proper pairs of algebraically closed fields is a particular case of a more general construction due to Poizat [18], who showed that the common theory of *belles paires* of models of a stable theory T is again stable whenever T *does not have the finite cover property (*nfcp*)*. For a stable theory, nfcp is equivalent [21, Chapter II, Theorem 4.4] to the elimination of $\exists^\infty$ in T^{eq}. The theory of algebraically closed fields eliminates both imaginaries and the quantifier $\exists^\infty$, so it has nfcp. However, there are Noetherian theories with fcp, as the following example shows.

Remark 5.1. In the language $\mathcal{L}$ consisting of a single binary relation $E(x, y)$ for an equivalence relation, consider the theory of the $\mathcal{L}$-structures which have exactly one equivalence class of size n for every $1 \leq n$ in $\mathbb{N}$. This theory is ω-stable of Morley rank 2 and admits quantifier elimination after adding countably many constant symbols as canonical representatives of the finite equivalence classes. In particular, this theory is Noetherian, but does not eliminate $\exists^\infty$, witnessed by the formula $E(x, y)$.

In [1], Ben Yaacov, Pillay and Vassiliev generalised Poizat's *belles paires* of stable structures to pairs of models of a simple theory. Akin to the result of Poizat, when the corresponding theory of pairs is first-order, then it is again simple. Moreover, non-forking independence in the theory of the pair (which we will denote by $\mathop{\smile\!\!\!\!\!\!\mid}^{P}$) can be characterised in terms of the independence(s) in the $\mathcal{L}$-reduct of the sets as well as of the canonical bases over the predicate, which in our setting corresponds to taking λ-closures, up to interalgebraicity.

All throughout this section, work inside a sufficiently saturated proper pair (K, E) of the theory ACFP of proper pairs of algebraically closed fields. All subsets and tuples considered are small with respect to the saturation of (K, E).

Fact 5.2. ([1, Remark 7.2 & Proposition 7.3]) Let a be a finite tuple and $k \subset L$ be subfields of K. We have the following description of non-forking:

$$a \mathop{\smile\!\!\!\!\!\!\mid}_{k}^{P} L \quad \text{if and only if} \quad \begin{cases} k(a) \mathop{\smile\!\!\!\!\!\!\mid}_{E\cdot k} E\cdot L \\ \text{and} \\ \lambda(k(a)) \mathop{\smile\!\!\!\!\!\!\mid}_{\lambda(k)} \lambda(L) \end{cases}$$

From the above description of non-forking, we easily deduce the following consequence.

Lemma 5.3. *Assume $k \subset L$ are subfields of K. Given a tuple a in K, whenever*

$$k(a) \mathop{\smile\!\!\!\!\!\!\mid}_{E\cdot k} E\cdot L,$$

we have that $\lambda(L(a))$ is interalgebraic with $\lambda(k(a))$ over $\lambda(L)$. In particular, if $a \mathop{\smile\!\!\!\!\!\!\mid}_{k}^{P} L$, then $\lambda(L(a))$ and $\lambda(k(a))$ are interalgebraic over $\lambda(L)$.

Proof. Note first that $\lambda(k(a))$ is contained in $\lambda(L(a))$. By Remark 3.1, we need only show that

$$\big(\lambda(L)\cdot\lambda(k(a))\big)^{\text{alg}}\cdot L(a) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{ld}}_{\left(\lambda(L)\cdot\lambda(k(a))\right)^{\text{alg}}} E,$$

or equivalently,

$$\lambda(k(a))\cdot L(a) \mathop{\smile\!\!\!\!\!\!\mid}_{\lambda(L)\cdot\lambda(k(a))} E.$$

Now, the fields $k(a)$ and E are linearly disjoint over $\lambda(k(a))$, so

$$\lambda(k(a))\cdot k(a) \mathop{\smile\!\!\!\!\!\!\mid}_{\lambda(k(a))\cdot k} E\cdot k.$$

Together with the assumption

$$k(a) \mathop{\smile\!\!\!\!\!\!\mid}_{E\cdot k} E\cdot L,$$

we deduce by transitivity of non-forking independence that

$$\lambda(k(a))\cdot k(a) \mathop{\smile\!\!\!\!\!\!\mid}_{\lambda(k(a))\cdot k\cdot} E\cdot L,$$

and thus

$$\lambda(k(a))\cdot L(a) \mathop{\smile\!\!\!\!\!\!\mid}_{\lambda(k(a))\cdot L} E\cdot L. \tag{$\star$}$$

Remark 3.1 yields that the subfield $\lambda(L) \cdot L$ is linearly disjoint from E over $\lambda(L)$, and therefore by monotonicity

$$\lambda(k(a)) \cdot L \underset{\lambda(L)\cdot\lambda(k(a))}{\downarrow} E.$$

Together with ($\star$), we conclude by transitivity that

$$\lambda(k(a)) \cdot L(a) \underset{\lambda(L)\cdot\lambda(k(a))}{\downarrow} E,$$

as desired. $\square$

Notation. Given a tuple a and a subfield k of K, set

$$\mathrm{rm}(a/k) = \omega \cdot \mathrm{tr}(a/E \cdot k) + \mathrm{tr}(\lambda(k(a))/\lambda(k)).$$

Poizat's formula now translates as $\mathrm{U}(a/k) = \mathrm{RM}(a/k) = \mathrm{rm}(a/k)$. In order to show that these three ranks agree, we will first show that the rank rm controls non-forking.

Lemma 5.4. *The ordinal-valued rank* rm *witnesses non-forking: Given subfields $k \subset L$ and a tuple a of K, we have that*

$$a \overset{P}{\underset{k}{\downarrow}} L \text{ if and only if } \mathrm{rm}(a/k) = \mathrm{rm}(a/L).$$

Proof. Adding to a set the values of its λ-functions does not affect non-forking independence in ACFP, since the λ-functions are $\mathcal{L}_P$-definable. Moreover, none of the transcendence degrees occurring in $\mathrm{rm}(a/k)$ change when passing from k to the intermediate λ-closed field extension $k \subset \lambda(k) \cdot k \subset E \cdot k$, so we may assume that k, and analogously L, is λ-closed.

We prove first that the rank rm remains constant when passing to a non-forking extension. By the description of non-forking in Fact 5.2, we have that

$$k(a) \underset{E\cdot k}{\downarrow} E \cdot L \quad \text{and} \quad \lambda(k(a)) \underset{\lambda(k)}{\downarrow} \lambda(L), \tag{$\natural$}$$

so $\mathrm{tr}(a/E \cdot k) \overset{(\natural)}{=} \mathrm{tr}(a/E \cdot L)$. Now, Lemma 5.3 yields that $\lambda(L(a))$ is interalgebraic with $\lambda(k(a))$ over $\lambda(L)$, so

$$\mathrm{tr}(\lambda(k(a))/\lambda(k)) \overset{(\natural)}{=} \mathrm{tr}(\lambda(k(a))/\lambda(L)) = \mathrm{tr}(\lambda(L(a))/\lambda(L)).$$

Therefore,

$$\begin{aligned}\mathrm{rm}(a/k) = \omega \cdot \mathrm{tr}(a/E \cdot k) + \mathrm{tr}(\lambda(k(a))/\lambda(k)) = \\ = \omega \cdot \mathrm{tr}(a/E \cdot L) + \mathrm{tr}(\lambda(L(a))/\lambda(L)) = \mathrm{rm}(a/L),\end{aligned}$$

as desired.

Let us now prove the converse: If $a \not\downarrow^{P}_{k} L$, then $\mathrm{rm}(a/L) < \mathrm{rm}(a/k)$. Again by Fact 5.2, one of the two independences in ($\natural$) cannot hold. If $k(a) \not\downarrow_{E\cdot k} E \cdot L$, the leading coefficient of ω in $\mathrm{rm}(a/L)$ is strictly smaller than the coefficient of ω in $\mathrm{rm}(a/k)$, so we are done. We may thus assume that $k(a) \downarrow_{E\cdot k} E \cdot L$ and hence $\lambda(L(a))$ is interalgebraic with $\lambda(k(a))$ over $\lambda(L)$ by Lemma 5.3. However

$$\lambda(k(a)) \underset{\lambda(k)}{\not\downarrow} \lambda(L),$$

so $\operatorname{tr}(\lambda(L(a))/\lambda(L)) = \operatorname{tr}(\lambda(k(a))/\lambda(L)) < \operatorname{tr}(\lambda(k(a))/\lambda(k))$. We conclude that $\operatorname{rm}(a/L) < \operatorname{rm}(a/k)$, as desired. □

In this section, we will show Poizat's formula in two steps: First, we show that the rank rm is *connected* (cf. Lemma 5.5), so it must be bounded from above by Lascar rank, as both ranks witness non-forking. We will then show that every type over a λ-closed subfield can be isolated by a tame formula among types of larger rm-rank, which will then give that the rank rm is bounded from below by Morley rank. Now, the inequality $\mathrm{U}(p) \leq \mathrm{RM}(p)$ always holds for all types, so putting all together we obtain the equality of all three ranks.

Lemma 5.5. *Consider a subfield k of K and a finite tuple a. Assume that $\alpha < \operatorname{rm}(a/k)$ for some ordinal number α. Then there is some field extension $k \subset L$ with $\alpha \leq \operatorname{rm}(a/L) < \operatorname{rm}(a/k)$. It follows that $\operatorname{rm}(p) \leq \mathrm{U}(p)$ for every type p, by Lemma 5.4.*

Proof. As in the proof of Lemma 5.4, we may assume that k is λ-closed. The proof follows immediately by transfinite induction from the following two claims:

Claim 1. If $\operatorname{rm}(a/k) = \beta + 1$, then there is some $L \supset k$ with $\operatorname{rm}(a/L) = \beta$.

Proof of Claim 1. Write

$$\beta + 1 = \operatorname{rm}(a/k) = \omega \cdot \operatorname{tr}(a/E \cdot k) + \operatorname{tr}(\lambda(k(a))/\lambda(k)),$$

so $0 < \operatorname{tr}(\lambda(k(a))/\lambda(k)) = m + 1$ for some natural number m. In particular, there is a transcendental element e in $\lambda(k(a))$ over $\lambda(k)$. Set $L = k(e)$. Notice that $\lambda(L) = \lambda(k)(e)$ by Remark 3.1, since k and E are linearly disjoint over $\lambda(k)$. Analogously, we have that $\lambda(L(a)) = \lambda(k(a))(e)$. As $\operatorname{tr}(\lambda(k(a))/\lambda(k)(e)) = m$, we conclude that

$$\operatorname{rm}(a/L) = \omega \cdot \operatorname{tr}(a/E \cdot L) + \operatorname{tr}(\lambda(L(a))/\lambda(L)) = \omega \cdot \operatorname{tr}(a/L \cdot E) + m = \beta,$$

as desired. $\square_{\text{Claim 1}}$

Claim 2. If $\operatorname{rm}(a/k) = \omega \cdot (n+1)$ for some n in $\mathbb{N}$, then for every m in $\mathbb{N}$ there is some field extension $k \subset L$ with $\omega \cdot n + m \leq \operatorname{rm}(a/L) < \operatorname{rm}(a/k)$.

Proof of Claim 2. Since

$$\omega \cdot (n+1) = \operatorname{rm}(a/k) = \omega \cdot \operatorname{tr}(a/E \cdot k) + \operatorname{tr}(\lambda(k(a))/\lambda(k)),$$

we deduce that $\lambda(k(a))$ is algebraic over $\lambda(k)$, so $k(a)$ and E are algebraically independent over $\lambda(k)$. Moreover, the transcendence degree $\operatorname{tr}(a/E \cdot k)$ is strictly positive, so choose some element c in $k(a)$ transcendental over $E \cdot k$. By saturation, there are elements $e_0, \ldots, e_{m-1}$ in E transcendental over k. Set $b = \sum_{i=0}^{m-1} e_i c^i$ and notice that that b and c are interalgebraic over $E \cdot k$. Thus, the element b is transcendental over $E \cdot k$. It follows that $\operatorname{tr}(a/E \cdot k(b)) = n < \operatorname{tr}(a/E \cdot k)$. Set thus $L = k(b)$ and notice that

$$\operatorname{rm}(a/L) < \operatorname{rm}(a/k).$$

The field L is trivially linearly disjoint from $E \cdot k$ over k, since b is transcendental over $E \cdot k$. Therefore, the field $\lambda(L)$ equals $\lambda(k)$ by Remark 3.1 and transitivity, for k is λ-closed.

Both elements b and c belong to $L(a)$, so $e_0, \ldots, e_{m-1}$ lie in $\lambda(L(a))$. Hence,

$$\operatorname{tr}(\lambda(L(a))/\lambda(L)) \geq \operatorname{tr}(e_0, \ldots, e_{m-1}/\lambda(k)) = m.$$

Therefore

$$\operatorname{rm}(a/k) > \operatorname{rm}(a/L) = \omega \cdot \operatorname{tr}(a/E \cdot L) + \operatorname{tr}(\lambda(L(a))/\lambda(L)) \geq \omega \cdot n + m,$$

as desired. $\square_{\text{Claim 2}}$ $\square$

We are now left to bounding the Morley rank from above in terms of the rank rm. We will do so in terms of an explicit tame formula χ which will isolate the type p among those types rm-rank at least $\operatorname{rm}(p)$. However, the tame formula χ we exhibit need not be the minimal tame formula in the type p (as in Corollary 2.14).

Proposition 5.6. *Consider a finite tuple a and a λ-closed subfield k of K. There exists an instance χ in $\operatorname{tp}_P(a/k)$ of a tame formula such that $\operatorname{rm}(b/k) \leq \operatorname{rm}(a/k)$ for every realisation b of χ in K. Moreover,*

$$\operatorname{rm}(b/k) = \operatorname{rm}(a/k) \iff \operatorname{tp}_P(b/k) = \operatorname{tp}_P(a/k)$$

Proof. Let $n = \operatorname{tr}(a/E \cdot k)$. Using [11, Theorem III.8] we conclude from

$$a \mathop{\smile\!\!\!\!\!\!|}^{\text{ld}}_{\lambda(k(a))\cdot k} E \cdot k,$$

that $\mathrm{I}(a/E \cdot k)$ has generators $r_1(X), \ldots, r_N(X)$ with coefficients in the ring generated by $k \cup \lambda(k(a))$, after possibly clearing denominators. Write thus $r_i(X) = r_i(X, e_i)$, for polynomials $r_i(X, Z) \in k[X, Z]$ and tuples e_i in $\lambda(k(a))$. As in the proof of Proposition 4.1, we may assume that each r_i is linear in Z with $r_i(X, f_i)$ non-zero whenever the tuple f_i from E is non-zero.

Setting now $n = \operatorname{tr}\big(a/k \cdot \lambda(k(a))\big)$, we may assume that for each $n+1$-element subtuple of a, one of the r_i's witnesses that this subtuple is algebraically dependent over $\lambda(k(a)) \cdot k$.

Consider the tuple $\bar{e} = (e_1, \ldots, e_N)$ as an element of $\mathbb{P}^{|e_1|} \times \cdots \times \mathbb{P}^{|e_N|}$ and denote by $\gamma(Z_1, \ldots, Z_N)$ the locus of $\bar{e}$ over k, homogeneous in every Z_i. Then the $\mathcal{L}_P$-formula

$$\chi(x) = \exists \zeta_1 \in P \ldots \exists \zeta_N \in P \Big(\bigwedge_{i=1}^{N} \neg \zeta_i \doteq 0 \ \wedge \ \gamma(\zeta_1, \ldots, \zeta_N) \ \wedge \ \bigwedge_{i=1}^{N} r_i(x, \zeta_i) \doteq 0 \Big)$$

in $\operatorname{tp}_P(a/k)$ is equivalent to a tame formula, by Fact 3.8.

Given a realisation b of χ, let $\bar{f} = (f_1, \ldots, f_N)$ be non-trivial tuples in E with $r_1(b, f_1) = \cdots = r_N(b, f_n) = 0$. Since the $r_i(X, f_i)$ are non-zero, it follows that

$$\operatorname{tr}(b/E \cdot k) \leq \operatorname{tr}(b/k(\bar{f})) \leq n.$$

If $\operatorname{tr}(b/E \cdot k) < n$, we have $\operatorname{rm}(b/k) < \operatorname{rm}(a/k)$. So let us assume from now on that $\operatorname{tr}(b/E \cdot k) = n$, so $b \mathop{\smile\!\!\!\!\!\!|}_{k(\bar{f})} E$. Corollary 3.4 yields now that $\lambda(k(b)) \subset \operatorname{acl}\big(\lambda(k)(\bar{f})\big)$, so $\operatorname{tr}(\lambda(k(b))/\lambda(k)) \leq \operatorname{tr}(\bar{f}/\lambda(k))$. Analogously, we have that

$$\operatorname{tr}(\lambda(k(a))/\lambda(k)) = \operatorname{tr}(\bar{e}/\lambda(k)),$$

since the tuple $\bar{e}$ belongs to $\lambda(k(a))$. Since $\bar{f}$ satisfies $\gamma(\bar{Z})$, we have

$$\operatorname{tr}(\bar{f}/\lambda(k)) \leq \operatorname{tr}(\bar{e}/\lambda(k)).$$

If the inequality is strict, we deduce that $\operatorname{rm}(b/k) < \operatorname{rm}(a/k)$. Assume therefore that $\operatorname{tr}(\bar{f}/\lambda(k)) = \operatorname{tr}(\bar{e}/\lambda(k))$. We want to show that b and a have the same $\mathcal{L}_P$-type over k. First, observe that $\operatorname{tr}(\bar{f}/k) = \operatorname{tr}(\bar{e}/k)$, so $\bar{f}$ and $\bar{e}$ have the same type over k, as a sequence of homogeneous tuples. By Fact 3.5, using that k is λ-closed, we deduce that $\bar{f}$ and $\bar{e}$ have the same $\mathcal{L}_P$-type over k. Hence, there exists a tuple a' in K such that $(a', \bar{e})$ has the same $\mathcal{L}_P$-type over k as $(b, \bar{f})$. In particular, the tuple a' is a solution of $\mathrm{I}(a/k \cdot E)$ with $\mathrm{tr}(a'/k \cdot E) = n$. This implies that a' and a have the same type over $E \cdot k$, and thus the same $\mathcal{L}_P$-type over k by Fact 3.5, as desired. □

Corollary 5.7. *Given a subfield k of K and a finite tuple a, we have that* $\mathrm{RM}(a/k) \leq \mathrm{rm}(a/k)$.

Proof. Morley rank does not change working over independent parameters and neither does rm by Lemma 5.4. Thus, we may assume that k is an $\aleph_0$-saturated elementary substructure of the pair (K, E) and in particular λ-closed.

For the inequality $\mathrm{RM}(a/k) \leq \mathrm{rm}(a/k)$, it suffices to show inductively on α that $\alpha \leq \mathrm{rm}(a/k)$ whenever $\alpha \leq \mathrm{RM}(a/k)$. We need only consider the case $\alpha = \beta + 1$. Let χ be a tame formula in $\mathrm{tp}_P(a/k)$ as in Proposition 5.6. If $\beta + 1 \leq \mathrm{RM}(a/k)$, the type $\mathrm{tp}_P(a/k)$ is an accumulation point of types over k of Morley rank at least β. We may thus assume that there is a type $\mathrm{tp}_P(b/k) \neq \mathrm{tp}_P(a/k)$ of Morley rank at least β containing the formula χ, so

$$\mathrm{rm}(b/k) \overset{5.6}{<} \mathrm{rm}(a/k).$$

By induction on $\beta \leq \mathrm{RM}(b/k)$, we have that $\beta \leq \mathrm{rm}(b/k)$, and thus $\alpha = \beta + 1 \leq \mathrm{rm}(a/k)$, as desired. □

We deduce immediately from Lemma 5.5 as well as Corollary 5.7 that Poizat's formula holds for proper pairs of algebraically closed fields.

Corollary 5.8. *In every sufficiently saturated proper pair (K, E) of algebraically closed fields, Morley and Lascar rank agree: Given a finite tuple a and a subfield k of K,*

$$\mathrm{U}(a/k) = \mathrm{RM}(a/k) = \omega \cdot \mathrm{tr}(a/E \cdot k) + \mathrm{tr}(\lambda(k(a))/\lambda(k)).$$

By Fact 3.5, Propositions 3.11 (Claim 1) and 5.6 as well as Corollaries 2.30 and 5.8, we deduce the following result:

Corollary 5.9. *Given a type p over an arbitrary subset A of parameters of (K, E), there exists an instance of a tame formula in p which isolates it among all types over A of rank at least* $\mathrm{RM}(p)$. *In particular, the theory* ACFP *has Noetherian isolation.*

6. Minimal formulae and Hilbert schemes

Corollary 2.14 and Corollary 4.2 together yield that every type contains a minimal tame formula. The goal of this section, which can be seen as a complement, is to provide an explicit description of the minimal tame formula in a given type $\mathrm{tp}_P(a/k)$ whenever k is λ-closed. (It follows from Proposition 3.11 (Claim 1)) that the parameter set being λ-closed is not an actual obstacle). For this, we will use Grothendieck's Hilbert Scheme. Since the classical references we consulted do not explicitly provide the construction of the Hilbert scheme as a projective variety, we have decided to exhibit the construction here, in Theorem 6.3, keeping the exposition as elementary as possible. Most of the results from algebraic geometry in this section can be found in [5, 15], unless explicitly stated.

Fix a base field F, write X for the sequence of variables $X_0, \ldots, X_n$. Given a homogeneous ideal J of $F[X]$, let J_d be the collection of all homogeneous polynomials in J of degree d. It was shown by Hilbert [7] that the codimension of J_d is given in terms of a (unique) numerical polynomial $Q_J(T)$, i.e. having rational coefficients, yet taking integer values when evaluated on $\mathbb{Z}$, called the *Hilbert polynomial* of J: there exists a degree d_0 such that $Q_J(d) = \operatorname{codim}_F(J_d, F[X]_d)$ for $d \geq d_0$. It follows that

$$\dim_F J_d = \binom{d+n}{n} - Q_J(d)$$

is also a numerical polynomial in d for $d \geq d_0$.

Definition 6.1. A homogeneous ideal I of $F[X]$ is *saturated* if the homogeneous polynomial $p(X)$ belongs to I whenever $X_i^D \cdot p$ belongs to I for all $0 \leq i \leq n$ and D large enough.

The homogeneous vanishing ideal over F of a non-zero tuple in some field extension of F is clearly saturated. Given a homogeneous ideal J of $F[X]$, the smallest saturated ideal $\operatorname{Sat}(J)$ containing J consists of all homogeneous polynomials $p(\bar{X})$ such that for some D all the products $X_i^D \cdot p$ with $0 \leq i \leq n$ belong to J.

Remark 6.2. Let us point out that J_d and $\operatorname{Sat}(J)_d$ are equal for large enough d, and thus J and $\operatorname{Sat}(J)$ have the same Hilbert polynomial. Indeed, the ideal $\operatorname{Sat}(J)$ is finitely generated by the polynomials $p_1, \ldots, p_m$. In particular, there exists some $D \geq 1$ such that $X_i^D \cdot p_j$ belongs to J for all $0 \leq i \leq n$. Let d be greater than

$$(n+1)(D-1) + \max_{1 \leq j \leq m} \deg(p_j).$$

If p belongs to $\operatorname{Sat}(J)_d$, then p is a linear combination over F of products of the form $M_\alpha(X) \cdot p_j$ for some monomials $M_\alpha(X)$ with $\deg(M_\alpha) = d - \deg p_j > (n+1)(D-1)$. It follows that $M_\alpha(X)$ contains a factor of the form X_i^r with $r \geq D$ for some $0 \leq i \leq n$, so each $M_\alpha(X) \cdot p_j$, and hence p, belongs to J, as desired. »»»> master

Furthermore, two saturated ideals J and J' are equal whenever $J_d = J'_d$ for infinitely many $d's$: Indeed, if p belongs to J, then $X_i^D \cdot p$ is in J_d for $d \geq \deg(p)$ with $D = d - \deg(p)$.

Theorem 6.3 (Grothendiek's Hilbert scheme of $\mathbb{P}^n$)**.** *For every n and every numerical polynomial Q the collection of saturated ideals in $F[X]$ with Hilbert polynomial Q is in bijection with a projective variety $\mathcal{H}^n_Q(F) \subset \mathbb{P}^N(F)$ such that, denoting by $I^{(\eta)}$ the corresponding to a tuple η of $\mathcal{H}^n_Q(F)$ (in homogeneous coordinates), we have the following:*

(1) There is a finite set of polynomials $S(X, \eta)$ with integer coefficients, homogeneous both in X and in η which generate $I^{(\eta)}$ as a saturated ideal, that is, if $\langle S(X, \eta)\rangle$ denotes the ideal generated by $S(X, \eta)$, then $I^{(\eta)} = \operatorname{Sat}(\langle S(X, \eta)\rangle)$.

(2) For every degree d there is a finite set $S_d(Y, \eta)$ of polynomials with integer coefficients, homogeneous both in Y and in η, such that

$$\sum_{|\beta|=d} c_\beta X^\beta \in I^{(\eta)} \Leftrightarrow S_d(c, \eta) = 0$$

for every tuple $c = (c_\beta)$ in F.

The variety $\mathcal{H}_Q^n$ as well as the sets S and S_d are all defined over $\mathbb{Z}$ and do not depend on the field F.

Proof. We fix n and Q as in the statement. By a result of Mumford ([5, Theorem III-55 & Page 263] & [15, Lecture 14]) there is a degree d_0 such that for every *saturated* ideal I of $F[X]$ with Hilbert polynomial Q we have

$$\dim_F I_d = \binom{d+n}{n} - Q(d) \quad \text{for all } d \geq d_0.$$

Moreover, the ideal $\bigoplus_{d \geq d_0} I_d$ is generated by I_{d_0}. This has the following consequence.

Claim 1. The assignment $I \mapsto U = I_{d_0}$ defines a bijection between all saturated ideals I of $F[X_0, \ldots, X_n]$ with Hilbert polynomial Q and the set H_Q^n of all subspaces U of $F[X_0, \ldots, X_n]_{d_0}$ satisfying

$$\dim_F \langle U \rangle_d = \binom{d+n}{n} - Q(d) \quad \text{for all } d \geq d_0, \tag{$\bigstar$}$$

where $\langle U \rangle$ is the ideal generated by U.

Note that all elements of H_Q^n have the same dimension, namely $N_0 = \binom{d_0+n}{n} - Q(d_0)$.

Proof of Claim 1. If I is saturated with Hilbert polynomial Q, then $U = I_{d_0}$ belongs to $\mathcal{H}_Q^n$, since $\langle U \rangle_d = I_d$ for all $d \geq d_0$. This also shows that $I = \mathrm{Sat}\langle U \rangle$ is uniquely determined by U. Conversely, if U belongs to $\mathcal{H}_Q^n$, it follows that $I = \mathrm{Sat}(\langle U \rangle)$ has Hilbert polynomial Q, by Remark 6.2. In particular,

$$\dim_F I_{d_0} = N_0 = \dim_F U,$$

whence $I_{d_0} = U$, as desired. $\square_{\text{Claim 1}}$

We will see below that the collection of N_0-dimensional subspaces of $F[X]_{d_0}$ form a projective variety $\mathrm{Gr}_{N_0}(F[X]_{d_0})$, called the N_0^{th}-*Grassmannian.* Let $\mathcal{H}_Q^n$ be the subset of $\mathrm{Gr}_{N_0}(F[X]_{d_0})$ which corresponds to H_Q^n. Together with Claim 1, Claim 2 below yields that $\mathcal{H}_Q^n$ is a Zariski closed subset of $\mathrm{Gr}_{N_0}(F[X]_{d_0})$ definable without parameters (or defined over $\mathbb{Z}$ in algebraic terms). It is easy to see that every suitable choice of the degree d_0 yields the same variety $\mathcal{H}_Q^n$, up to canonical isomorphism.

Fix some s in $\mathbb{N}$. To view the set of all r-dimensional subspaces V of the vector space F^s as a projective variety, we will encode V by the exterior product $v_1 \wedge \cdots \wedge v_r$, where $v_1, \ldots, v_r$ is some basis of V. Up to a non-zero scalar factor this exterior product only depends on V, so it determines a unique element of the projective space $\mathbb{P}(\bigwedge^r F^s)$. Its (homogeneous) coordinates are the *Plücker coordinates* $\mathrm{Pk}(V)$ of V. Given Plücker coordinates $\mathrm{Pk}(V) = v_1 \wedge \cdots \wedge v_r$ in $\mathbb{P}(\bigwedge^r F^s)$, we recover V as the set of all vectors v in F^s such that $v \wedge (v_1 \wedge \cdots \wedge v_r) = 0$. The collection $\mathrm{Gr}_r(F^s)$ of Plücker coordinates η is given by the quadratic equations $\eta \wedge (e^* \lrcorner \eta) = 0$, where e^* runs through some basis of $\bigwedge^{r-1}(F^s)^*$ and the map

$$\lrcorner : \bigwedge^{r-1}(F^s)^* \times \bigwedge^r (F^s) \to F^s$$

is the (bilinear) inner product of exterior algebra (see [4, Page 182]). Thus, the r^{th}-Grassmannian $\mathrm{Gr}_r(F^s)$ of F^s is a projective variety.

Note that $\binom{s}{r} = \dim \bigwedge^r(F^s)$, so $\mathrm{Gr}_{N_0}(F[X]_{d_0})$ is a subvariety of $\mathbb{P}(F^N)$ with $N = \binom{\binom{d_0+n}{n}}{N_0}$.

We will now use the following result of Macaulay [13] (of which Sperner gave a simplified proof in [22]): *If Q is the Hilbert polynomial of some homogeneous ideal in $n+1$ variables and d_1 is large enough, then for any homogeneous ideal J of $F[X]$ with*

$$\dim J_{d_1} \geq \binom{d_1+n}{n} - Q(d_1),$$

it follows that

$$\dim J_d \geq \binom{d+n}{n} - Q(d) \quad \textit{for all } d \geq d_1.$$

Claim 2. For d_0 large enough, the set $\mathcal{H}^n_Q$ as defined above is a Zariski closed subset of $\mathrm{Gr}_{N_0} F[X]_{d_0}$, definable without parameters.

Proof of Claim 2. If no ideal of $F[X]$ has Q as Hilbert polynomial, then $\mathcal{H}^n_Q$ is empty and thus Zariski closed. Otherwise, we may assume that Q satisfies Macaulay's property with respect to d_1, which we may assume to be equal to d_0 as in Mumford's result. It now follows that an N_0-dimensional subspace U of $F[X]_{d_0}$ satisfies the condition ((✈)) of Claim 1 if and only if

$$\dim_F \langle U\rangle_d \leq \binom{d+n}{n} - Q(d) \quad \text{for all } d \geq d_0.$$

If η are the Plücker coordinates of U, the polynomials $e^* \lrcorner\, \eta$, where e^* runs among the elements of the canonical basis of $\bigwedge^{N_0-1}(F[X]_{d_0})^*$, generate U as an F–vector space [4, Résultats d'Algèbre (A,44) p.183, IX]. Thus, the graded component $\langle U\rangle_d$ of the ideal $\langle U\rangle$ is generated as an F–vector space by the polynomials $M_\alpha \cdot (e^* \lrcorner\, \eta)$, where the M_α's enumerate all monomials in the variables X of degree $d-d_0$. Hence, the Plücker coordinates η belong to $\mathcal{H}^n_Q$ if and only if for all $d \geq d_0$ the dimension of the vector space generated by the $M_\alpha \cdot (e^* \lrcorner\, \eta)$ is bounded by $\binom{d+n}{n} - Q(d)$. The last condition can be expressed by determinantal equations in the coefficients of η, as desired. $\square_{\text{Claim 2}}$

Property (1) follows easily from the proof of the last claim, if one defines $S(X, \eta)$ as the set of all $(e^* \lrcorner\, \eta)$, where $e^* = e^*(X)$ are elements of the canonical basis of $\bigwedge^{N_0-1}(F[X]_{d_0})^*$.

For property (2), fix some degree d and a polynomial $h(c, X) = \sum_{|\beta|=d} c_\beta X^\beta$ of degree d. If $d < d_0$, note that $h(c, X)$ belongs to $I^{(\eta)}$ if and only if all $X_i^{d_0} \cdot h(c, X)$ belong to $I^{(\eta)}$, for $I^{(\eta)}$ is saturated. Therefore, we may assume that the polynomial $h(c, X)$ has degree $d \geq d_0$, so $h(c, X)$ belongs to $I^{(\eta)}$ if and only if the F-vector space generated by $h(x, X)$ and all $M_\alpha \cdot (e^* \lrcorner\, \eta)$ where the M_α are monomials of degree $d-d_0$ has dimension at most $\binom{d+n}{n} - Q(d)$. Again, the latter can be expressed by determinantal equations $S_d(c, \eta) = 0$ in c and η. $\square$

We will see now how Theorem 6.3 produces the minimal tame formula in a given type $\mathrm{tp}_P(a/k)$, where k is a λ-closed subfield of a sufficiently saturated model (K, E) and $a = (a_1, \ldots, a_n)$ is a tuple in K.

The homogeneous vanishing ideal I of the tuple $(1, a_1, \ldots, a_n)$ over the subfield $F = E \cdot k$ is saturated, so denote by Q its Hilbert polynomial. By Theorem 6.3, let $\mathcal{H}^n_Q$ be the corresponding Hilbert scheme as well as S and S_d the associated finite sets of polynomials. The ideal I is of the form $I = I^{(\eta)}$ for some η in $\mathcal{H}^n_Q$. Clearing denominators, we may assume that the coefficients of η belong to the ring generated by $E \cup k$. As in the proof of Proposition 4.1, write $\eta = \eta(e)$ for some is a tuple $\eta(Z)$ of linear forms over k and a non-trivial tuple e in E. Rewriting the coefficients of η with respect to a fixed basis of k over $\lambda(k)$, which remains linearly independent over E by linear disjointness, we may assume that $\eta(f) \neq 0$ for every non-zero tuple f in E.

Denoting by $\gamma(\zeta)$ the homogeneous locus of e over k, we have the following:

Theorem 6.4. *The tame formula*

$$\varphi(x) = \exists \zeta \in P \left(\neg\, \zeta \doteq 0 \;\wedge\; \gamma(\zeta) \;\wedge\; \mathcal{H}^n_Q(\eta(\zeta)) \;\wedge\; S(1, x_1, \ldots, x_n, \eta(\zeta)) \doteq 0 \right)$$

is the minimal tame formula in $\mathrm{tp}_P(a/k)$.

Proof. It is clear that the tame formula $\varphi(x)$ belongs to $\mathrm{tp}_P(a/k)$, setting $\zeta = e$, since $S(X, \eta(e))$ is contained in $I = I^{\eta(e)}$.

Assume now that we are given an instance of a tame formula

$$\psi(x) = \exists\, \zeta' \in P \left(\neg\, \zeta' \doteq 0 \;\wedge\; \bigwedge_{j=1}^{m} q_j(1, x, \zeta') \doteq 0 \right)$$

in $\mathrm{tp}_P(a/k)$ for some polynomials q_ℓ in $k[X, Z']$, homogeneous both in X and in Z'. By Theorem 6.3 (2) there is for every degree $\deg(q_j)$ a finite set $S_{q_j}(Z', Z)$ of polynomials over k, homogeneous both in Z and in Z', such that for all tuples f and f' in E with $\eta(f)$ in $\mathcal{H}^n_Q$, then

$$q_j(X, f') \in I^{\eta(f)} \;\Leftrightarrow\; S_{q_j}(f', f) = 0.$$

Now, the tuple a realises ψ, so there exists a non-trivial tuple e' in E such that $q_j(1, a, e') = 0$ for all $1 \leq j \leq m$. Therefore, each polynomial $q_j(X, e')$ belongs to $I^{(\eta(e))}$, so $S_{q_j}(e', e) = 0$, by the above. Hence, the tuple e satisfies the tame formula

$$\rho(\zeta) = \exists\, \zeta' \in E \left(\neg\, \zeta' \doteq 0 \;\wedge\; \bigwedge_{j=1}^{m} S_{q_j}(\zeta', \zeta) \doteq 0 \right).$$

As k is λ-closed, Lemma 3.10 yields that for realisations in E the formula $\rho(\zeta)$ is equivalent to a finite system of equations $\sigma(\zeta)$ with coefficients in k, homogeneous in ζ. In particular, every solution in E of the homogeneous locus $\gamma(\zeta)$ of e over k is a solution of the system $\sigma(\zeta)$, and thus satisfies $\rho(\zeta)$.

Given now a realisation b of $\varphi(x)$, we need to show that b also satisfies $\psi(x)$, which gives that φ is the minimal tame formula in $\mathrm{tp}_P(a/k)$. Since b realises $\varphi(x)$, there is non-zero tuple f in E which is a solution of $\gamma(\zeta)$ such that $\eta(f)$ lies in $\mathcal{H}^n_Q$ and $S(1, b, \eta(f)) = 0$.

As f is a solution of $\gamma(\zeta)$, the above discussion yields that f realises $\rho(\zeta)$, so there is a non-zero tuple f' in E such that

$$\bigwedge_{j=1}^{m} S_{q_j}(f', f) = 0.$$

We deduce that every polynomial $q_j(X, f')$ belongs to $I^{\eta(f)}$. Since $S(1, b, \eta(f)) = 0$ and the polynomials in $S(X, \eta(f)$ generate $I^{\eta(f)}$ as a saturated ideal, it follows that $q_j(1, b, f') = 0$ for all $1 \leq j \leq m$, so b realises $\psi(x)$, as desired. □

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