# A Remark on Morley Rank

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## 1 Introduction

Let $f : D \to E$ be a definable map between definable classes. The following theorem is well known:

**Theorem 1 ([1], [2, V 6.8])** *If $E$ has Morley rank $\beta$ and the Morley rank of all fibers $f^{-1}(e)$ is bounded by $\alpha$. Then*

1. *If $\alpha = 0$ $D$ has Morley rank at most $\beta$.*
2. *If $\alpha > 0$ the Morley rank of $D$ is bounded by $\alpha(\beta + 1)$.*

It seems to be less well known that this theorem gives the optimal bound. We will prove:

**Lemma 2** *For all $\alpha > 0$ and all $\beta$ there is a theory $T$ and (in the monster model of $T$) a definable map $f : D \to E$ such that*

*a) $E$ has Morley rank $\beta$*

*b) the Morley rank of all fibers of $f$ is $\alpha$*

*c) $D$ has Morley rank $\alpha \cdot (\beta + 1)$.*

In section 4 we discuss a bound for the Morley rank of $D$ if the Morley rank of all fibers of $f$ is smaller than a limit ordinal $\alpha$.

In the sequel let $\mathrm{R}(F)$ denote the (Morley) rank of the definable set $F$.

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## 2 Proof of Theorem 1

We include a proof of Theorem 1 for the convenience of the reader.

Let $E$ have rank $\alpha$ and $A$ be a set of parameters. We call an element $e$ of $E$ generic over $A$ if it is not contained in any $A$–definable set of smaller rank than $\alpha$. $E$ has always generic elements (in the monster model). Note that all generics of $E$ have the same type over $A$ if $E$ has degree one.

First we handle case 1, where $\alpha = 0$. Let $f : D \to E$ have finite fibers. We prove

$$\mathrm{R}(D) \leq \mathrm{R}(E)$$

by induction on $\beta = \mathrm{R}(E)$.
We may assume that $E$ has degree 1. Let $D_i$ be an infinite family of disjoint definable subsets of $D$. We have to show that almost all of them have smaller rank than $\beta$. Let $e \in E$ be generic over the parameters over which $f$, $D$, $E$ and the $D_i$ are defined. Almost all of the $D_i$ do not intersect $f^{-1}(e)$. So these $f(D_i)$ do not contain $e$ and have therefore smaller rank than $\beta$. So by induction almost all $D_i$ have smaller rank than $\beta$.

For the case 2 we need a lemma. If $E$ has Morley degree one and $e \in E$ is generic over the relevant parameters we call the (possibly empty) set $f^{-1}(e)$ a *generic* fiber of $f$.

**Lemma 3** *Let $E$ have Morley degree one and $\alpha$ be the rank of the generic fiber of $f$. If*

$$\gamma + \alpha < \mathrm{R}(D),$$

*$D$ has a definable subset $D'$ such that $f \restriction: D' \to E$ has finite generic fiber and*

$$\gamma < \mathrm{R}(D').$$

PROOF: We may assume $\alpha > 0$. $D$ contains then an infinite family $D_i$ of definable disjoint sets having at least rank $\gamma + \alpha$. Let $e \in E$ be generic. Then for one index $i$ the rank of $D_i \cap f^{-1}(e)$ is smaller than $\alpha$. By induction on $\alpha$ $D_i$ contains a $D'$ as required. QED.

We prove case 2 of the theorem by induction on $\beta$. We may assume that $E$ has degree one. If

$$\alpha \cdot \beta + \alpha < \mathrm{R}(D),$$

by the last lemma, $D$ contains a definable $D'$ of rank bigger than $\alpha \cdot \beta$ such that the generic fiber of $f \restriction D'$ has finitely many, say $k$ many, elements. For

$$E^* = \left\{ e \in E \mid D' \cap f^{-1}(e) \text{ has cardinality } k \right\},$$

the complement $E \setminus E^*$ has a rank $\beta' < \beta$. Since (by case 1) $D' \cap f^{-1}(E^*)$ has at most rank $\beta$, the rank of $D'' = D' \cap f^{-1}(E \setminus E^*)$ is bigger than $\alpha \cdot \beta \geq \alpha(\beta' + 1)$. This contradicts the induction hypothesis applied to $f \restriction D'' : D'' \to E \setminus E^*$.

## 3 Proof of Lemma 2

We deal only with countable $\alpha$ and $\beta$. (The proof in the uncountable case is essentially the same.) So in the sequel *infinite* means *countably infinite.*

For a fixed $\alpha > 0$ and for all $\beta$ we will construct models

$$\mathfrak{M}_\beta = (D_\beta, E_\beta, f_\beta),$$

which consist of a two sorts $D_\beta$ and $E_\beta$, a map $f_\beta : D_\beta \to E_\beta$ and unary predicates on $D_\beta$ and $E_\beta$ such that

a) $E_\beta$ has Morley rank $\beta$

b) the Morley rank of all fibers of $f_\beta$ is $\alpha$

c) $D_\beta$ has Morley rank $\alpha \cdot (\beta + 1)$.

d) $\mathfrak{M}_\beta$ is saturated and has quantifier elimination.

We start with a structure $\mathfrak{A} = (A, P_i)_{i \in I}$, where $A$ is an infinite set and the $P_i$ are unary predicates which ensure that $\mathfrak{A}$ has rank $\alpha$ (and is saturated). For the model $\mathfrak{M}_0$ we take $(A, E_0, f_0, P_i)_{i \in I}$, where $E_0$ consists of one point and $f_0$ is the constant map.

We give the following case a special treatment: Assume that $\alpha$ is finite and $\beta$ is a limit cardinal. We take for $E_\beta$ any set with unary predicates giving it rank $\beta$. Choose a surjection $f_\beta : D_\beta \to E_\beta$ with infinite fibers and sets $X^a \subset D_\beta$ ,$(a \in A)$, which intersect each fiber of $f_\beta$ in exactly one point. From the predicates $P_i$ we define the predicates $Q_i = \bigcup_{a \in P_i} X^a$. This is our $\mathfrak{M}_\beta$. The sets $X^a$ inherit rank $\beta$ from $E_\beta$. Whence $D_\beta$ has rank $\beta + \alpha$, which in our case equals $\alpha(\beta + 1)$.

Now assume that $\alpha$ is infinite or $\beta$ is a successor ordinal. Also assume that for all $\beta' < \beta$ the structures $\mathfrak{M}_{\beta'}$ are constructed. Let $\alpha'$ be such that $1 + \alpha' = \alpha$ and $\mathfrak{A}' = (A', P'_i)_{i \in I'}$ be the $\alpha'$–version of $\mathfrak{A}$[1]. To construct $\mathfrak{M}_\beta$ we take infinite sets $D_\beta$ and and $E_\beta$ and a surjective map $f_\beta : D_\beta \to E_\beta$ with infinite fibers.

On $D_\beta$ and $E_\beta$ we choose two families $(X^a)_{a \in A'}$ and $(E^{a,i})_{a \in A', i \in \omega}$ of disjoint subsets (and introduce predicates for them) such that

1. All intersections $X^a \cap f_\beta^{-1}(e)$ and the differences $f_\beta^{-1}(e) \setminus \bigcup_{a \in A'} X^a$ are infinite.

2. The difference $E_\beta \setminus \bigcup_{a \in A', i \in \omega} E^{a,i}$ is infinite. The cardinality of the $E^{a,i}$ will be specified later.

[1] If $\alpha' = 0$ $A'$ is just any finite set

From the predicates $P_i'$ we define also the predicates

$$Q_i = \bigcup_{a \in P_i'} X^a.$$

Let $(\beta_i)_{i\in\omega}$ be an enumeration of the ordinals $\beta' < \beta$ where all $\beta'$ occur infinitely often. In our last step, for each $a \in A'$ and $i \in \omega$ we introduce new predicates for subsets of $E^{a,i}$ and for subsets of

$$D^{a,i} = f_\beta^{-1}(E^{a,i}) \cap X^a$$

such that with the new predicates the structure

$$f_\beta \restriction D^{a,i} : D^{a,i} \to E^{a,i}$$

looks like $\mathfrak{M}_{\beta_i}$. This also tells us the right cardinality of the $E^{\alpha,i}$. This completes the construction of $\mathfrak{M}_\beta$.

It is easy to check that $\mathfrak{M}_\beta$ has quantifier elimination and is saturated.
Since the $E_{a,i}$ have rank $\beta_i$ $E_\beta$ has rank $\beta$.
Without the structure imprinted on the $D_{a,i}$ the fibers look like $\mathfrak{A}'$ with each point blown up to an infinite set and have therefore rank $1 + \alpha' = \alpha$. The structure on $D_{a,i}$ adds one set of rank $\alpha$ on the fiber. Whence the fibers have rank $\alpha$.

Each $D^{a,i}$ has rank $\alpha \cdot (\beta_i + 1)$. We have to distinguish two cases:

1. $\beta$ is a successor ordinal: Then $X^a$ has at least rank $\alpha \cdot \beta + 1$ and $D_\beta$ has at least rank $\alpha \cdot \beta + 1 + \alpha' = \alpha(\beta + 1)$.

2. $\beta$ is a limit ordinal and $\alpha$ is infinite: Then $X^\alpha$ has at least rank $\alpha \cdot \beta$ and $D_\beta$ the rank $\alpha \cdot \beta + \alpha' = \alpha \cdot \beta + \alpha = \alpha \cdot (\beta + 1)$.

# 4 Fiber rank smaller than a limit ordinal

The following problem is left open by Theorem 1: Let $\alpha$ be a limit ordinal. If $E$ has rank $\beta$ and the ranks of all fibers of $f : D \to E$ are smaller than $\alpha$ can we say more about $\mathrm{R}(D)$ than just $\mathrm{R}(D) \leq \alpha(\beta + 1)$ ? The answer is *yes*:

**Theorem 4** *Let $\alpha$ be a limit ordinal and $\beta$ be arbitrary*

1. *Let $f : D \to E$ be a definable map between definable classes: Assume $E$ has Morley rank $\beta$ and that the Morley rank of all fibers $f^{-1}(e)$ is smaller than $\alpha$. Then the Morley rank of $D$ is smaller than $\alpha(\beta + 1)$.*

2. *For all $\gamma < \alpha$ and all $\beta$ there is a theory $T$ and (in the monster model of $T$) a definable map $f : D \to E$ such that*

*a) E has Morley rank $\beta$*

*b) the Morley rank of all fibers of $f$ is smaller than $\alpha$*

*c) D has Morley rank $\alpha \cdot \beta + \gamma$.*

Part 1 has the same proof as Theorem 1. But part 2 needs a modification of the construction in Lemma 2.

Again we deal only with countable $\alpha$ and $\beta$. By recursion on $\beta$ we construct models

$$\mathfrak{M}^{\gamma}_{\beta} = (D^{\gamma}_{\beta}, E^{\gamma}_{\beta}, f^{\gamma}_{\beta})$$

such that

a) $E^{\gamma}_{\beta}$ has Morley rank $\beta$

b) the Morley rank of all fibers of $f^{\gamma}_{\beta}$ is smaller that $\alpha$

c) $D^{\gamma}_{\beta}$ has Morley rank $\alpha \cdot \beta + \gamma$.

d) $\mathfrak{M}^{\gamma}_{\beta}$ is saturated and has quantifier elimination.

We construct $\mathfrak{M}^{\gamma}_{0}$ as in the proof of Lemma 2 from a structure $\mathfrak{A} = (A, P_i)_{i \in I}$ of rank $\gamma$. If $\beta > 0$ assume that for all $\beta' < \beta$ (and all $\gamma < \alpha$) the structures $\mathfrak{M}^{\gamma}_{\beta'}$ are constructed. Take infinite sets $D^{\gamma}_{\beta}$ and and $E^{\gamma}_{\beta}$ and a surjective map $f^{\gamma}_{\beta} : D^{\gamma}_{\beta} \to E^{\gamma}_{\beta}$ with infinite fibers. Then choose two families $(X^a)_{a \in A}$ and $(E^{a,i})_{a \in A, i \in \omega}$ of disjoint subsets (and introduce predicates for them) such that

1. All intersections $X^a \cap (f^{\gamma}_{\beta})^{-1}(e)$ and the differences $(f^{\gamma}_{\beta})^{-1}(e) \setminus \bigcup_{a \in A} X^a$ are infinite.
2. The difference $E^{\gamma}_{\beta} \setminus \bigcup_{a \in A, i \in \omega} E^{a,i}$ is infinite.

Define again the predicates $Q_i = \bigcup_{a \in P_i} X^a$.

Finally we introduce new unary predicates on $E^{a,i}$ and

$$D^{a,i} = (f^{\gamma}_{\beta})^{-1}(E^{a,i}) \cap X^a$$

such that with the new predicates the structure

$$\mathfrak{N}^{a,i} = \left(D^{a,i}, E^{a,i}, f^{\gamma}_{\beta} \restriction D^{a,i}\right)$$

looks as follows:

Case 1: $\beta = \beta' + 1$ is a successor.
Then choose an enumeration $(\gamma_i)_{i \in \omega}$ of the ordinals below $\alpha$ and let $\mathfrak{N}^{a,i}$ look like $\mathfrak{M}^{\gamma_i}_{\beta'}$.

Case 2: $\beta$ is a limit ordinal.

Let $(\gamma_i)_{i\in\omega}$ enumerate the ordinals below $\beta$ and let $\mathfrak{N}^{a,i}$ look like $\mathfrak{M}^0_{\beta_i}$.

In the successor case $D^{a,i}$ has rank $\alpha \cdot \beta' + \gamma_i$, $X^a$ has rank $\alpha \cdot \beta' + \alpha = \alpha \cdot \beta$. In the limit case $D^{a,i}$ has rank $\alpha \cdot \beta_i$ and it follows again that $X^a$ has rank $\alpha \cdot \beta$. This implies that $D^\gamma_\beta$ has rank $\alpha \cdot \beta + \gamma$.

## 5 A better bound

The following theorem implies both Theorem 1 (2) and Theorem 4 (1):

**Theorem 5 ([3, Exercise 6.4.4])** *If $E$ has Morley rank $\beta$, Morley rank of all fibers $f^{-1}(e)$ is bounded by $\alpha > 0$ and the Morley rank of the generic fibers is bounded by $\alpha^{\mathrm{gen}}$, then the Morley rank of $D$ is bounded by*

$$\alpha\beta + \alpha^{\mathrm{gen}}.$$

The proof is a slight variation of the proof of Theorem 1. One proves similarly:

**Remark 6** *If $\beta$ is a limit ordinal, $\beta < \alpha\beta$, and $\alpha^{\mathrm{gen}}$ is finite, then the Morley rank of $D$ is smaller than $\alpha\beta + \alpha^{\mathrm{gen}}$.*

Slight modifications of the constructions above show that this bounds are optimal: If $\beta$ and $0 \le \alpha^{\mathrm{gen}} \le \alpha$ are given, there are two cases:

1. If the conditions of Remark 6 are not satisfied, there is an example $D$ whith Morley rank $\alpha\beta + \alpha^{\mathrm{gen}}$.
2. If the conditions of Remark 6 are satisfied, for every $\gamma$ smaller than $\alpha\beta + \alpha^{\mathrm{gen}}$ there is an example whith at least Morley rank $\gamma$.

## References


[1] M. M. Erimbetov. Complete theories with 1-cardinal formulas. *Algebra i Logika*, 14:245–257, 1975.

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[3] K. Tent and M. Ziegler. *A Course in Model Theory.* ASL Lecture Note Series. Cambridge University Press, 2012.