The 116 reducts of $(\mathbb{Q}, <, a)$*

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Abstract

This article aims to classify those reducts of expansions of $(\mathbb{Q}, <)$ by unary predicates which eliminate quantifiers, and in particular to show that, up to interdefinability, there are only finitely many for a given language. Equivalently, we wish to classify the closed subgroups of $\mathrm{Sym}(\mathbb{Q})$ containing the group of all automorphisms of $(\mathbb{Q}, <)$ fixing setwise certain subsets. This goal is achieved for expansions by convex predicates, yielding expansions by constants as a special case, and for the expansion by a dense, co-dense predicate. Partial results are obtained in the general setting of several dense predicates.

1 Introduction

In this article, we study expansions of $(\mathbb{Q}, <)$ by unary predicates that have quantifier elimination. Our goal is to classify the reducts and to show that, up to interdefinability, there are only finitely many such. Here, a *reduct* for us is a structure with domain $\mathbb{Q}$ in some relational language each of whose basic relations are $\emptyset$-definable in the original structure. We will consider two interdefinable reducts as the same structure. E.g. $\mathbb{Q}$ with the circular ordering coming from $<$ is a reduct of $(\mathbb{Q}, <)$, and considered to be the same reduct as $\mathbb{Q}$ with the reverse circular ordering.

All structures under consideration here are $\aleph_0$-categorical, and so we can identify a structure with its automorphism group, viewed as a permutation group on $\mathbb{Q}$. A reduct then corresponds to a closed subgroup of the full symmetric group on $\mathbb{Q}$ containing the automorphism group of the original structure. Having finitely many reducts thus is implies that there are only finitely many closed subgroups of the symmetric group on $\mathbb{Q}$ containing the corresponding automorphism group.

First we give a proof classifying the well-known reducts of $(\mathbb{Q}, <)$, which we believe to be simpler than the existing ones. Then we show that there are only finitely many reducts of the expanded structure by an explicit classification in the following two cases: Expansions of $(\mathbb{Q}, <)$ by convex subsets (which include the case of expansion by constants), and expansions by a dense and co-dense predicate. We have partial results for several dense predicates, and indicate how the general case of an expansion of $(\mathbb{Q}, <)$ by unary predicates that eliminates quantifiers reduces to the case of dense predicates. Our results support Simon Thomas' conjecture (shown

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in [T1] to hold for the random graph and for the homogeneous universal K_m-free graphs, and in [T2] for random k-graphs):

Conjecture *If $\mathfrak{M}$ is a structure with quantifier elimination in a finite relational language, then $\mathfrak{M}$ has, up to interdefinability, only finitely many reducts.*

The condition is necessary: [AZ] describes infinitely many reducts of a doubled infinite-dimensional projective space over $\mathbb{F}_2$. That structure is totally categorical and axiomatisable in a finite relational language, but does not have quantifier elimination in a finite relational language.

In [T1], there is an example due to Cherlin and Lachlan of a structure having quantifier elimination in a finite relational language and a reduct of this structure that does not have quantifier elimination in a finite relational language. This is never the case in our situation.

In this context, the following questions seem to be open:

Question 1 *Are the following properties invariant under bi-interpretability?*

- *to have quantifier elimination in a finite relational language*
- *to have finitely many reducts*

Question 2 *Do non-$\aleph_0$-categorical theories always have infinitely many reducts?*

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1.1 Notations, conventions, and some explanations

We will switch freely between the model theoretic and the group theoretic aspects. Most of the model theory used in this paper can be looked up in a standard textbook, e.g. [Ho] or [P2]. Following a suggestion of one of the referees, we have provided group theoretic translations of some model theoretic statements, so that at least the propositions and theorems should become understandable for non model theorists. All structures in this paper will be $\aleph_0$-categorical, i.e. have an oligomorphic automorphism group. Some of the definitions are only possible, and some of the group theoretic translations are only valid in this context.

We always consider structures *up to interdefinability* (also called "definitional equivalence" in [Ho]), which allows us to identify an $\aleph_0$-categorical structure with its automorphism group. Hence we identify two reducts $\mathfrak{R}_1$, $\mathfrak{R}_2$ of a structure $\mathfrak{M}$ if they have the same $\emptyset$-definable sets, or, equivalently, if $\mathrm{Aut}(\mathfrak{R}_1) = \mathrm{Aut}(\mathfrak{R}_2)$. Different reducts may still be isomorphic as structures. This happens when the automorphism groups are isomorphic as permutation groups on M, which is the same as to say that they are conjugate in the symmetric group on M. In this case, we call $\mathfrak{R}_1$ and $\mathfrak{R}_2$ *equivalent up to isomorphism.* If $\mathfrak{R}$ is a reduct of $\mathfrak{M}$, we call $|\mathrm{Aut}(\mathfrak{R}) : \mathrm{Aut}(\mathfrak{M})|$ the *index* of $\mathfrak{R}$ in $\mathfrak{M}$.

An equivalence relation is called *finite* if it has finitely many classes. If $\mathfrak{M}$ is $\aleph_0$-categorical, we let $E_{\mathfrak{M}}$ be the finest $\emptyset$-definable finite equivalence relation on M, i.e. the finest $\mathrm{Aut}(\mathfrak{M})$-invariant equivalence relation with finitely many classes. This is the relation "having the same strong type over $\emptyset$", or "having the same type over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$". We denote by $\mathfrak{M}_C$ the expansion of $\mathfrak{M}$ by predicates for the $E_{\mathfrak{M}}$-classes.

(In our context, "$\emptyset$-definable" means the same as "invariant under $\mathrm{Aut}(\mathfrak{M})$", and the elements of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ are equivalence classes of finite $\mathrm{Aut}(\mathfrak{M})$-invariant equivalence relations on some M^n, where $\mathrm{Aut}(\mathfrak{M})$ acts naturally on M^n. Invariance of a set or a relation is always meant as setwise invariance, unless otherwise specified. A (complete) type over a set corresponds to an orbit of the pointwise stabilizer of that set in the automorphism group.)

$\mathrm{Sym}(X)$ denotes the symmetric group on a set X, $\mathrm{Sym}(n)$ or S_n the symmetric group on $\{1, \ldots, n\}$, A_n the alternating group. For a finite permutation group, we often write its order as a superscript and the number of elements it acts on as a subscript. If $\bar{x} = (x_1, \ldots, x_n) \in X^n$ and $\sigma \in S_n$, then $\bar{x}^\sigma$ stands for $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

A type or a formula $p(x_1, \ldots, x_n)$ will be called *strict* if $p \vdash \bigwedge_{i \neq j} x_i \neq x_j$. A tuple is strict if its type is strict. Partitions are usually meant to be given by subsets or predicates, not by an equivalence relation. If P is a subset of the domain of a structure and R an n-ary relation, then R_P denotes the restriction of R to P^n, and $\neg P$ the complement of P.

1.2 The setting: expansions of $(\mathbb{Q}, <)$ with quantifier elimination

An expansion of $(\mathbb{Q}, <)$ by finitely many unary predicates is quantifier-free interdefinable with an expansion by unary predicates partitioning $\mathbb{Q}$: Replace the predicates by the atoms of the generated boolean algebra. Then the first structure eliminates quantifiers if and only if the second does.

If, for example, we add to $(\mathbb{Q}, <)$ a predicate for $[0, +\infty)$, then the point 0 becomes definable as the minimum of the new predicate, but it is not quantifier-free definable.

Lemma 1.1 *Let $\mathbb{Q}$ be partitioned by unary predicates $P_1, \ldots, P_n$. Then $(\mathbb{Q}, <, P_1, \ldots, P_n)$ has quantifier elimination if and only if each P_i is either a singleton or infinite without maximum and minimum, and for all i, j, either $P_i < P_j$, or $P_j < P_i$, or P_i and P_j are dense in each other. Moreover, the structure is determined up to isomorphism by these data.*

Proof: Suppose the structure eliminates quantifiers. A maximum or a minimum of P_i is definable, but not quantifier-free if P_i is not a singleton. If neither $P_i < P_j$ nor $P_j < P_i$, then we get $x_1 < y < x_2$ with $x_1, x_2 \in P_i$ and $y \in P_j$ (or vice versa). If follows that P_i is infinite, and that P_j is dense in P_i because all ordered pairs in P_i have the same quantifier-free type, hence the same type as (x_1, x_2). But then we also get the symmetric situation with i and j interchanged.

Conversely, with the usual back and forth techniques it is easy to construct an automorphism of the structure between two tuples having the same quantifier-free type, or to construct an isomorphism between two structures with the same data. □

Note that there are $\aleph_0$-categorical expansions of $(\mathbb{Q}, <)$ by unary predicates that do not eliminate quantifiers.

The rest of the paper is organised as follows: In Section 2 we treat the case $n = 0$ of $(\mathbb{Q}, <)$ without predicates, and in Section 3 two other very special cases: expansions of $\mathbb{Q}$ by predicates without ordering, and reducts of the general case but with a unique strong 1-type. In Section 4, we study definable orderings in expansion of $(\mathbb{Q}, <)$ by unary predicates which eliminate quantifiers. These preparations allow us to give an explicit classification of all reducts of expansions by convex sets in Section 5. In Section 6, we consider the case of dense predicates, where a complete proof of Thomas' conjecture is obtained for the case $n = 2$. Finally, Section 7 indicates how the general case follows from the dense case.

2 Reducts of the dense linear order $(\mathbb{Q}, <)$

First we introduce the three non trivial relations definable from a dense linear order, and convenient notation for some of them.

- *The unoriented linear order* or *betweenness relation* "betw":

$$\mathrm{betw}(x, y, z) \ :\Longleftrightarrow \ y \in \mathrm{conv}(x, z) \ \Longleftrightarrow \ (x < y < z) \vee (z < y < x)$$

  where "conv" stands for the convex hull.

- *The cyclic order* "cyc":

$$\mathrm{cyc}(x, y, z) \ \Longleftrightarrow \ \overrightarrow{xyz} \ :\Longleftrightarrow \ (x < y < z) \vee (y < z < x) \vee (z < x < y)$$

- *The unoriented cyclic order* or *separation relation* "sep":

$$\mathrm{sep}(w, x, y, z) \ \Longleftrightarrow \ \overleftrightarrow{wxyz} \ :\Longleftrightarrow \ \overrightarrow{wxyz} \vee \overrightarrow{zyxw}$$

- We extend our notation for the cyclic orders by defining

$$\overrightarrow{x_1 \dots x_n} \ :\Longleftrightarrow \bigwedge_{1 \leqslant i < j < k \leqslant n} \overrightarrow{x_i x_j x_k} \quad \text{and} \quad \overleftrightarrow{x_1 \dots x_n} \ :\Longleftrightarrow \bigwedge_{1 \leqslant i < j < k < l \leqslant n} \overleftrightarrow{x_i x_j x_k x_l}$$

Note that there are always two interdefinable linear and two interdefinable cyclic orderings: an ordering and its reversed ordering; whereas no other betweenness or separation relation is $\emptyset$-definable from a dense linear order.

In the rest of this section, we present a proof of the following theorem:

Theorem 2.1 *Up to interdefinability, there are exactly five reducts of $(\mathbb{Q}, <)$, namely:*

1. *the dense linear order $(\mathbb{Q}, <)$ itself;*
2. *the betweenness relation coming from $<$;*
3. *the dense cyclic order coming from $<$;*
4. *the separation relation coming from $<$;*
5. *and the infinite set $\mathbb{Q}$ without structure.*

This follows immediately from a theorem of Cameron classifying the highly homogeneous permutation groups on a countable set, see [C2] theorem 3.10. That result was first proved using group-theoretic methods in [C1]. A model theoretic approach is given in [HLS]). On of the referees pointed out that the theorem was first proved by Frasnay in [F]. We reprove Theorem 2.1 here by another method. We subsequently learned of the approach of Higman in [Hi], which is very similar to ours. We are able here to combine this approach with some basic model theory to provide an alternate proof of Cameron's theorem.

2.1 The method

Let $\mathfrak{R}$ be a reduct of $(\mathbb{Q},<)$. It is determined by the groups

$$G_n^{\mathfrak{R}} := \big\{\sigma \in S_n \;\big|\; (x_1,\ldots,x_n) \equiv (x_{\sigma(1)},\ldots,x_{\sigma(n)}) \text{ in } \mathfrak{R} \\ \text{for some (equivalently each) } x_1 < x_2 < \cdots < x_n\big\},$$

because they determine the orbits of strict n-tuples. We call $(G_n^{\mathfrak{R}})_{n\in\mathbb{N}}$ the *structure sequence* of $\mathfrak{R}$. When is a sequence of groups a structure sequence? We need some terminology: For $i \in \{1,\ldots,n+1\}$, let

- $\beta_i : \{1,\ldots,n\} \to \{1,\ldots,i-1,i+1,\ldots,n+1\}$ be the order preserving bijection,
- $\pi_i : S_{n+1} \to S_n$ be the map $\sigma \mapsto \beta_{\sigma(i)}^{-1} \circ \sigma \circ \beta_i$.

We say that G_n *extends* to G_{n+1}, and write $G_n \sqsubset G_{n+1}$, if $\pi_i[G_{n+1}] = G_n$ for all i.

Lemma 2.2 *A sequence $(G_n)_{n\in\mathbb{N}}$ with $G_n \leqslant S_n$ is a structure sequence iff G_n extends to G_{n+1} for all n.*

Proof: "$\Rightarrow$" is clear: choose a tuple $x_1 < \cdots < x_{n+1}$ and consider all extracted n-tuples.
"$\Leftarrow$": the automorphism group of the ($\aleph_0$-categorical) structure is given by all bijections acting like an element of G_n on ordered n-tuples, and the condition ensures that this automorphism group acts exactly like G_n. $\square$

Remark 2.3 *(a) Since $\pi_{\sigma(i)}(\tau) \circ \pi_i(\sigma) = \pi_i(\tau\sigma)$, we have $\bigcap_{i=1}^{n+1} \pi_i[G_{n+1}] \leqslant S_n$.*
It follows that G_{n+1} extends some group $G_n \leqslant S_n$ iff all $\pi_i[G_{n+1}]$ are equal.

(b) If $G_n \sqsubset G_{n+1}$, then $|G_{n+1}| \geqslant |G_n|$.

(c) If $G_n \sqsubset G_{n+1}$ and $|G_{n+1}| = |G_n|$, then all maps π_i must be bijective.

2.2 The proof

We start to determine all possible structure sequences. We need the following subgroups of S_n:

- the identity group I_n;
- the cyclic "swap group" of order 2 $\;Z_n^2 := \big\langle\, \big(1\; n\big)\big(2\; (n-1)\big) \cdots \big([\tfrac{n}{2}]\; [\tfrac{n+3}{2}]\big) \,\big\rangle$
- the cyclic "cycle group" of order n $\;Z_n^n := \big\langle (1\,2\,\ldots\,n) \big\rangle$;
- and the dihedral "square group" $D_n^{2n} := Z_n^2 \ltimes Z_n^n$.

2.2.1 The different orders

Lemma 2.4 *(a) (I_n) is the structure sequence of the dense order, and obviously I_{n+1} is the only possible extension of I_n if $n \geqslant 2$.*

(b) (Z_n^2) is the structure sequence of the betweenness relation, and it is the only possible structure sequence containing Z_3^2.

(c) (Z_n^n) *is the structure sequence of the cyclic order, and it is the only possible structure sequence containing* Z_3^3*.*

(d) (D_n^{2n}) *is the structure sequence of the separation relation, and it is the only possible structure sequence containing* D_4^8*.*

PROOF: In each case, it is rather obvious that the sequences are the structure sequences of the given relations. For example, in case (d) it is clear that D_n^{2n} conserves the separation relation, hence that it is a subgroup of the n^{th} structure group. D_n^{2n} acts transitively, thus it is sufficient to show that the one-point stabilizers of the structure group equals those of D_n^{2n}, which are the "swap group around that point". But after fixing one point, the betweenness relation, whose structure group is the swap group, is definable from the separation relation.

(a) is clear anyway. For (b), if a structure sequence (G_n) contains Z_3^2, it has to preserve the betweenness relation, hence $G_n \leqslant Z_n^2$. By induction, G_n has to have at least 2 elements, whence $G_n = Z_n^2$.

Analogously for (c) and (d): if (G_n) contains Z_3^3 or D_4^8, we get $(G_n) \leqslant Z_n^n$ and $(G_n) \leqslant D_n^{2n}$ and by induction $|G_n| \geqslant n-1$ and $|G_n| \geqslant 2n-2$ respectively, whence equality. □

2.2.2 The full symmetric group

Clearly, (S_n) is the structure sequence of the infinite set.

Lemma 2.5 $\pi_i : G_{n+1} \to S_n$ *is not injective for some* i *iff* G_{n+1} *contains a cycle* ζ *of consecutive elements.*

PROOF: If $\pi_i(\sigma) = \pi_i(\tau)$ and $\sigma \neq \tau$, then $\sigma(i) \neq \tau(i)$. Suppose $\sigma(i) < \tau(i)$. Then $\sigma \circ \tau^{-1} = \big(\sigma(i)\ (\sigma(i)+1) \ldots \tau(i)\big)$ is the desired cycle. Conversely, suppose $\zeta = \big(c\ (c+1) \ldots d\big) \in G_{n+1}$ and let $\sigma = \zeta \circ \tau$ for arbitrary τ. Then $\pi_i(\sigma) = \pi_i(\tau)$ for $i = \sigma^{-1}(c)$. □

Lemma 2.6 *(a)* S_2 *only extends to* Z_3^2*,* Z_3^3 *or* S_3*.*

(b) S_3 *only extends to* A_4*,* S_4 *or one of the three dihedral Sylow-2-subgroups of* S_4*.*

(c) If $n \geqslant 4$*, then* S_n *only extends to* A_{n+1} *or* S_{n+1}*.*

PROOF: (a) S_3 has five subgroups with at least two elements. By Remark 2.3 (c) and Lemma 2.5, S_2 does not extend to two of the one-point stabilizers.

(b),(c) If $S_n \sqsubset G_{n+1}$, then G_{n+1} has at least $n!$ elements, hence index $m \leqslant n+1$ in S_{n+1}. The action of S_{n+1} on the cosets of G_{n+1} provides a homomorphism $S_{n+1} \to S_m$ whose image has at least m elements. If $n \geqslant 4$, then A_{n+1} is simple. Hence either $m = 1$ and $G_{n+1} = S_{n+1}$, or $m = 2$ and $G_{n+1} = A_{n+1}$, or $m = n+1$, and then the homomorphism above is an isomorphism and G_{n+1} the one-point stabilizer of itself, hence $G_{n+1} \cong S_n$. If $n = 3$, the only further subgroups of S_4 of index at most 4 are the three Sylow-2-subgroups of index 3. This proves (b).

If $G_{n+1} \cong S_n$, G_{n+1} is a one-point stabilizer, except for $n = 5$, since S_6 has non-trivial outer automorphisms and therefore six "exotic" subgroups $S_6^{\text{ex1}}, \ldots, S_6^{\text{ex6}}$ isomorphic to S_5 (given by the action of S_5 on its six Sylow-5-subgroups). By Remark 2.3 (c) and Lemma 2.5, the one-point stabilizers are excluded. Inspection of the exotic groups shows, in some numbering, $(123456) \in$

$S_6^{\text{ex}1}$; $(2345) \in S_6^{\text{ex}2}$; $(1234), (3456) \in S_6^{\text{ex}3}$; $(1234), (23456) \in S_6^{\text{ex}4}$; $(12345), (3456) \in S_6^{\text{ex}5}$; $(2345) \in S_6^{\text{ex}6}$. (This can also be deduced somewhat lengthily from the way the non-trivial outer automorphisms act on the conjugation classes of S_6). Hence $S_6^{\text{ex}i}$ does not extend S_5 by Lemma 2.5.

This proves the "only"-parts of the lemma. One can check without difficulties that the groups really extend to the given ones, but this is not necessary for the proof of the theorem. □

2.2.3 The remaining cases

Finally we show that the remaining cases — two dihedral groups and the alternating groups — can't occur.

Lemma 2.7 *$D'_4 = \langle (12), (1324) \rangle$ does not extend S_3.*

Proof: It is straightforward to check that $\pi_4[D'_4] = \{\mathrm{id}_3, (12), (13), (123)\} \neq S_3$. □

Lemma 2.8 *$D''_4 = \langle (14), (1243) \rangle$ does not extend to any group.*

Proof: The 4-ary relation $R(a, b, c, d)$ induced by D''_4 is the following: "the unoriented intervals (a, d) and (b, c) lie one in the other". Given five elements $a < b < c < d < e$, R allows to identify a and e up to "swap" as the two elements such that $\models R(a, x, y, e)$ for each choice $x, y \in \{b, c, d\}$. Then d is determined from a by $R(a, x, y, d)$ with $x, y \neq e$, analogously b from e. Thus all the elements are identifiable up to swap, the betweenness relation is definable from R and the structure sequence of $(\mathbb{Q}, R)$ must be (Z_n^2). □

Lemma 2.9 *A_n has no extension if $n \geqslant 4$.*

Proof: Consider all $\sigma \in S_{n+1}$ such that $\pi_5(\sigma) = (123)$. We show that some $\pi_i(\sigma)$ is odd. Note that σ is determined by $\sigma(5)$. 1st case: $\sigma(5) \geqslant 3$. Then $\pi_3(\sigma)$ has the same parity as σ and $\pi_1(\sigma)$ has a different parity, whence one is odd. 2nd case: $\sigma(5) = 2$. Then $\sigma = (1452)$ and $\pi_3(\sigma) = (1342)$ is odd. 3rd case: $\sigma(5) = 1$. Then $\sigma = (145)$ and $\pi_2(\sigma) = (1234)$ is odd. □

This proves the theorem. The information might be put together in a final picture as follows:

$$
\begin{array}{cc|c|c|c|c|c|cl}
I_1 & \sqsubset & I_2 \;\sqsubset & I_3 \;\sqsubset & I_4 \;\sqsubset & I_5 \;\sqsubset & I_6 \;\sqsubset & \ldots & \text{linear order} \\
 & & S_2 \;\sqsubset & Z_3^2 \;\sqsubset & Z_4^2 \;\sqsubset & Z_5^2 \;\sqsubset & Z_6^2 \;\sqsubset & \ldots & \text{betweenness} \\
 & & & Z_3^3 \;\sqsubset & Z_4^4 \;\sqsubset & Z_5^5 \;\sqsubset & Z_6^6 \;\sqsubset & \ldots & \text{cyclic order} \\
 & & & S_3 \;\not\sqsubset & D'_4 & & & & \\
 & & & \sqsubset & D_4^8 \;\sqsubset & D_5^{10} \;\sqsubset & D_6^{12} \;\sqsubset & \ldots & \text{separation} \\
 & & & & D''_4 \;\not\sqsubset & & & & \\
 & & & & A_4 \;\not\sqsubset & & & & \\
 & & & & S_4 \;\sqsubset & A_5 \;\not\sqsubset & & & \\
 & & & & & S_5 \;\not\sqsubset & \text{exotic } S_5 & & \\
 & & & & & \sqsubset & A_6 \;\not\sqsubset & & \\
 & & & & & & S_6 \;\sqsubset & \ldots & \text{infinite set}
\end{array}
$$

2.3 Additional remarks

The proof of Theorem 2.1 also shows that the dense linear order, cyclic order, betweenness and separation relations eliminate quantifiers in their natural languages, since in each case, if n is the arity of the relation, the n^{th} group in the structure sequence already determines the whole sequence.

2.3.1 The lattice

The reducts of a given structure $\mathfrak{M}$ form a complete lattice. The infimum of a family $(\mathfrak{R}_i)_{i \in I}$ is the structure whose basic relations are those which are $\emptyset$-definable in all structures $\mathfrak{R}_i$; the supremum is the structure generated by the $\emptyset$-definable sets of all structures.

Let the *normalised structure* $\mathfrak{M}'$ be the structure with domain M and $2n$-ary basic relations $\mathrm{tp}^{\mathfrak{M}}(\bar{x}) = \mathrm{tp}^{\mathfrak{M}}(\bar{y})$ for each n. Note that $\mathfrak{M}''$ can be different from $\mathfrak{M}'$. If $\mathfrak{M}$ is $\aleph_0$-categorical, then $\mathfrak{M}'$ is a reduct of $\mathfrak{M}$, and if $\mathfrak{M}$ eliminates quantifiers in a finite relational language, then $\mathfrak{M}'$ has finite index in $\mathfrak{M}$. The group $\mathrm{Perm}(\mathfrak{M})$ of *permorphisms* of $\mathfrak{M}$ consists of the permutations of M inducing a bijection on the definable sets. If $\mathfrak{M}$ is $\aleph_0$-categorical, then $\mathrm{Aut}(\mathfrak{M}') = \mathrm{Perm}(\mathfrak{M}) = \mathrm{N}_{\mathrm{Sym}(M)}(\mathrm{Aut}(\mathfrak{M}))$.

The lattice of reducts of an $\aleph_0$-categorical structure carries the indices and the *normaliser function* $\mathfrak{R} \mapsto \mathfrak{R}'$ as additional information. A general problem would be to classify all (finite) lattices with indices and normaliser function occurring as lattices of reducts of $\aleph_0$-categorical structures. Since $\mathrm{Sym}(\aleph_0)$ has no proper closed subgroups of finite index, the index over the smallest element $(M, \emptyset)$ is always ∞.

The lattice for $(\mathbb{Q}, <)$ looks as follows:

$$
\begin{array}{ccl}
 & (\mathbb{Q},<) & \\
\infty\ / & & \backslash\ 2 \\
(\mathbb{Q},\mathrm{cyc}) & & (\mathbb{Q},\mathrm{betw}) = (\mathbb{Q},\mathrm{betw})' = (\mathbb{Q},<)' \\
2\ \backslash & & /\ \infty \\
 & (\mathbb{Q},\mathrm{sep}) = (\mathbb{Q},\mathrm{sep})' = (\mathbb{Q},\mathrm{cyc})' & \\
 & \mid\ \infty & \\
 & (\mathbb{Q},\emptyset) &
\end{array}
$$

The numbers are the indices. $\mathrm{Aut}(\mathbb{Q}, \mathrm{betw}) \cong \mathrm{Aut}(\mathbb{Q}, <) \rtimes Z^2$ and $\mathrm{Aut}(\mathbb{Q}, \mathrm{sep}) \cong \mathrm{Aut}(\mathbb{Q}, \mathrm{cyc}) \rtimes Z^2$, where in both cases Z^2 is generated by any of the order-reversing automorphisms of order 2. With other words, the relations $<$ and cyc are exchanged with their complements on strict tuples. $\mathrm{Aut}(\mathbb{Q}, \mathrm{cyc})$ is generated by $\mathrm{Aut}(\mathbb{Q}, <)$ and bijections β_c moving irrational cuts c to the cut "at infinity". That is, β_c is composed of order-preserving bijections $(c, +\infty) \to (-\infty, c)$ and $(-\infty, c) \to (c, +\infty)$, and reverses the ordering between the two parts. $\mathrm{Aut}(\mathbb{Q}, <)$ acts transitively, $\mathrm{Aut}(\mathbb{Q}, \mathrm{betw})$ and $\mathrm{Aut}(\mathbb{Q}, \mathrm{cyc})$ act 2-transitively, and $\mathrm{Aut}(\mathbb{Q}, \mathrm{sep})$ 3-transitively on $\mathbb{Q}$.

The random graph has exactly the same lattice as $(\mathbb{Q}, <)$, see [T1], as does the random tournament, see [B]. There is a surprising analogy between the reducts: the index 2 steps correspond to semi-direct products with an anti-isomorphism (exchanging edges and non-edges on strict pairs); the *switching group* corresponding to $\mathrm{Aut}(\mathbb{Q}, \mathrm{cyc})$ is generated by bijections β_Y which are isomorphisms on a subset Y of vertices and on its complement, and anti-isomorphisms between Y and its complement. In particular, the groups act as transitively as do their counterparts.

2.3.2 Imaginaries

We consider the following two equivalence relations, definable in $(\mathbb{Q},<)$. To simplify the terminology, we assume that they are only defined on strict tuples.

- The "equally ordered relation" $E^{<}\bar{x}\bar{y} :\iff \mathrm{tp}(x_1,x_2)=\mathrm{tp}(y_1,y_2)$,
- and the "equally circled relation"

$$E^{\mathrm{cyc}}\bar{x}\bar{y} :\iff \bigvee_{\sigma\in A_3} \mathrm{tp}((x_1,x_2,x_3)^{\sigma})=\mathrm{tp}(y_1,y_2,y_3)$$

- Both are special cases of a general "equally oriented relation" between strict m-tuples $\bar{x}$ and strict n-tuples $\bar{y}$, which is $\sigma\in A_m \iff \tau\in A_n$ where $\sigma\in S_m$ and $\tau\in S_n$ are such that $x_{\sigma(1)}<\cdots<x_{\sigma(m)}$ and $y_{\tau(1)}<\cdots<y_{\tau(n)}$.

$E^{<}$ is interdefinable with the betweenness relation and E^{cyc} with the separation relation, since by definition they are the normalised structures of $(\mathbb{Q},<)$ and $(\mathbb{Q},\mathrm{cyc})$. Alternatively, $b\in\mathrm{conv}(a,c) :\iff E^{<}abbc$ and $\overleftrightarrow{abcd} :\iff E^{\mathrm{cyc}}abccda$, and $E^{<}$ and E^{cyc} do not define the linear and cyclic ordering respectively, hence by the classification of the reducts $(\mathbb{Q},E^{<})=(\mathbb{Q},\mathrm{betw})$ and $(\mathbb{Q},E^{\mathrm{cyc}})=(\mathbb{Q},\mathrm{sep})$.

Both equivalence relations have exactly two classes (on strict tuples). When $E^{<}$ is definable, then the $E^{<}$-classes and the E^{cyc}-classes are pairwise interdefinable as they are fixed by the same automorphisms, namely those not turning around the orientation. Thus is it sufficient to consider E^{cyc}.

In $(Q,<)$ and in (Q,cyc), the E^{cyc}-classes are definable, whereas in $(\mathbb{Q},\mathrm{sep})$ and in $(\mathbb{Q},\mathrm{betw})$ the two E^{cyc}-classes are conjugate. Thus the two conjugate E^{cyc}-classes correspond to the two linear (or cyclic) orderings inducing the betweenness (or separation) relation: $\mathrm{Aut}(\mathbb{Q},<)$ equals the stabiliser of the E^{cyc}-classes in $\mathrm{Aut}(\mathbb{Q},\mathrm{betw})$, and $\mathrm{Aut}(\mathbb{Q},\mathrm{cyc})$ in $\mathrm{Aut}(\mathbb{Q},\mathrm{sep})$.

Lemma 2.10 $\mathrm{acl}^{\mathrm{eq}}(\emptyset)=\mathrm{dcl}^{\mathrm{eq}}(\emptyset)$ *holds in each of* $(\mathbb{Q},<)$, $(\mathbb{Q},\mathrm{cyc})$ *and* $(\mathbb{Q},\emptyset)$, *and* $\mathrm{acl}^{\mathrm{eq}}(\emptyset)=\mathrm{dcl}^{\mathrm{eq}}(\mathbb{Q}/E^{\mathrm{cyc}})$ *in* $(\mathbb{Q},\mathrm{betw})$ *and* $(\mathbb{Q},\mathrm{sep})$.

(Group theoretic translation: An equivalence class of a finite Aut-invariant equivalence relation on $\mathbb{Q}^n$ is itself invariant (as a set) in the first three structures, and invariant under those automorphisms fixing the E^{cyc}-classes in the other two structures.)

Proof: $\mathrm{acl}(\emptyset)=\emptyset$ in all five structures. Thus the result holds for $(\mathbb{Q},<)$ and $(\mathbb{Q},\emptyset)$ because they weakly eliminate imaginaries (easy, or see [P1]).

Elements of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ correspond, up to interdefinability, to closed subgroups of finite index of the automorphism group (by [P1]). Suppose H is a proper closed subgroup of $\mathrm{Aut}(\mathbb{Q},\mathrm{cyc})$ of finite index. According to the classification of the reducts of $(\mathbb{Q},<)$, H is not a supergroup of $\mathrm{Aut}(\mathbb{Q},<)$, thus $H\cap\mathrm{Aut}(\mathbb{Q},<)$ is a subgroup of finite index of $\mathrm{Aut}(\mathbb{Q},<)$: contradiction.

The same argument now shows that $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Aut}(\mathbb{Q},\mathrm{cyc})$ are the only proper closed subgroups of finite index of $\mathrm{Aut}(\mathbb{Q},\mathrm{betw})$ and $\mathrm{Aut}(\mathbb{Q},\mathrm{sep})$ respectively. $\square$

3 Two extreme cases

Let $\mathfrak{P} = (\mathbb{Q}, <, P_1, \ldots, P_n)$ be an expansion of $(\mathbb{Q}, <)$ by a partition of $\mathbb{Q}$ such that $\mathfrak{P}$ eliminates quantifiers, as described in Lemma 1.1. $E_{\mathfrak{P}}$ has been defined to be the relation "having same strong type over $\emptyset$", which is the finest $\emptyset$-definable finite equivalence relation in $\mathfrak{P}$. Two elements $x < y$ of same type have the same Lascar strong type, since they have the same type over the elementary submodel $\mathbb{Q} \setminus [x, y]$, hence $E_{\mathfrak{P}}xy = \bigwedge_{i=1}^{n}(x \in P_i \leftrightarrow y \in P_i)$.

Let $\mathfrak{R}$ be an reduct of $\mathfrak{P}$. We consider two somehow opposite extreme cases: the case where there is no ordering at all, and the case where there is a unique strong 1-type.

3.1 No ordering

Recall that $\mathfrak{R}_C$ denotes the expansion of $\mathfrak{R}$ by predicates for the $E_{\mathfrak{R}}$-classes. As $E_{\mathfrak{R}}$ is a coarsening of $E_{\mathfrak{P}}$, each $E_{\mathfrak{R}}$-class is a union of predicates P_j.

Theorem 3.1 *The reducts $\mathfrak{R}$ of $\mathfrak{P}^- := (\mathbb{Q}, P_1, \ldots, P_n)$ are, up to interdefinability, determined by $E_{\mathfrak{R}}$ and the finite structure $\mathbb{Q}/E_{\mathfrak{R}}$ given by the action of* $\mathrm{Aut}(\mathbb{Q}/E_{\mathfrak{R}}) = \mathrm{Aut}(\mathfrak{R})/\mathrm{Aut}(\mathfrak{R}_C)$ *on the $E_{\mathfrak{R}}$-classes.*

PROOF: Let $C_1, \ldots, C_k$ be the $E_{\mathfrak{R}}$-classes. It is easy to see that $E_{\mathfrak{R}} = E_{\mathfrak{R}_C}$. Thus 1-types in $\mathfrak{R}_C$ are strong types and hence stationary. Now we work in $\mathfrak{R}_C$, which has Morley rank 1 because $\mathfrak{P}^-$ does. Therefore forking is determined by the algebraic closure operator, which is trivial. Suppose x and y have the same type p, and assume $\bar{a}$ is disjoint from x and y. Then $\bar{a}$ is independent from x, and from y. It follows that $\mathrm{tp}(x/\bar{a}) = \mathrm{tp}(y/\bar{a})$, as they are the unique non-forking extension of p. This implies immediately $\mathrm{Aut}(\mathfrak{R}_C) = \mathrm{Sym}(C_1) \times \cdots \times \mathrm{Sym}(C_k)$ and $\mathfrak{R}_C = (\mathbb{Q}, C_1, \ldots, C_k)$. It follows that $\mathfrak{R}$ is determined by $\mathrm{Aut}(\mathbb{Q}/E_{\mathfrak{R}})$ as $\mathrm{Aut}(\mathfrak{R})$ consists of all permutations of $\mathbb{Q}$ respecting $E_{\mathfrak{R}}$ and having image in $\mathrm{Aut}(\mathbb{Q}/E_{\mathfrak{R}})$. $\square$

Clearly, all coarsenings of $E_{\mathfrak{P}^-} = E_{\mathfrak{P}}$ occur as $E_{\mathfrak{R}}$. If k is the total number of $E_{\mathfrak{R}}$-classes and e the number of infinite (i.e. non-singleton) classes, all subgroups of $S_e \times S_{k-e}$ are possible as $\mathrm{Aut}(\mathbb{Q}/E_{\mathfrak{R}})$.

Moreover, it follows from the theorem that $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ in $\mathfrak{R}$ consists, up to definable closure, of the $E_{\mathfrak{R}}$-classes, that is $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset)) = \mathrm{Aut}(\mathbb{Q}/E_{\mathfrak{R}})$. This is because we may assume $\mathfrak{R} = \mathfrak{R}_C$, hence without loss of generality $\mathfrak{R} = \mathfrak{P}^-$. There the claim holds since all types are stationary.

3.2 Only one strong type

Theorem 3.2 *Let $\mathfrak{R}$ be a reduct of $\mathfrak{P}$ with a unique strong 1-type over $\emptyset$, i.e. without non-trivial finite* $\mathrm{Aut}(\mathfrak{R})$*-invariant equivalence relations on R. Then $\mathfrak{R}$ is a reduct of $(\mathbb{Q}, <')$ where $<'$ is a dense linear order on $\mathbb{Q}$.*

PROOF: An n-ary relation S is called *symmetric* if $\models S(\bar{x}) \iff \models S(\bar{x}^\sigma)$ for all $\sigma \in S_n$ where $(x_1, \ldots, x_n)^\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Claim Let S be an n-ary symmetric relation that is $\emptyset$-definable in $\mathfrak{P}$. Then S is $\emptyset$-definable in $\mathfrak{P}^- := (\mathbb{Q}, P_1, ..., P_n)$.

Proof: Suppose two n-tuples $\bar{x}$ and $\bar{y}$ have the same type in $\mathfrak{P}^-$. Then for some permutation $\sigma \in S_n$, $\bar{x}^\sigma$ and $\bar{y}$ have the same type in $\mathfrak{P}$. It follows $\mathfrak{P} \models S(\bar{y}) \iff \mathfrak{P} \models S(\bar{x}^\sigma) \iff \mathfrak{P} \models S(\bar{x})$. $\Diamond$

Claim Let $\mathfrak{R}$ be as in the theorem. Given two strict n-tuples $\bar{x}$ and $\bar{y}$, there is some $\bar{x}^\sigma$ having the same $\mathfrak{R}$-type as $\bar{y}$.

Proof: For fixed $\bar{y}$, we define

$$S(\bar{z}) :\iff \bar{z} \text{ is strict and } \bar{z}^\sigma \equiv^{\mathfrak{R}} \bar{y} \text{ for some } \sigma \in S_n$$

Since S is symmetric, it is $\emptyset$-definable in $\mathfrak{P}^-$ by the first claim. On the other hand, S is $\emptyset$-definable in $\mathfrak{R}$, hence $(\mathbb{Q}, S)$ does not define a non-trivial finite equivalence relation. It follows from Theorem 3.1 that S is $\emptyset$-definable in the trivial structure $(\mathfrak{P}, \emptyset)$. Since it holds for $\bar{y}$, it holds for all strict n-tuples, in particular for $\bar{x}$. $\Diamond$

Now by Cameron's theorem (Theorem 6.1 in [C1], or Theorem 3.3 below), $\mathfrak{R}$ is a reduct of a dense linear order on $\mathbb{Q}$. $\square$

We will see in Proposition 4.3 that $<'$ can be chosen to be $\emptyset$-definable in $\mathfrak{P}$.

3.3 A proof of the remaining part of Cameron's theorem

In this section, we prove

Theorem 3.3 (Cameron) *Let $\mathfrak{M}$ be a countable structure such that* $\mathrm{Aut}(\mathfrak{M})$ *acts transitively on $[M]^n$ for all n. Then $\mathfrak{M}$ is isomorphic to a reduct of* $(\mathbb{Q}, <)$.

Cameron's original *Main Theorem 6.1* in [C1] is equivalent to Theorems 2.1 and 3.3 here. The latter follows easily from the following two model theoretic lemmas.

Lemma 3.4 *If $\mathfrak{M}$ is a structure which is indiscernible with respect to an ordering $<$, then there is an $\mathfrak{M}' \equiv \mathfrak{M}$, which is indiscernible of same type with respect to some open dense linear ordering $<'$.*

Note that the orderings $<$, $<'$ need not to be definable in $\mathfrak{M}$, $\mathfrak{M}'$ respectively.

Proof: Choose (by the usual compactness and Ramsey arguments) an $\mathfrak{M}'' \equiv \mathfrak{M}$ containing an infinite indiscernible sequence $(I, <')$ of same type as $(M, <)$, where $<'$ is an open dense linear order. "Same type" here means that for all n, all tuples $m_1 < \ldots < m_k$ in M and $i_1 <' \ldots <' i_k$ in I and all formulas φ

$$\mathfrak{M} \models \varphi(m_1, \ldots, m_k) \iff \mathfrak{M}'' \models \varphi(i_1, \ldots, i_k).$$

We show with Tarski's test that I is an elementary substructure of $\mathfrak{M}''$:

Assume $\mathfrak{M}'' \models \exists x\, \varphi(x, \bar{b}')$ with $\bar{b}' \in I$. Then for a tuple $\bar{b}$ in $\mathfrak{M}$ of same $<$-order type as the $<'$-order type of $\bar{b}'$, we have $\mathfrak{M} \models \exists x\, \varphi(x, \bar{b})$, that is $\mathfrak{M} \models \varphi(a, \bar{b})$ for some a. Choose $a' \in I$ such that $(a', \bar{b}')$ has same order type as $(a, \bar{b})$, which exists because $<'$ is dense and open. It follows that $\mathfrak{M}'' \models \varphi(a', \bar{b}')$. Hence $I \preccurlyeq \mathfrak{M}''$ and we let $\mathfrak{M}' = I$. $\square$

Lemma 3.5 *Let $\mathfrak{M}$ be a countable structure. Equivalent are:*

(a) For all n, the automorphism group of $\mathfrak{M}$ acts transitively on $[M]^n$.

(b) M is an indiscernible sequence with respect to some linear ordering $<$.

Proof: b) $\Longrightarrow$ a): Given two elements of $[M]^n$, then ordered as ascending strict n-tuples they have the same type.

a) $\Longrightarrow$ b): $\mathfrak{M}$ is $\aleph_0$-categorical, because $\mathrm{Aut}(\mathfrak{M})$ has finitely many orbits on M^n for each n. Hence one can find an infinite sequence I of pairwise distinct indiscernibles in $\mathfrak{M}$. Up to permutation, each finite A has the same type as any subset of I with the same cardinality as A. Hence A admits an ordering as indiscernibles. But this is enough (by compactness or by König's lemma) to get an indiscernible ordering on M. $\square$

Proof of Theorem 3.3: Let $\mathfrak{M}$ be as in the theorem. Then $\mathfrak{M}$ is $\aleph_0$-categorical and admits an indiscernible ordering $<$ by Lemma 3.5, which by Lemma 3.4 and the $\aleph_0$-categoricity can be chosen to be dense and open. Indiscernibility implies $\mathrm{Aut}(M,<) \subseteq \mathrm{Aut}(\mathfrak{M})$.

(In fact, from the classification of the reducts one sees that the original ordering $<$ is already dense if $\mathfrak{M}$ is not trivial, but it might have a maximum or a minimum in case $\mathfrak{M}$ is a cyclic ordering or a separation relation.) $\square$

4 Definable generalised orderings

Terminology We call any structure isomorphic to one of the five reducts of $(\mathbb{Q},<)$ a *generalised ordering*, linear and cyclic orderings are *oriented*, betweenness and separation relations are *unoriented orderings*, and an infinite set without structure is called the *trivial case*.

Let again $\mathfrak{P} = (\mathbb{Q},<,P_1,\ldots,P_n)$ be an expansion of $(\mathbb{Q},<)$ by a partition of $\mathbb{Q}$ and eliminating quantifiers. Our aim is to describe all $\emptyset$-definable generalised orderings in $\mathfrak{P}$.

Let $\mathcal{B} = (B_1,\ldots,B_k)$ be an ordered partition of $\mathbb{Q}$ such that each B_i is a union of predicates P_j that are pairwise dense in each other. Let $\triangleleft\!\!\!+$ denote the ordered disjoint sum of orders, let $<^{-1}$ be the reversed ordering of $<$, and $<^1 = <$. Then in $\mathfrak{P}$ there are definable linear orderings

$$(\mathbb{Q},<_{\mathcal{B},\chi}) \;:=\; (B_1,<_{B_1}^{\chi(1)}) \triangleleft\!\!\!+ \cdots \triangleleft\!\!\!+ (B_k,<_{B_k}^{\chi(k)})$$

on $\mathbb{Q}$ for every such $\mathcal{B}$ and every $\chi:\{1,\ldots,k\}\to\{-1,1\}$.

Two such orderings $<_{\mathcal{B},\chi}$ and $<_{\mathcal{B}',\chi'}$ are equal if and only if $\mathcal{B}=\mathcal{B}'$ and χ and χ' coincide on the infinite B_i; and interdefinable iff equal or reversed. An ordering $<_{\mathcal{B},\chi}$ is dense and open if and only if each singleton B_i is put between two infinite blocks B_{i-1} and B_{i+1}.

Lemma 4.1 *$\mathfrak{P}$ eliminates imaginaries, and* $\mathrm{dcl}^{\mathrm{eq}}(\emptyset) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ *holds.*

(Cf. the explanation after Lemma 2.10; elimination of imaginaries means that the stabilizer of an element in $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ equals the pointwise stabilizer of some tuple of elements in $\mathfrak{P}$.)

Proof: Clearly, $\mathfrak{P}$ has the following property: if $\bar a \equiv_{\bar c} \bar b$, then there is a sequence $\bar a = \bar a_0 \equiv_{\bar c} \bar a_1 \equiv_{\bar c} \bar a_2 \cdots \equiv_{\bar c} \bar a_k = \bar b$ such that $\bar a_{i+1}$ differs from $\bar a_i$ by just one element. This shows that $\mathrm{Aut}_{A\cap B}(\mathfrak{P}) = \langle \mathrm{Aut}_A(\mathfrak{P}), \mathrm{Aut}_B(\mathfrak{P})\rangle$ for finite A,B, which is Lascar's criterion for weak elimination of imaginaries. This implies $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}(\emptyset))$. But $\mathrm{acl}(\emptyset)=\mathrm{dcl}(\emptyset)$ in $\mathfrak{P}$. Finally, weak elimination implies full elimination because $\mathfrak{P}$ is a totally ordered structure. $\square$

Corollary 4.2 *In $\mathfrak{P}$, there is no non-trivial $\emptyset$-definable finite equivalence relation on the realisation set of a complete type (i.e. on an orbit of* $\mathrm{Aut}(\mathfrak{P})$ *on some P^n).*

Proposition 4.3 *Any $\emptyset$-definable generalised ordering in $\mathfrak{P}$ is a reduct of some $<_{\mathcal{B},\chi}$.*

We have defined a generalised ordering to be dense. Proposition 4.3 holds and is proved for the wider class of substructures of generalised ordering.

Proof: Let $<'$ be a $\emptyset$-definable linear order in $\mathfrak{P}$. Obviously, there are only two linear orders $\emptyset$-definable in $(\mathbb{Q}, <)$, namely $<$ and the reversed order $<^{-1}$. Thus the restriction of $<'$ to P_i must either equal $<$ or $<^{-1}$, and as in the proof of Lemma 1.1, quantifier elimination implies that for $i \neq j$, either $P_i <' P_j$ or $P_j <' P_i$, or P_i and P_j are dense in each other with respect to $<'$ (where the last case can only happen when P_i and P_j are dense in each other w.r.t. $<$). Density w.r.t. $<'$ is an equivalence relation on $\{P_1, \ldots, P_n\}$ and defines a partition $\mathcal{B}$ such that $<'$ equals some $<_{\mathcal{B},\chi}$.

Now let cyc' be a $\emptyset$-definable cyclic order in $\mathfrak{P}$. As above, cyc' restricted to P_i must be either cyc or cyc^{-1}. Assume first that some P_i is dense in $\mathbb{Q}$ with respect to cyc'. Then $<$ on P_i allows to define a cut in cyc', namely the cut at infinity in P_i, and therefore yields a $\emptyset$-definable linear order inducing cyc'. If no P_i is dense, then there is a cut in cyc' determined by two predicates P_i, P_j, which again allows to define without parameters a linear ordering inducing cyc'.

Assume finally that O is a $\emptyset$-definable betweenness or separation relation, and let E^{cyc} be the equivalence relation which is interdefinable with the separation relation definable from O, as in Section 2.3.2. Let C be an E^{cyc}-class. Then C is definable in $\mathfrak{P}$ by Lemma 4.1, so $(\mathbb{Q}, O, C)$ is a reduct of $\mathfrak{P}$ and defines the two linear (cyclic respectively) orderings inducing O. Hence O is a reduct of some $<_{\mathcal{B},\chi}$. $\square$

For further reference in Section 5.3, we note a special case:

Remark 4.4 *If all predicates P_i are convex and k is the number of finite P_i, then there are, up to interdefinability, $\binom{n-k-1}{k} \cdot k! \cdot (n-k)! \cdot 2^{n-k-1}$ dense and open $\emptyset$-definable linear orderings, and $\binom{n-k}{k} \cdot k! \cdot (n-k-1)! \cdot 2^{n-k-1}$ dense $\emptyset$-definable cyclic orderings.*

5 Reducts of $(\mathbb{Q}, <)$ with a partition into convex sets

In this section we examine expansions of $(\mathbb{Q}, <)$ by a partition into *convex* sets. We will describe the reducts in Theorem 5.1 and Proposition 5.8. As usual, we assume that the expansion has quantifier elimination, i.e. that it is of the form $\mathfrak{Q} := (\mathbb{Q}, <, Q_1, \ldots, Q_n)$ where $Q_1 < Q_2 < \cdots < Q_n$ and the Q_i are either singletons, or infinite without maximum and minimum. (Note that *any* expansion of $(\mathbb{Q}, <)$ by convex sets is interdefinable with such an expansion, but possibly with quantifiers).

First we look at two special cases of this situation:

(1) Let $(\mathbb{Q}, <, R_0, \ldots, R_n)$ be an expansion where $R_0 < R_1 < \cdots < R_n$ are all infinite convex sets without maximum and minimum. Let $<_i := <\restriction_{R_i}$. Then $(\mathbb{Q}, <, R_0, \ldots, R_n)$ is interdefinable with the "free union of $n+1$ dense linear orders" $(\mathbb{Q}, R_0, <_0, \ldots, R_n, <_n)$. In particular

$$\begin{aligned} \mathrm{Aut}(\mathbb{Q}, <, R_0, \ldots, R_n) &= \mathrm{Aut}(R_0, <_0) \times \cdots \times \mathrm{Aut}(R_n, <_n) \\ &\cong \mathrm{Aut}(\mathbb{Q}, <) \times \cdots \times \mathrm{Aut}(\mathbb{Q}, <). \end{aligned}$$

(2) The expansion by constants $(\mathbb{Q},<,a_1,\dots,a_n)$ is interdefinable with $(\mathbb{Q},<,Q_0,\dots,Q_{2n})$ where $Q_{2i-1}=\{a_i\}$ and $Q_{2i}=(a_i,a_{i+1})$ with $a_0=-\infty$ and $a_{n+1}=+\infty$. It is clear that for a choice of predicates $R_0,\dots,R_n$ as above,

$$\begin{aligned}\mathrm{Aut}(\mathbb{Q},<,a_1,\dots,a_n) &= \mathrm{Aut}(\mathbb{Q}\setminus\{a_1,\dots,a_n\},<,Q_0,Q_2,\dots,Q_{2n})\\ &\cong \mathrm{Aut}(\mathbb{Q},<,R_0,\dots,R_n).\end{aligned}$$

It follows that the reducts of $(\mathbb{Q},<,a_1,\dots,a_n)$ *in which the constants are definable* are in one-to-one correspondence with the reducts of $(\mathbb{Q},<,R_0,\dots,R_n)$.

5.1 Classification of the reducts

Let $\mathfrak{R}$ be a reduct of $\mathfrak{Q}$ with $E_{\mathfrak{R}}$-classes $C_1,\dots,C_k$; say $C_1,\dots,C_e$ are infinite and $C_{e+1},\dots,C_k$ finite, hence singletons. Recall that each C_i is a union of predicates Q_j. Unless otherwise specified, $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ will always be meant in $\mathfrak{R}$, i.e. $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ is the group of permutations of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ induced by $\mathrm{Aut}(\mathfrak{R})$. Let $\mathrm{Aut}_C(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ be the normal subgroup of $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ fixing all classes C_i setwise.

The aim now is to prove:

Theorem 5.1 *$\mathfrak{R}$ is determined by $E_{\mathfrak{R}}$, the action of $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$, and the generalised orderings O_i induced on the infinite classes C_i by $\mathfrak{R}_C$.*

These generalised orderings O_i will be introduced in Lemma 5.2; they are completely known by Proposition 4.3. An analysis of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ comes in Proposition 5.6. In particular there will be only finitely many possibilities for the data above. A description of how $\mathfrak{R}$ is determined by these data follows in Proposition 5.8.

Lemma 5.2 *The induced structure on an infinite C_i by $\mathfrak{R}_C$ is a generalised ordering O_i.*

Note that O_i is only determined up to orientation, but otherwise unique.

PROOF: Since $C_i\in\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, the strong types in $\mathfrak{R}_C=(\mathfrak{R},C_1,\dots,C_k)$ are the same as in $\mathfrak{R}$, therefore $E_{\mathfrak{R}_C}=E_{\mathfrak{R}}$. It follows that the induced structure $\mathfrak{R}_C\restriction_{C_i}$ has a unique strong 1-type. Now $\mathfrak{R}_C\restriction_{C_i}$ is a reduct of $\mathfrak{Q}\restriction_{C_i}$, hence a generalised ordering as it satisfies the hypotheses of Theorem 3.2. □

Unlike the situation in Section 3.1, fixing $E_{\mathfrak{R}}$-classes can induce additional structure on the other classes:

Example Let A be an infinite predicate together with a dense betweenness relation betw_A, and an equivalence relation E with two infinite classes without structure on the complement of A. We add additional structure such that the automorphisms are exactly the maps that respect betw_A and E and exchange the two E-classes iff they change the orientation of the order on A. Then fixing the E-classes induces a linear order on A. □

Let $\mathfrak{R}_C^-$ be the reduct $(\mathbb{Q},C_1,\dots,C_k,O_1,\dots,O_e)$ of $\mathfrak{R}_C$, i.e. the free union of the structures $(C_1,O_1),\dots,(C_e,O_e)$ plus the singletons $C_{e+1},\dots,C_k$. Furthermore, let $\mathfrak{R}_C^+$ be the expansion of $\mathfrak{R}_C$ by the (imaginary) elements in $C_i/E_{C_i}^{\mathrm{cyc}}$ for all i for which O_i is an unoriented ordering (where $E_{C_i}^{\mathrm{cyc}}$ is the equivalence relation interdefinable with O_i as defined in Section 2.3.2).

Because $\mathfrak{R}_C^+$ is an expansion of $\mathfrak{R}_C$ by finitely many elements of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, we get on the one hand that $\mathfrak{R}_C^+$ has finite index over $\mathfrak{R}_C$, on the other hand that $E_{\mathfrak{R}_C^+} = E_{\mathfrak{R}}$. This implies with the proof of Lemma 5.2 that the structure induced on C_i by $\mathfrak{R}_C^+$ is a generalised ordering O_i^+ expanding O_i. We will see in Proposition 5.6 that $\mathfrak{R}_C^+$ already the expansion by all of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

Lemma 5.3 *O_i^+ is the oriented version of an unoriented ordering O_i, and equals O_i in the other cases.*

PROOF: As $\mathfrak{R}_C^+$ has finite index in $\mathfrak{R}_C$, the index $\left|\mathrm{Aut}(\mathfrak{R}_C)\restriction_{C_i} : \mathrm{Aut}(\mathfrak{R}_C^+)\restriction_{C_i}\right|$ is finite. Then the index of the completions $\left|\mathrm{Aut}(C_i, O_i) : \mathrm{Aut}(C_i, O_i^+)\right|$ is finite, too. Hence O_i^+ equals O_i if the latter is trivial or already oriented, and is the oriented version of O_i in the other cases because we have added the "orientations" $C_i/E_{C_i}^{\mathrm{cyc}}$. $\square$

Lemma 5.4
$$\begin{aligned}\mathrm{Aut}(\mathfrak{R}_C^-) &\cong \mathrm{Aut}(C_1, O_1) \times \cdots \times \mathrm{Aut}(C_e, O_e)\\ \mathrm{Aut}(\mathfrak{R}_C^+) &\cong \mathrm{Aut}(C_1, O_1^+) \times \cdots \times \mathrm{Aut}(C_e, O_e^+)\end{aligned}$$
The isomorphisms are the natural isomorphisms as permutation groups.

PROOF: The first statement is by definition of $\mathfrak{R}_C^-$.

If $\mathfrak{M}$ is a structure and $A \subseteq M$, we let $\mathfrak{M}(A)$ be the expansion of $\mathfrak{M}$ by constants for the elements of A.

Claim Any relation definable on C_i in $\mathfrak{R}_C^+(\mathbb{Q} \setminus C_i)$ is a generalised ordering.

PROOF: Assume that new structure on C_i is definable with parameters $\bar{a} \in \mathbb{Q}\setminus C_i$. The structure $\mathfrak{R}_C^+(\bar{a})$ is a reduct of $\mathfrak{Q}(\bar{a})$, which is also an expansion of $(\mathbb{Q}, <)$ by convex sets. Hence we get the result by Theorem 3.2 if we show that the induced structure on C_i by $\mathfrak{R}_C^+(\bar{a})$ has a unique strong 1-type. We will work in $\mathfrak{R}_C^+$ and show that fixing pointwise $\mathbb{Q} \setminus C_i$ does not induce new strong 1-types on C_i. For this, refine $E_{\mathfrak{R}}$ on C_i by the the equivalence relation "having same strong type over the union of the other $E_{\mathfrak{R}}$-classes". It is $\emptyset$-definable because of the $\aleph_0$-categoricity and has finitely many classes because even in $\mathfrak{Q}$ there are only finitely many strong types in C_i over $\mathbb{Q} \setminus C_i$. Hence it equals $E_{\mathfrak{R}}$. $\Diamond$

Claim $\mathfrak{R}_C^+(\mathbb{Q} \setminus C_i)$ does not induce more structure on C_i than $\mathfrak{R}_C^+$.

PROOF: If $\bar{a}, \bar{b} \in \mathbb{Q} \setminus C_i$ have the same type in $\mathfrak{Q}$, there is $\alpha \in \mathrm{Aut}_{C_i}(\mathfrak{Q})$ mapping $\bar{a}$ onto $\bar{b}$. Hence $\bar{a}$ and $\bar{b}$ induce the same generalised ordering S on C_i. Since $\mathfrak{Q}$ has only finitely many types of given length, S has finite orbit under $\mathrm{Aut}(\mathfrak{R}_C^+)$. This implies that the stabiliser of S has finite index in $\mathrm{Aut}(\mathfrak{R}_C^+)$. As in the proof of Lemma 5.3 we can conclude from the finiteness of $\left|\mathrm{Aut}(\mathfrak{R}_C^+)\restriction_{C_i} : \mathrm{Aut}(\mathfrak{R}_C^+(\bar{a}))\restriction_{C_i}\right|$ that $\left|\mathrm{Aut}(C_i, O_i^+) : \mathrm{Aut}(C_i, S)\right|$ is also finite. Because O_i^+ is already oriented, $O_i^+ = S$. $\Diamond$

The second claim shows that every automorphism of (C_i, O_i^+) can be extended by the identity to an automorphism of $\mathfrak{R}_C^+$, which proves the second statement. $\square$

Corollary 5.5 *C_i is stably embedded in $\mathfrak{R}_C$, i.e.* $\mathrm{Aut}(C_i, O_i) = \mathrm{Aut}(\mathfrak{R}_C)\restriction_{C_i}$.

PROOF: If $\alpha \in \mathrm{Aut}(C_i, O_i)$ preserves O_i^+, it can be extended by the identity to an automorphism of $\mathfrak{R}_C^+$. In the other case, choose $\beta \in \mathrm{Aut}(\mathfrak{R}_C)$ which also changes the orientation on C_i. Then extend $\beta\restriction_{C_i} \circ \alpha$ and compose with β^{-1}. $\square$

Let O_i^- be the unoriented version of an oriented O_i, and equal O_i in the other cases. Let $C^{\mathrm{or}} \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ consist of the classes $C_1, \ldots, C_k$ and the imaginary elements in $C_i/E_{C_i}^{\mathrm{cyc}}$ for the infinite C_i for which O_i^- is not trivial.

Proposition 5.6 *In $\mathfrak{R}$,* $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(C^{\mathrm{or}})$.

(That is: An automorphism of $\mathfrak{R}$ fixing the equivalence classes in C^{or} setwise, fixes all the elements of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$.)

Proof: By definition of $\mathfrak{R}_C^+$, $\mathrm{Aut}(\mathfrak{R}_C^+)$ is the stabiliser of the algebraic elements in C^{or}. Now an algebraic element not definable over those would correspond to a proper closed subgroup of finite index. But according to Lemma 5.4, $\mathrm{Aut}(\mathfrak{R}_C^+)$ is a direct product of groups without proper closed subgroups of finite index, hence itself a group without proper closed subgroups of finite index. □

Remark: This proposition clarifies what is meant with "the action of $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$" in Theorem 5.1, as an automorphism of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ is determined by its action on the finitely many imaginary elements in C^{or}. For Proposition 5.6, one could make C^{or} even smaller by throwing away the elements $C_i/E_{C_i}^{\mathrm{cyc}}$ for which O_i is oriented.

Corollary 5.7 $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ *is finite.* □

Proposition 5.8 $\mathrm{Aut}(\mathfrak{R})$ *consists of all* $\alpha \in \mathrm{Sym}(\mathbb{Q})$ *such that*

- *α respects $E_{\mathfrak{R}}$* (which allows to define $C_{\alpha(i)} := \alpha[C_i]$)*;*
- *$\alpha \restriction_{C_i} : (C_i, O_i^-) \to (C_{\alpha(i)}, O_{\alpha(i)}^-)$ is an isomorphism for all $i \leqslant e$;*
- *the induced map $\bar{\alpha} \in \mathrm{Sym}(C^{\mathrm{or}})$ is in* $\mathrm{Aut}(C^{\mathrm{or}})$.

Proof: Each of the conditions makes sense by the previous one. In particular, the induced map exists because isomorphisms $(C_i, O_i^-) \to (C_j, O_j^-)$ map the orientations $C_i/E_{C_i}^{\mathrm{cyc}}$ onto $C_j/E_{C_j}^{\mathrm{cyc}}$.

It is clear that automorphisms of $\mathfrak{R}$ have to satisfy the conditions. (For the second, note that $\alpha[O_i^-]$ is definable from $O_{\alpha(i)}$, therefore $\alpha[O_i^-]$ and $O_{\alpha(i)}^-$ are equal as unoriented generalised orderings of same type).

Conversely, assume that α satisfies the conditions. Then there is $\beta \in \mathrm{Aut}(\mathfrak{R})$ inducing the same map $\bar{\beta} = \bar{\alpha}$ on C^{or}. Then $\gamma := \beta^{-1} \circ \alpha$ still satisfies the conditions, but fixes C^{or}. Hence for infinite C_i, $\gamma \restriction_{C_i}$ is not only an O_i^--isomorphism, but an O_i^+-isomorphism. Thus $\gamma \in \mathrm{Aut}(\mathfrak{R}_C^+) \leqslant \mathrm{Aut}(\mathfrak{R}_C)$ by Lemma 5.4. □

This proves Theorem 5.1 and Thomas' conjecture for $\mathfrak{Q}$, since $\mathrm{Aut}(\mathfrak{R})$ is determined by finitely many data.

5.2 Complements

(1) We consider C^{or} as a two-sorted finite structure $\mathfrak{C}^{\mathrm{or}}$: the first sort consists of the classes $C_1, \ldots, C_k$ and is partitioned by four colours, one for each type of induced generalised ordering O_i^-, plus one for singleton classes; and there is a function mapping the elements of $C_i/E_{C_i}^{\mathrm{cyc}}$ (of second sort) onto C_i. Then Theorem 5.8 shows that a reduct $\mathfrak{R}$ is determined by $\mathfrak{C}^{\mathrm{or}}$ and a sub-permutation group of $\mathrm{Aut}(\mathfrak{C}^{\mathrm{or}})$. All subgroups occur.

(2) It is possible to find sections of the canonical epimorphism $\mathrm{Aut}(\mathfrak{R}) \to \mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$. Choose

- an order-reversing involution ι_i on each C_i for which O_i^- is not trivial, and
- an isomorphism $\alpha_{ij} : (C_i, O_i^-) \to (C_j, O_j^-)$ for all (i,j) for which C_i and C_j are conjugate under $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$,

such that $\alpha_{ii} = \mathrm{id}$, $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$, and $\iota_j \circ \alpha_{ij} = \alpha_{ij} \circ \iota_i$, and construct automorphisms from these maps as described in the proof of Proposition 5.8. Then $\mathrm{Aut}(\mathfrak{R})$ is a generalised semi-direct product

$$\mathrm{Aut}(\mathfrak{R}_C) \rtimes_{\mathrm{Aut}_C(\mathrm{acl}^{\mathrm{eq}}(\emptyset))} \mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$$

as in the following fact.

Fact 5.9 (generalised semi-direct product) *Let H, N be groups and $\alpha : H \to \mathrm{Aut}(N)$ a morphism, let $K_N \leqslant N$, $K_H \leqslant H$ and $\varphi : K_N \to K_H$ an isomorphism such that*

(1) $\varphi(k)^h = \varphi(k^{\alpha(h)})$ for $k \in K_N$ and $h \in H$, and
(2) $n^{\alpha(\varphi(k))} = n^k$ for $k \in K_N$ and $n \in N$.

The generalised semi-direct product $N \rtimes_K H$ is the group $(N \rtimes H)/K$ where $K := \{(n^{-1}, \varphi(n)) \mid n \in K_N\}$.

Inner version: If $N \lhd G$ and $H \leqslant G$ such that $G = HN$ and $K = H \cap N$, then G is isomorphic to the generalised semi-direct product $N \rtimes_K H$ where α is conjugation and φ the identity.

If $N \cong M^n$ and $H \leqslant S_n$ acts on the coordinates of the product, we get a generalised wreath product $M \wr_K H$.

5.3 The case $n = 2$: reducts of $(\mathbb{Q}, <, Q_1, Q_2)$ and $(\mathbb{Q}, <, a)$

Theorem 5.10 *Let Q_1 be a proper initial segment of $\mathbb{Q}$ without supremum, Q_2 its complement. $\mathfrak{Q} = (\mathbb{Q}, <, Q_1, Q_2)$ has 53 reducts up to interdefinability, and 32 reducts up to isomorphism.*

Proof: According to our analysis, we get:

(A) 29 reducts $\mathfrak{R}$ where Q_1 is definable:

(a) 25 reducts (15 up to isomorphism) with $\mathrm{Aut}(\mathfrak{R}) \cong \mathrm{Aut}(Q_1, O_1) \times \mathrm{Aut}(Q_2, O_2)$ for generalised orderings O_i.

(b) 4 (3) reducts with $\mathrm{Aut}(\mathfrak{R}) \cong \big(\mathrm{Aut}(Q_1, O_1) \times \mathrm{Aut}(Q_2, O_2)\big) \rtimes Z^2$ for oriented generalised orderings O_i (where Z^2 is generated by an order-reversing involution).

According to Section 5.2 (1), they can also be counted as 9 reducts where a trivial structure is involved, plus $2 \cdot 2 \cdot 5$ other reducts: on each $C_i = Q_i$ two possible types of non-oriented orderings O^- times the number of subgroups of $Z^2 \times Z^2$, the automorphism group of two fixed two-element sets.

(B) 11 reducts where Q_1 is not definable, but $E_{\mathfrak{Q}}$:

(a) 7 (5) reducts with $\mathrm{Aut}(\mathfrak{R}) \cong \mathrm{Aut}(C_i, O) \wr Z^2$ for a generalised ordering O (where Z^2 is generated by an involution ι exchanging the two $E_{\mathfrak{Q}}$-classes. In case O is oriented, there are two non-interdefinable possibilities: either ι changes or respects the orientation, i.e. ι maps O_1 onto O_2 or onto O_2^{-1}.)

(b) 2 (2) reducts with $\mathrm{Aut}(\mathfrak{R}) \cong \mathrm{Aut}(C_i, O) \wr_{Z^2} (Z^2 \times Z^2)$ and 2 (2) with $\mathrm{Aut}(\mathfrak{R}) \cong \mathrm{Aut}(C_i, O) \wr_{Z^2} Z^4$ for an oriented ordering O (generalised wreath product as after Fact 5.9; the common Z^2 is generated by an order-reversing involution fixing the classes; its complement in $Z^2 \times Z^2$ exchanges the classes; and Z^4 is generated by an automorphism exchanging the classes and reversing the order on one side).

Alternatively: 1 reduct with trivial induced structure, plus $2 \cdot (10-5)$ other reducts: two possible non-trivial O^- times the five of the ten subgroups of D_4^8, i.e. the automorphism group of two two-elements sets, exchanging the two sets, i.e. not counted under (A).

(C) 13 REDUCTS WHERE $E_{\mathfrak{Q}}$ IS NOT DEFINABLE:

Theorem 3.2 applies and provides 13 (5) generalised orderings with Remark 4.4.

Clearly, no isomorphisms are possible between reducts listed under different items. □

Theorem 5.11 *Let $a \in \mathbb{Q}$ be a constant. Then $(\mathbb{Q}, <, a)$ has 116 reducts up to interdefinability and 56 reducts up to isomorphism.*

PROOF: With the notations from the beginning of Section 5, the structure can be written as $(\mathbb{Q}, <, Q_0, Q_1, Q_2)$ with $Q_1 = \{a\}$. There are four cases; the first three reduce to Theorem 5.10 (A) and (B), the fourth to Theorem 3.2 and Proposition 4.3:

- Reducts, where a is definable: they are in 1-1-correspondence with the reducts of $(\mathbb{Q}, <, Q_1, Q_2)$ of Theorem 5.10: 53 reducts (32 up to isomorphism).
- Reducts where Q_0 is definable and a has the same type as the elements in Q_2. Then the induced structure on $Q_2 \cup \{a\}$ can't be a linear order or a betweenness relation. Theorem 5.10 (A) yields 15 (12) reducts in case (a) and 2 (2) in case (b). So we have 17 (14) reducts altogether.

  Analogously with Q_0 and Q_2 exchanged. We get again 17 reducts, but no new isomorphism classes.
- Reducts with a unique 1-type, but where the equivalence relation with classes Q_0 and $Q_2 \cup \{a\}$ is definable. The induced structure on each class is the cyclic order, the separation relation or nothing. Theorem 5.10 (B) gives us 4 (3) reducts in case (a) and 2 (2) in case (b), altogether 6 (5) reducts.

  Analogously with Q_0 and Q_2 exchanged. We get 6 reducts, but no new isomorphism classes.
- Reducts with a unique strong 1-type are by Theorem 3.2 reducts of a definable linear order. By Remark 4.4, these are 4 linear orders, 4 betweenness relations, 4 cyclic orderings, 4 separation relations, and the trivial structure: 17 reducts (5 up to isomorphism).

Summa summarum: $53 + 17 + 17 + 6 + 6 + 17 = 116 \quad (32 + 14 + 5 + 5 = 56)$ □

6 Reducts of $(\mathbb{Q}, <)$ with dense predicates

In this section, we study the structure $\mathfrak{P} = (\mathbb{Q}, <, P_1, \ldots, P_n)$ where the P_i form a partition of $\mathbb{Q}$ into dense subsets ($n > 1$). We conjecture that the reducts are given by the following data, and hence that there are only finitely many possibilities (with Theorem 5.1 and Proposition 4.3):

- the equivalence relation $E_{\mathfrak{R}}$ with classes $C_1, \ldots, C_k$;
- a reduct $\mathfrak{R}'$ of $(\mathbb{Q}, C_1, <'_1, \ldots, C_n, <'_n)$ where $<'_i$ is a linear order on C_i that is $\emptyset$-definable in $\mathfrak{P}$;
- an equivalence relation $E^{\mathrm{dep}}_{\mathfrak{R}}$, coarser than $E_{\mathfrak{P}}$ (as defined in Definition 6.13).

The reduct $\mathfrak{R}$ should then be generated by $\mathfrak{R}'$ and the separation relation on each $E^{\mathrm{dep}}_{\mathfrak{R}}$-class, except on those formed of predicates P_i with trivial induced structure. Theorem 3.2 shows the conjecture in the case $E_{\mathfrak{R}}$ is trivial, and we will show in Theorem 6.14 the conjecture for the case $n = 2$, or more generally for $E_{\mathfrak{R}} = E_{\mathfrak{P}}$.

The following is an example that is not covered by our methods:

Example 6.1 *Let $n = 4$ and $\mathfrak{R} = (\mathbb{Q}, <_{P_1 \cup P_2} \triangleleft <^{-1}_{P_3 \cup P_4}, P_1 \cup P_3, P_2 \cup P_4)$. In this example, the $E_{\mathfrak{R}}$-classes are $P_1 \cup P_3$ and $P_2 \cup P_4$, the $E^{\mathrm{dep}}_{\mathfrak{R}}$-classes are $P_1 \cup P_2$ and $P_3 \cup P_4$.*

6.1 Induced structure

Let $\mathfrak{R}$ be a reduct of $\mathfrak{P}$.

Lemma 6.2 *Let C be an $E_{\mathfrak{R}}$-class. If the induced structure on C is trivial, then the induced structure on C over $\neg C$ is trivial, and C does not induce any additional structure on $\neg C$.*

Proof: Otherwise there are strict k-tuples $\bar{x}, \bar{x}'$ in C, an l-tuple $\bar{y}$ in $\neg C$ and a formula φ such that $\models \varphi(\bar{x}, \bar{y})$ and $\models \neg\varphi(\bar{x}', \bar{y})$. Moving $\bar{x}$ and $\bar{x}'$ slightly without changing their type in the original structure $\mathfrak{P}$, we find pairwise disjoint strict k-tuples $\bar{x}_1, \ldots, \bar{x}_m, \bar{x}'_1, \ldots, \bar{x}'_m$ for arbitrarily large m, such that $\models \varphi(\bar{x}_i, \bar{y})$ and $\models \neg\varphi(\bar{x}'_i, \bar{y})$. Since the induced structure on C is trivial, the formula $\exists \bar{y} \big(\bigwedge_{i_1}^m \varphi(\bar{x}_i, \bar{y}) \wedge \bigwedge_{i_1}^m \neg\varphi(\bar{x}'_i, \bar{y}) \big)$ is realised by all strict tuples in C^{2km}.

Now, chose $\bar{x}_1 < \bar{x}'_1 < \bar{x}_2 < \cdots < \bar{x}'_m$. Since $\bar{x}_i$ and $\bar{x}'_i$ are of different type over $\bar{y}$, some component of $\bar{x}_i$ and its corresponding component of $\bar{x}'_i$ are separated by a component of $\bar{y}$. This is not possible if $m > l$.

The second part of the assertion is a consequence of the first. $\square$

6.2 Independent predicates

Definition 6.3 *In $\mathfrak{R}$, $\emptyset$-definable subsets $D_1, \ldots, D_l$ of $\mathbb{Q}$ are called* independent *if whenever $\bar{x}_i, \bar{x}'_i \in D_i$ have the same type in $\mathfrak{P}$, then $(\bar{x}_1, \ldots, \bar{x}_l)$ and $(\bar{x}'_1, \ldots, \bar{x}'_l)$ have the same type in $\mathfrak{R}$.*

It follows that, if $\mathbb{Q}$ is *partitioned* into $\emptyset$-definable independent $D_1, \ldots, D_l$, then $\mathfrak{R}$ is a reduct of $(\mathbb{Q}, <_{D_1}, \ldots, <_{D_l}, P_1, \ldots, P_n)$. Furthermore, Lemma 6.2 shows that if the induced structure on some $E_{\mathfrak{R}}$-class C is trivial, then C is independent from its complement $\neg C$.

6.2.1 Dependence gives the separation relation

Proposition 6.4 *Let $\mathfrak{R}$ be a reduct of $\mathfrak{P}$, where P_1 and P_2 are strong types. Then P_1 and P_2 are dependent if and only if the separation relation is definable on $\mathbb{Q}$.*

Beginning of the proof: The 'if'-direction is obvious. For the converse, we may assume $n = 2$ because $\mathfrak{R} \restriction_{P_1 \cup P_2}$ is (trivially) a reduct of $\mathfrak{P} \restriction_{P_1 \cup P_2}$. Furthermore we will assume in the sequel that P_1, P_2 are definable. We will see that the definition of sep is invariant under exchanging P_1 and P_2. Therefore the result will also hold in the more general case.

Thus we have the following situation: $\mathfrak{R}$ is a reduct of $(\mathbb{Q}, <, P_1, P_2)$, the two predicates are definable and dependent. It follows that $\mathrm{sep}_{P_1}, \mathrm{sep}_{P_2}$ are definable, because the induced structure on each P_i is non-trivial. Also it follows from Lemma 6.8 below that there is more than one strong 1-type in P_1 over P_2, and conversely.

For $\bar{b} = (b_1, \ldots, b_m) \in P_2$, we let $E_{\bar{b}}$ be the equivalence relation defined on P_1 "to have same strong type over $\bar{b}$". We may assume $b_1 < \cdots < b_m$, and we let $b_0 = -\infty$ and $b_{m+1} = +\infty$. The $E_{\bar{b}}$-classes are unions of intervals $(b_i, b_{i+1}) \cap P_1$. We say that $E_{\bar{b}}$ *jumps* at b_i if $(b_{i-1}, b_i) \cap P_1$ and $(b_i, b_{i+1}) \cap P_1$ are in different $E_{\bar{b}}$-classes.

Fact $E_{\bar{b}}$ jumps at every $b_i \in \bar{b}$ if $\bar{b}$ is long enough.

(This is intuitively clear. Corollary 6.10 will provide a proof in a more general situation.)

Lemma 6.5 *Assume the betweenness relation is not definable on P_1, and let $\bar{b}$ be long enough. Then $(-\infty, b_1) \cap P_1$ and $(b_m, +\infty) \cap P_1$ are in the same class.*

Proof: Choose for every $i = 0, \ldots, m$ an $x_i \in (b_i, b_{i+1}) \cap P_1$, and a $y \in P_1$ with $x_m < y$. Then $\neg E_{\bar{b}} x_i x_{i+1}$, but $E_{\bar{b}} x_m y$.

Assume x_0 and y are not equivalent. As the betweenness relation is not definable on P_1, there is a $y' < x_0$ such that $(\bar{x}, y)$ and $(\bar{x}, y')$ have the same type in $\mathfrak{R}$. Thus there are $b'_1, \ldots, b'_m \in P_2$ such that consecutive elements in the sequence $y', x_0, \ldots, x_m$ are not $E_{b'_1 \ldots b'_m}$-equivalent. But the m elements $b'_1, \ldots, b'_m$ can't cause jumps at these $m+1$ consecutive pairs in the sequence: contradiction. $\square$

Now we are able to finish the proof of Proposition 6.4:

First case: The order is definable on P_2. Then for $a \in P_1$ and $b_1 \in P_2$

$$a < b_1 \iff \exists y \in P_1 \big(a \neq y \wedge \forall b_2 \ldots b_m \in P_2 \; (b_1 < b_2 < \cdots < b_m \rightarrow E_{\bar{b}} ay)\big).$$

Then the order is definable on $P_1 \cup P_2$. Also it follows that $m = 1$ and that $E_b xy \iff b \notin \mathrm{conv}(x, y)$, and clearly the situation is symmetric in P_1 and P_2: $E_a xy$ for $a \in P_1$ and $x, y \in P_2$ also defines the negation of the betweenness relation. Hence the definition of the separation relation is invariant under exchanging P_1 and P_2.

Second case: The betweenness relation is definable on P_2. As $<\restriction_{P_2}$ is algebraic over $\mathfrak{R}$, the finest definable finite equivalence relation in $(\mathfrak{R}, <\restriction_{P_2}, b)$ equals the finest definable finite equivalence relation in $(\mathfrak{R}, b)$. Hence $b \notin \mathrm{conv}(x, y)$ is also definable in $\mathfrak{R}$, and we get the global betweenness relation on $P_1 \cup P_2$.

As the assumptions for the proposition are symmetric in P_1 and P_2, we get the same result if the betweenness relation is definable on P_1.

Third case: The cyclic ordering is definable on P_2 and the betweenness relation is not definable on P_1. Let $x, y \in P_1$ and $b_1, \ldots, b_m \in P_2$ be pairwise distinct. Lemma 6.5 shows that the minimal m such that $E_{b_1 \ldots b_m}$ is not trivial is at least 2. Moreover, if $m = 2$, then $\overleftrightarrow{xb_1yb_2} \iff \neg E_{b_1b_2}xy$, and if $m > 2$, then

$$\overrightarrow{b_1xb_2} \iff \exists y \in P_1 \big(x \neq y \wedge \forall b_3 \ldots b_m \in P_2 \; (\overrightarrow{b_1b_2b_3 \ldots b_m} \to E_{\bar{b}}xy)\big).$$

Again, the global cyclic order on $P_1 \cup P_2$ is definable from this. Hence we see that $m = 2$ and that $E_{b_1b_2}xy$ defines the negation of the separation relation. In particular, the same holds for P_1 and P_2 exchanged.

Fourth case: The separation relation is definable on P_2. As above, the finest definable finite equivalence relation in $(\mathfrak{R}, \mathrm{cyc} \upharpoonright_{P_2}, b_1, b_2)$ equals the finest definable finite equivalence relation in $(\mathfrak{R}, b_1, b_2)$. Hence the separation relation between P_1 and P_2 is also definable in $\mathfrak{R}$. $\square$ of 6.4

Remark 6.6 *It is easy to see directly the following:*

(a) If the separation relation between P_1 and P_2 is definable, i.e. $\overleftrightarrow{x_1y_1x_2y_2}$ for $x_i \in P_1$ and $y_i \in P_2$, then the separation relation on $P_1 \cup P_2$ is definable.

(b) If the separation relation on $P_1 \cup P_2$ is definable and a generalised ordering O defined on P_1, then O extends definably on $P_1 \cup P_2$.

6.2.2 Dependence is trivial as a matroid

Throughout Section 6.2.2, we will assume that all P_i are definable in $\mathfrak{R}$.

Proposition 6.7 *If $P_1, \ldots, P_k$ are pairwise independent, then they are independent.*

Beginning of the proof: We prove Proposition 6.7 by induction on k. We may therefore assume that each selection of $k-1$ out of $P_1, \ldots, P_k$ is independent. The proof will include two lemmas and will be completed at the end of Section 6.2.2.

By definition, independence of $P_{i_1}, \ldots, P_{i_l}$ implies that there is only one strong 1-type in P_{i_1} over $P_{i_2} \cup \cdots \cup P_{i_l}$.

Lemma 6.8 *If there is only one strong 1-type in P_1 over $P_2 \cup \cdots \cup P_k$, then $P_1, P_2, \ldots, P_k$ are independent.*

Proof: By assumption and Theorem 3.2, any $\bar{c} \in C := P_2 \cup \cdots \cup P_k$ induces a generalised ordering on P_1 (determined up to interdefinability). A longer tuple induces a compatible generalised ordering, hence there is some $\bar{c} \in C$ with maximal induced structure on P_1 given by a relation $O_{\bar{c}}$. If $\bar{d}$ has the same type as $\bar{c}$ in the structure $\mathfrak{S} := (\mathbb{Q} \setminus P_1, P_2, \ldots, P_k, <_{P_2}, \ldots, <_{P_k})$, then, because $P_2, \ldots, P_k$ are independent, $O_{\bar{c}}$ and $O_{\bar{d}}$ can only differ by orientation. The equivalence relation $E\bar{x}\bar{y} :\iff O_{\bar{x}} = O_{\bar{y}}$ on $P_2 \cup \cdots \cup P_k$ is definable in $\mathfrak{R} \upharpoonright_{P_2 \cup \cdots \cup P_k}$, hence in $\mathfrak{S}$ since the former is a reduct of the latter. As in Lemma 4.1 and Corollary 4.2, one sees that $\mathfrak{S}$ eliminates imaginaries, $\mathrm{acl}(\emptyset) = \emptyset$, thus there is no non-trivial finite $\emptyset$-definable equivalence relation on a complete type in $\mathfrak{S}$. Therefore $O_{\bar{c}} = O_{\bar{d}}$, and $O_{\bar{c}}$ is $\emptyset$-definable in $\mathfrak{R}$.

Now, if $\bar{x}, \bar{y} \in P_1$ have the same $\mathfrak{R}$-type, they have the same $O_{\bar{c}}$-type, and therefore the same $\mathfrak{R}$-type over every tuple in C. This shows independence. $\square$

Now we assume $P_1, P_2, \ldots, P_k$ to be dependent. As a consequence of Lemma 6.2, the induced structure on each P_i is non-trivial. By Lemma 6.8, there is more than one strong 1-type in P_1 over its complement. We will show that this leads to a contradiction.

We consider finite subsets B of $P_2 \cup \cdots \cup P_k$ such that there is more than one strong type in P_1 over B. Due to the induction hypothesis, B intersects each $P_2, \ldots, P_k$. Write E_B for the equivalence relation on P_1 "to have same strong type over B".

Lemma 6.9 *If B is minimal with non-trivial E_B, then E_B jumps at each parameter from B.*

Proof: Assume E_B does not jump at $b \in B$ and let $b \in P_2$. Set $B' = B \setminus \{b\}$ and $B'' = B' \cap P_2$. Consider the equivalence relation $F(x, y)$ on P_2 defined by $E_{xB'} = E_{yB'}$. Since $P_2, \ldots, P_k$ are independent, F is B''–definable. Since E_B does not jump at $b \in B$, a whole neighbourhood of b in P_2 belongs to the same F-class.

We conclude from this that the F-class A of b is B''–definable. This can easily be checked through the four cases of generalised orderings O induced on P_2. There are only two cases where one has to recur to the special nature of E. The first case is, when O is the betweenness relation and $B'' = \{b''\}$ is a singleton: Assume $b'' < b$, then all $y \in P_2$ with $b'' < y$ are in A. With the notation $B''' = B \cap (P_3 \cup \cdots \cup P_k)$ this means that $E_{b''yB'''} = E_{b''bB'''}$. Since $P_2, \ldots, P_k$ are independent this implies $E_{x''yB'''} = E_{x''xB'''}$ for all three distinct $x'', x, y \in P_2$ such that x'' is not between x and y. This implies that $E_{xyB'''} = E_{x'y'B'''}$ for *all* distinct x, y and x', y' in P_2. So we have that $A = P_2 \setminus \{b''\}$. The second case, where O is separation and $|B''| = 2$, is treated similarly.

Therefore, if q is the O-type of b over B'', E_B is also defined by the formula

$$\exists x \in P_2 \left(\mathrm{tp}^O(x/B'') = q \wedge E_{xB'}\right),$$

contradicting the minimality of B. $\square$

Corollary 6.10 *If E_B is non-trivial, then it jumps at each of its parameters.*

Proof: Let $B_0 \subset P_2 \cup \cdots \cup P_k$ be minimal with non–trivial E_{B_0}. The independence of $P_2, \ldots, P_k$ implies that a set B_0' is minimal iff $|B_0' \cap P_i| = |B_0 \cap P_i|$ for $i = 2, \ldots, k$. So, if b is any element of B, we find a minimal subset B_0' of B such that $b \in B_0'$. E_B jumps at b since $E_{B_0'}$ does. $\square$

To complete the proof of Proposition 6.7, choose a non-trivial E_B such that B intersects P_2 and P_3 each in at least 3 elements. Write $B \cap P_2 = \{b_0, \ldots, b_m\}$ for $b_0 < \cdots < b_m$, $B \cap P_3 = \{c_0, \ldots, c_l\}$ for $c_0 < \cdots < c_l$ and set $B' = B \cap (P_4 \cup \cdots \cup P_k)$.

Since sep_{P_2} is definable, we can define the interval $(b_0, b_m) \cap P_2$ over $\bar{b}$, similarly $(c_0, c_l) \cap P_3$ is definable over $\bar{c}$. E_{xyB} jumps at x and y for all x, y. Thus

$$\forall a \neq a' \in P_1 \quad \exists x \in (b_0, b_m) \cap P_2 \quad \exists y \notin P_3 \setminus (c_0, c_l) \quad \neg E_{xyB}(a, a')$$

iff $(c_0, c_l) \subset (b_0, b_m)$. This shows that there are at least two possible types of $\bar{b}\bar{c}$ over B', which contradicts the independence of $P_2, \ldots, P_k$. $\square$ of 6.7

6.2.3 Dependence defines an equivalence relation

Lemma 6.11 *If* sep *is definable on $P_1 \cup P_2$ and on $P_2 \cup P_3$, then also on $P_1 \cup P_2 \cup P_3$.*

PROOF: It is sufficient to show that sep is definable on $P_1 \cup P_3$. If $x_1, x_2 \in P_1$ and $y_1, y_2 \in P_3$,

$$\overleftrightarrow{x_1 y_1 x_2 y_2} \iff \exists x_i^- x_i^+ \in P_2 \left(\overleftrightarrow{x_1^- x_1 x_1^+ x_2^- x_2 x_2^+} \wedge \overleftrightarrow{x_1^- x_1^+ y x_2^- x_2^+ y_2} \right)$$

$\square$

The lemma shows that dependence is an equivalence relation on the predicates. Next we show that the equivalence classes are independent.

Proposition 6.12 *Let D_i be the union of all P_j which are dependent from P_i. If $P_1, \dots, P_k$ are independent, then $D_1, \dots, D_k$ also.*

PROOF: Consider two long tuples $\bar{x}, \bar{y}$ with entries from $D_1 \cup \dots \cup D_k$ and same order type in each D_i. We construct new tuples of doubled length by replacing each element v of D_i in $\bar{x}, \bar{y}$ with two elements $v^-, v^+ \in P_i$ such that $v \in (v^-, v^+)$ and such that the intervals (v^-, v^+) are pairwise disjoint. The new tuples have the same type by independence of the P_i, and an automorphism moving one onto the other moves x_i on some element $y_i' \in (y_i^-, y_i^+)$ because the separation relation is definable on D_i (Proposition 6.4) and the tuples are long. But then there is an automorphism of $\mathfrak{P}$ fixing all y_i^-, y_i^+ and moving y_i' on y_i. $\square$

Definition 6.13 *Let $E_{\mathfrak{R}}^{\mathrm{dep}}$ be the equivalence relation on $\mathbb{Q}$ induced by dependence: $x \in P_i$ and $y \in P_j$ are equivalent if and only if P_i and P_j are dependent in $\mathfrak{R}$.*

The picture is now as follows: Let $\mathfrak{R}$ be a reduct with $E_{\mathfrak{R}} = E_{\mathfrak{P}}$, and D an $E_{\mathfrak{R}}^{\mathrm{dep}}$-class. Then either D consists of a single predicate with trivial induced structure, or D is a union of possibly several predicates which are dense in each other in $\mathfrak{R}$, and the induced structure on D is one of the four non-trivial reducts of $(D, <_D)$. The classes are independent, but the induced generalised orderings on different classes are possibly linked.

6.3 The proof of the conjecture in case $E_{\mathfrak{R}} = E_{\mathfrak{P}}$

Let $\mathfrak{R}$ be a reduct of $\mathfrak{P}$ with $E_{\mathfrak{R}} = E_{\mathfrak{P}}$, i.e. all P_i are strong types. Let $D_1, \dots, D_l$ be the classes of $E_{\mathfrak{R}}^{\mathrm{dep}}$. Define $\mathfrak{R}' := \mathfrak{R} \cap (\mathbb{Q}, P_1, <_{P_1}, \dots, P_n, <_{P_n})$, and $\mathfrak{R}^< := \mathfrak{R} \cap (\mathbb{Q}, D_1, <_{D_1}, \dots, D_l, <_{D_l})$. We will prove:

Theorem 6.14 $\mathfrak{R}$ *is of the form $\mathfrak{R}' \cup \mathfrak{R}^<$ (with $\cup$ in the lattice of reducts).*

PROOF: First we need the following remark: If P_1, P_2 are in the same $E_{\mathfrak{R}}^{\mathrm{dep}}$-class D_i, then the equivalence relations $E_{P_1}^{\mathrm{cyc}}$ and $E_{P_2}^{\mathrm{cyc}}$ are interdefinable. This is clear as two triples in P_1 are equally oriented if and only if intercalated triples of P_2 are equally oriented. Therefore the orientations $P_1/E_{P_1}^{\mathrm{cyc}}$ and $P_2/E_{P_2}^{\mathrm{cyc}}$ are interdefinable, and interdefinable with $D_i/E_{D_i}^{\mathrm{cyc}}$. We let $P^{\mathrm{or}} \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ consist of the predicates $P_1, \dots, P_n$ and the orientations $D_i/E_{D_i}^{\mathrm{cyc}}$ for all D_i with non-trivial induced structure. Recall that $\mathfrak{R}(A)$ denotes the expansion by constants for the elements of A.

We have to show that each automorphism α of $\mathfrak{R}' \cup \mathfrak{R}^<$ is an automorphism of $\mathfrak{R}$ (the converse is trivially true). It is not hard to see that $\mathrm{Aut}(\mathfrak{R}) \restriction_{P_i \to P_j, P^{\mathrm{or}}}$ is dense in $\mathrm{Aut}(\mathfrak{R}') \restriction_{P_i \to P_j, P^{\mathrm{or}}}$, i.e. an automorphism of $\mathfrak{R}'$ coincide on finite tuples of P_i with automorphisms of $\mathfrak{R}$ permuting P^{or}

in the same way. This is true because $\mathrm{Aut}(\mathfrak{R}')$ equals the completion of the group generated by $\mathrm{Aut}(\mathfrak{R})$ and $\mathrm{Aut}(\mathbb{Q}, P_1, <_{P_1}, \dots, P_n, <_{P_n})$.

As a special case we have that $\mathfrak{R}$ and $\mathfrak{R}'$ induce the same structure on P^{or}. So we may assume that α leaves every P_i and the orientations on it invariant. We will now show that α coincides on every finite part of each D_i with some $\beta_i \in \mathrm{Aut}_{P^{\mathrm{or}}}(\mathfrak{R})$ that is the identity outside D_i. Then $\alpha = \beta_1 \circ \dots \beta_l \in \mathrm{Aut}(\mathfrak{R})$, which will prove the theorem.

If the induced structure on D_i is trivial, this follows immediately from Lemma 6.2.

If the induced structure on D_i is non-trivial, we consider long tuples $\bar{x}$ whose elements come alternately from $P_1, \dots, P_n$, at least four out of each P_j. Say $P_1 \subseteq D_i$. Then there is $\beta_i \in \mathrm{Aut}_{P^{\mathrm{or}}}(\mathfrak{R})$ coinciding with α on $\bar{x} \cap P_1$. Since α respects the separation relation on D_i and $\bar{x} \cap P_1$ is long enough, we can find $\beta_i' \in \mathrm{Aut}(\mathfrak{P})$ fixing $\alpha(\bar{x}) \cap P_1$ and moving the remaining D_i-part of $\beta_i(\bar{x})$ on $\alpha(\bar{x}) \cap D_i$. Thus α and $\beta_i' \circ \beta_i$ coincide on $\bar{x} \cap D_i$.

It is now sufficient to show that $\mathfrak{R}(P^{\mathrm{or}}, \mathbb{Q} \setminus D_i)$ does not induce more structure on D_i as $\mathfrak{R}(P^{\mathrm{or}})$ does. The proof works as for the second claim of Lemma 5.4:

Independence of the D_i implies that there is only one strong 1-type in P_1 over $\mathbb{Q} \setminus D_i$ (as in the remark preceding Lemma 6.8). Therefore $\mathbb{Q} \setminus D_i$ induces a generalised ordering on P_1. Again because of the independence, tuples in $\mathbb{Q} \setminus D_i$ of same type induce the same structure. Since there are only finitely many types, the induced structure is of finite index. But the orientations are already fixed, hence there is no new induced structure on P_1. Now the induced structure on P_1 together with the separation relation on D_i completely determines the induced structure on D_i (Remark 6.6). $\square$

In particular, this settles the conjecture for $n = 2$, as either $E_{\mathfrak{R}} = E_{\mathfrak{P}}$ or $E_{\mathfrak{R}}$ is trivial.

In fact, the proof of the theorem shows that $\mathfrak{R}$ is of the form $\mathfrak{R}'' \cup \mathfrak{R}^{<}$ where $\mathfrak{R}'' := \mathfrak{R} \cap (\mathbb{Q}, P_1, \dots, P_n, P_1/E_{P_1}^{\mathrm{cyc}}, \dots, P_n/E_{P_n}^{\mathrm{cyc}})$.

6.3.1 Ideas for the remaining cases

One possibility to prove the conjecture completely would be to develop a theory of dependence for $E_{\mathfrak{R}}$-classes. Another possibility would be to show the following: If $\mathfrak{R}$ is a reduct where P_1 and P_2 live in the same strong type, then $\mathfrak{R}$ is a reduct of $(\mathbb{Q}, <', P_1 \cup P_2, P_3, \dots, P_n)$ for some linear order $<'$ on $\mathbb{Q}$ that is $\emptyset$-definable in $\mathfrak{P}$. Example 6.1 might help to illustrate the difficulties.

7 The general case

We sketch a proof of a lower bound for the number of reducts in the general case.

Remark *Let $\mathfrak{P} = (\mathbb{Q}, <, P_1, \dots, P_n)$ be an expansion of $(\mathbb{Q}, <)$ by a partition of $\mathbb{Q}$, eliminating quantifiers, and let $\mathfrak{P}' = (\mathbb{Q}, <, P_1', \dots, P_n')$ be an expansion by a partition of dense predicates. Then $\mathfrak{P}$ has at most as many reducts as $\mathfrak{P}'$.*

Sketch of a proof: This is clear if all the P_i are infinite, since then $\mathfrak{P}$ is isomorphic to some reduct $(\mathbb{Q}, <_{\mathcal{B},\mathcal{X}}, P_1', \dots, P_n')$ of $\mathfrak{P}'$. The idea now is to "blow up" singletons.

Let $\mathfrak{R}$ be a reduct of $\mathfrak{P}$. With each $E_{\mathfrak{R}}$-class $C = P_{i_1} \cup \dots \cup P_{i_l}$ we associate $C' := P_{i_1}' \cup \dots \cup P_{i_l}'$, and call $\mathfrak{P}'/E_{\mathfrak{R}}$ the set of these C'. Fix order preserving bijections $\alpha_i : P_i \to P_i'$ for the infinite predicates P_i, and choose for each C' a $\emptyset$-definable generalised ordering O' with the following properties: (1) O' is trivial if C is finite; (2) O' is of the same type as the generalised ordering O induced by $\mathfrak{R}$ on C; (3) O and O' are compatible via the α_i, i.e. if $\beta \in \mathrm{Aut}_{\{P_i\}}(\mathfrak{R})$, then $\alpha \circ \beta \restriction_{P_i}$ extends to an automorphism of O'.

Then we define a reduct $\mathfrak{R}'$ of $\mathfrak{P}'$ by letting $\mathrm{Aut}(\mathfrak{R}')$ the closed subgroup generated by all permutations of $\mathbb{Q}$ that act on $\mathfrak{P}'/E_{\mathfrak{R}}$ like an automorphism of $\mathfrak{R}$ and respect the generalised orderings O' on the C'. This defines an injection of the reducts of $\mathfrak{P}$ to the reducts of $\mathfrak{P}'$. $\square$

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