Compactly expandable dense linear orderings

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Definition 0.1. *Let M be an L-structure, $L \subseteq L'$ and Σ a set of L'-sentences. We say that Σ is* satisfiable in M *if some expansion of M satisfies Σ. We say that Σ is* finitely satisfiable in M *if every finite subset of Σ is satisfiable in M.*

We say that the L-structure M is κ-compactly expandable *if for all L' every set of L'-sentences of cardinality $< \kappa$ which is finitely satisfiable in M is satisfiable in M. It is* κ-expandable *if every set of sentences of cardinality $< \kappa$ which is consistent with the complete theory of M is satisfiable in M. Finally, if $\kappa = |M|$ is the cardinality of M, we say that M is* compactly expandable *if it is κ^+-compactly expandable, and it is* expandable *if it is κ^+-expandable .*

Remark 0.2. *Let $(A, <)$ be compactly expandable dense linear order of cardinality ω_1.*

1. *η_1 is embeddable in $(A, <)$, if it exists.*

2. *All intervals in $(A, <)$ are uncountable.*

Let S_i be a binary predicate symbol for every natural number i, let f be a unary function symbol and consider the language $L = \{<, f\} \cup \{S_i \mid i \in \omega\}$. Let T be the L-theory given by the axioms of dense linear order without endpoints and the following axioms, for all i:

1. $\forall xy\, (S_i(x, y) \to x < y)$, for all i,

2. $\exists xy\, S_0(x, y)$

3. $\forall xy\, \Big(S_i(x, y) \to \forall z\, \exists x'y'\, \big(x < x' < y' < y \wedge S_{i+1}(x', y') \wedge \forall w\, (x' < w < y' \to f(w) > z)\big)\Big)$

Proposition 0.3. *The countable theory T is finitely satisfiable in $(\eta_1 \times \omega, <)$ but is not satisfiable in this model.*

Proof. We check first that no expansion of $\eta_1 \times \omega$ satisfies T. Assume M is such an expansion. Choose a cofinal increasing sequence $(c_i \mid i < \omega)$ and choose $a_0 < b_0$ such that $S_0(a_0, b_0)$. Inductively find $a_i < a_{i+1} < b_{i+1} < b_i$ for $i < \omega$ such that $S_i(a_i, b_i)$ and $f(x) > c_i$ for every x in the open interval (a_{i+1}, b_{i+1}). Every interval in $\eta_1 \times \omega$ is an η_1-order and hence there is some c such that $a_i < c < b_i$ for every $i < \omega$. But then, $f(c) > c_i$ for every i, in contradiction with the choice of $(c_i \mid i < \omega)$ as a cofinal sequence.

We finish the proof checking that T is finitely satisfiable in $\eta_1 \times \omega$. We build an expansion that satisfies the two basic axioms and the first n axioms of the third kind. Choose $(a_s, b_s \mid s \in \omega^{\leq n})$ such that $a_s < b_s$ and $\{(a_{s^\frown i}, b_{s^\frown i}) \mid i < \omega\}$ are pairwise disjoint intervals contained in the interval (a_s, b_s). For every $i \leq n$, define S_i as the set of all pairs a_s, b_s with length$(s) = i$. Now choose a cofinal increasing sequence $(c_i \mid i < \omega)$ and define inductively f as indicated in the interval $(a_\emptyset, b_\emptyset)$ (and arbitrarily everywhere else):

1. If $s \in \omega^{n-1}$ and $x \in (a_{s^\frown i}, b_{s^\frown i})$, then $f(x) = c_{n+i}$
2. If $s \in \omega^{m-1}$ with $m < n$ and $x \in (a_{s^\frown i}, b_{s^\frown i}) \setminus \bigcup_{j<\omega}(a_{s^\frown ij}, b_{s^\frown ij})$, then $f(x) = c_{m+i}$

Note than $s \in \omega^m$ implies $f(x) \geq c_m$ for every $x \in (a_s, b_s)$ □

Corollary 0.4. *The linear order $\eta_1 \times \omega$ is not compactly expandable, even for countable sets of sentences.*

Proof. By Proposition 0.3. □

Remark 0.5. *1. T is finitely satisfiable in every dense linear order of cofinality ω.*

2. T is satisfiable in a dense linear order of cofinality ω if and only if the order contains an (ω, ω)-Dedekind cut.

Proof. 1. This is what really gives the proof of Proposition 0.3.

2. On the one hand, if there are not (ω, ω)-Dedekind cuts, we can reproduce the proof of non satisfiablity of Proposition 0.3. Assume now there is such a cut and choose $a < b$ such that the interval (a, b) contains the cut. Let the increasing sequence $(a_i \mid i < \omega)$ and the decreasing sequence $(b_i \mid i < \omega)$

with $a < a_i < b_j < b$ define the cut and let $(c_i \mid i < \omega)$ be an increasing cofinal sequence. We define S_i as the set of all pairs $a_j b_j$ with $j \geq i$. If $x \in (a_i, b_i) \smallsetminus (a_{i+1}, b_{i+1})$ we put $f(x) = c_i$ and we define $f(x)$ arbitrarily everywhere else. This expansion satisfies T. □

The following theory T' is a parametrized version of T. The language is $L = \{<, f\} \cup \{S_i \mid i \in \omega\}$, where each S_i is now a 4-ary predicate symbol and f is a 3-ary function symbol. We write $S_i^{uv}(x, y)$ and $f^{uv}(x)$ for $S_i(u, v, x, y)$ and $f(u, v, x)$ respectively. The axioms of T' are the axioms of dense linear order without endpoints and:

1. $\forall uv\, xy\, (S_i^{uv}(x, y) \to u < x < y < v)$
2. $\forall uv\, \exists xy\, S_0^{uv}(x, y)$
3. $\forall uv\, xy\, \Big(S_i^{uv}(x, y) \to \forall z\, \exists x'y'\, \big(x < x' < y' < y \wedge S_{i+1}^{uv}(x', y') \wedge \forall w\, (x' < w < y' \to f^{uv}(w) > z)\big)\Big)$

Proposition 0.6. *The countable theory T' is finitely satisfiable in $\eta_1 \times \eta_0$ but is not satisfiable in this model.*

Proof. Finite satisfiablility is like in the proof of Proposition 0.3, but relativized to every interval. Assume that T' is satisfiable, choose $a < b$ in a copy of η_1 and relativize the proof of the first part of Proposition 0.3 to the interval (a, b). □

Corollary 0.7. *The linear order $\eta_1 \times \eta_0$ is not compactly expandable, even for countable sets of sentences.*

Proof. By Proposition 0.6. □

Remark 0.8.

1. *T' is finitely satisfiable in every dense linear ordering of cofinality ω.*
2. *T' is satisfiable in a dense linear order of cofinality ω if and only if the (ω, ω)-Dedekind cuts are dense.*
3. *In any compactly expandable dense linear order of cofinality ω, the (ω, ω)-Dedekind cuts are dense.*

Proof. *1.* This is what really is used in the proof of Proposition 0.6.

2. If $a < b$ are choosen in such a way that the interval (a, b) does not contain (ω, ω)-Dekedind cuts, then we can reproduce the first part of the proof of Proposition 0.3 relativized to the interval (a, b). The other direction is a relativized version of the proof of *2* of Remark 0.5.

3. follows from *1* and *2.* □