SIMPLICITY OF THE AUTOMORPHISM GROUPS OF GENERALISED METRIC SPACES

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Abstract. Tent and Ziegler proved that the automorphism group of the Urysohn sphere is simple and that the automorphism group of the Urysohn space is simple modulo bounded automorphisms. A key component of their proof is the definition of a stationary independence relation (SIR). In this paper we prove that the existence of a SIR satisfying some extra axioms is enough to prove simplicity of the automorphism group of a countable structure. The extra axioms are chosen with applications in mind, namely homogeneous structures which admit a "metric-like amalgamation", for example all primitive 3-constrained metrically homogeneous graphs of finite diameter from Cherlin's list.

1. Introduction

In 2011, Macpherson and Tent [MT11] proved that the automorphism groups of Fraïssé limits of free amalgamation classes are simple. This was followed by two papers of Tent and Ziegler [TZ13b, TZ13a] where they prove that the isometry group of the Urysohn space (the unique complete separable homogeneous metric space universal for all finite metric spaces) modulo bounded isometries (i.e. isometries f with a finite bound on the distance between x and $f(x)$) is simple and that the isometry group of the Urysohn sphere is simple. Later, Evans, Ghadernezhad and Tent [EGT16] proved simplicity for automorphism groups of some Hrushovski constructions, and Li [Li18] proved simplicity for the structures from Cherlin's list of 26 primitive triangle-constrained homogeneous structures with 4 binary symmetric relations (see appendix of [Che98]).

More recently, Tent and Ziegler's method was generalised to asymmetric structures. Li [Li19] proved that the automorphism groups of some of Cherlin's asymmetric structures in the appendix of [Che98] are simple. The same result for non-trivial linearly ordered free homogeneous structures has been proved independently by Calderoni, Kwiatkowska and Tent [CKT20] and Li [Li20]. Also in [Li20], simplicity was proved for the automorphism groups of the universal n-linear orders for $n \geq 2$. Another recent example where (non-stationary) independence relations have been used to prove strong results about automorphism groups of structures is a paper by Kaplan and Simon [KS19].

In this paper, we adapt the methods of Tent and Ziegler and prove the following theorem (definitions and examples will be given in the upcoming paragraphs).

Theorem 1.1. *Let $\mathbb{F}$ be a transitive countable relational structure with a bounded 1-supported metric-like stationary independence relation $\mathop{\smile\!\!\!\!\!\!|}$. Then* $\mathrm{Aut}(\mathbb{F})$ *is simple.*

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As direct corollaries of Theorem 1.1, we get the following two more concrete results, for which the definitions will be given in Section 4.

Theorem 1.2. *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a finite archimedean partially ordered commutative semigroup with at least two elements and let $\mathbb{F}$ be a homogeneous $\mathfrak{M}$-metric space which realises all distances. Assume that $\mathbb{F}$ admits an $\mathfrak{M}$-shortest path independence relation $\mathop{\smile\!\!\!\!\!\!|}$ and that $\mathop{\smile\!\!\!\!\!\!|}$ is a 1-supported SIR. Then* $\mathrm{Aut}(\mathbb{F})$ *is simple.*

Theorem 1.3. *If G is a countably infinite metrically homogeneous graph which corresponds to one of the primitive 3-constrained finite-diameter classes from Cherlin's catalogue* [Che11]*, then* $\mathrm{Aut}(G)$ *is simple.*

1.1. Stationary independence relations.

The notion of stationary independence relations (Definition 1.4) was developed by Tent and Ziegler in their paper on the Urysohn space [TZ13b]. It has several generalisations (e.g. for structures with closures [EGT16]), but for our purposes the original variant suffices.

Let $\mathbb{F}$ be a relational structure and let $A, B \subseteq \mathbb{F}$ be finite subsets. We will identify them with the substructures induced by $\mathbb{F}$ on A and B respectively and by AB we will denote the union $A \cup B$ (and hence also the substructure induced by $\mathbb{F}$ on AB). If the set $A = \{a\}$ is singleton, we may write a instead of $\{a\}$. Uppercase letters will denote sets while lowercase will denote the elements of the structure, which we call *vertices* owing to the combinatorial background of part of the authors. As is usual in this area, if $A \subseteq \mathbb{F}$, we sometimes assume that it has some implicit enumeration. This is clear from the context and should not cause any confusion.

Let $A, X \subseteq \mathbb{F}$. By the *type of A over X* (denoted by $\mathrm{tp}(A/X)$) we mean the orbit of A under the action of the stabilizer subgroup of $\mathrm{Aut}(\mathbb{F})$ with respect to X. If $p = \mathrm{tp}(A/X)$, we say that $B \subseteq \mathbb{F}$ *realises* p (and denote it as $B \models p$) if B lies in p, in other words, if there is an automorphism of $\mathbb{F}$ fixing X pointwise which maps A to B. To simplify the notation, we write $\mathrm{tp}(A)$ for $\mathrm{tp}(A/\emptyset)$. Our types correspond to realised types in a (strongly) homogeneous structure in the standard model-theoretic terminology. In fact, we may assume that the language is chosen so that $\mathbb{F}$ is homogeneous, that is, partial automorphisms between finite substructures of $\mathbb{F}$ extend to automorphisms.

Definition 1.4 (Stationary Independence Relation)**.** Let $\mathbb{F}$ be a relational structure. A ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on finite subsets of $\mathbb{F}$ is called a *stationary independence relation* (*SIR*, with $A \mathop{\smile\!\!\!\!\!\!|}_C B$ being pronounced "A is independent from B over C") if the following conditions are satisfied:

- SIR1 *(Invariance).* The independence of finite subsets of $\mathbb{F}$ only depends on their type. In particular, for every automorphism f of $\mathbb{F}$, we have $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$.
- SIR2 *(Symmetry).* If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.
- SIR3 *(Monotonicity).* If $A \mathop{\smile\!\!\!\!\!\!|}_C BD$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$.
- SIR4 *(Existence).* For every A, B and C in $\mathbb{F}$, there is some $A' \models \mathrm{tp}(A/C)$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
- SIR5 *(Transitivity)* If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} B'$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B'$.
- SIR6 *(Stationarity)* If A and A' have the same type over C and are both independent over C from some set B then they also have the same type over BC.

Note that by an observation of [Bau16], these axioms are redundant as Monotonicity can be derived from the rest of them.

Stationary independence relations correspond to "canonical amalgamations" by putting $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if the canonical amalgamation of AC and BC over C

is isomorphic to ABC. The notion of canonical amalgamations can be formalised, see [ABWH+17c].

To make our proofs shorter, we will sometimes use Symmetry, Monotonicity and Existence implicitly. The following observation which follows from Invariance will be useful later.

Observation 1.5. *If $\mathbb{F}$ is a relational structure, $\mathop{\smile\!\!\!\!\!|}$ a SIR on $\mathbb{F}$ and $A \mathop{\smile\!\!\!\!\!|}_C B$, then $A \mathop{\smile\!\!\!\!\!|}_C BC$.*

Definition 1.6 (k-supported SIR)**.** Let k be a positive integer. We say that a SIR $\mathop{\smile\!\!\!\!\!|}$ is *k-supported* if for every a, b, C such that $a \mathop{\smile\!\!\!\!\!|}_C b$ there is $C' \subseteq C$ such that $|C'| \leq k$ and $a \mathop{\smile\!\!\!\!\!|}_{C'} b$.

Observation 1.7. *For $k = 1$, k-supportedness is equivalent to:*
(1-supportedness) *If $a \mathop{\smile\!\!\!\!\!|}_C b$ and $C = C_1 \cup C_2$ then $a \mathop{\smile\!\!\!\!\!|}_{C_1} b$ or $a \mathop{\smile\!\!\!\!\!|}_{C_2} b$.*

We say that a structure $\mathbb{F}$ is *transitive* if $\mathrm{tp}(a) = \mathrm{tp}(b)$ for every $a, b \in \mathbb{F}$.

Definition 1.8 (Metric-like SIR)**.** Let $\mathbb{F}$ be a relational structure with a SIR $\mathop{\smile\!\!\!\!\!|}$. We say that $\mathop{\smile\!\!\!\!\!|}$ is *metric-like* if the following conditions are satisfied:

(1) If $a \notin A$, then $a \not\mathop{\smile\!\!\!\!\!|}_A a$.
(2) For every $a \in \mathbb{F}$ there is $b \in \mathbb{F}$ such that $a \neq b$ and $a \not\mathop{\smile\!\!\!\!\!|}_\emptyset b$.
(3) *(Perfect triviality)* If $A \mathop{\smile\!\!\!\!\!|}_C B$ and $C \subseteq C'$ then $A \mathop{\smile\!\!\!\!\!|}_{C'} B$.

Lemma 1.9. *Let $\mathbb{F}$ be a relational structure with a SIR $\mathop{\smile\!\!\!\!\!|}$ which satisfies Perfect triviality. Then $\mathop{\smile\!\!\!\!\!|}$ satisfies*

(1) (Metricity) *If $A \mathop{\smile\!\!\!\!\!|}_{C_1C_2} B$ and $C_1 \mathop{\smile\!\!\!\!\!|}_D B$ then $A \mathop{\smile\!\!\!\!\!|}_{C_2D} B$.*
(2) (Triviality) *If $A \mathop{\smile\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!|}_B D$ then $A \mathop{\smile\!\!\!\!\!|}_B CD$.*

Proof. First assume that $A \mathop{\smile\!\!\!\!\!|}_{C_1C_2} B$ and $C_1 \mathop{\smile\!\!\!\!\!|}_D B$. By Perfect triviality, $C_1 \mathop{\smile\!\!\!\!\!|}_{C_2D} B$ and $A \mathop{\smile\!\!\!\!\!|}_{C_1C_2D} B$. Using Transitivity it follows that $A \mathop{\smile\!\!\!\!\!|}_{C_2D} B$, which proves Metricity.

Now assume that $A \mathop{\smile\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!|}_B D$. By Perfect triviality we get $A \mathop{\smile\!\!\!\!\!|}_{BC} D$ and by Observation 1.5 and Monotonicity it then follows that that $A \mathop{\smile\!\!\!\!\!|}_{BC} CD$. Using Transitivity together with $A \mathop{\smile\!\!\!\!\!|}_B C$ then implies $A \mathop{\smile\!\!\!\!\!|}_B CD$. □

In fact, Metricity is equivalent to Perfect triviality if $\mathop{\smile\!\!\!\!\!|}$ is a SIR. The following is a simple corollary of Triviality which will be useful later.

Corollary 1.10. *If $a \mathop{\smile\!\!\!\!\!|}_\emptyset x$ for every $x \in X$, then $a \mathop{\smile\!\!\!\!\!|}_\emptyset X$.*

Definition 1.11 (Geodesic sequence[1])**.** Let $\mathbb{F}$ be a relational structure with a SIR $\mathop{\smile\!\!\!\!\!|}$. We say that a sequence $a_1, \ldots, a_n \in \mathbb{F}$ of pairwise distinct vertices of $\mathbb{F}$ is *geodesic* if for every $1 \leq i < j < k \leq n$ it holds that $a_i \mathop{\smile\!\!\!\!\!|}_{a_j} a_k$.

Definition 1.12. Let $\mathbb{F}$ be a relational structure with a SIR $\mathop{\smile\!\!\!\!\!|}$. We say that $\mathop{\smile\!\!\!\!\!|}$ is *bounded* if it satisfies

(Boundedness) There exists an integer k_0 such that if $a_0, \ldots, a_k$ is a geodesic sequence with $k \geq k_0$, then $a_0 \mathop{\smile\!\!\!\!\!|}_\emptyset a_k$.

We denote the smallest such k_0 by $\| \mathop{\smile\!\!\!\!\!|} \|$.

The reader is encouraged to have the following examples in mind when reading this paper.

[1]We thank the anonymous referee for suggesting this definition.

Example 1. Let $\mathbb{F}$ be the Fraïssé limit of all finite metric spaces using only distances $\{0, 1, \dots, n\}$ for some fixed $n \geq 2$. Define $\mathop{\smile\!\!\!\!\!\!\mid}$ on $\mathbb{F}$ by putting $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if for every $a \in A$ and every $b \in B$ it holds that $d(a, b) = \min(\{n\} \cup \{d(a, c) + d(b, c) : c \in C\})$. It is straightforward to check that $\mathop{\smile\!\!\!\!\!\!\mid}$ is a bounded 1-supported metric-like SIR with $\| \mathop{\smile\!\!\!\!\!\!\mid} \| = n$.

For the Urysohn sphere, the only axiom which we do not have at hand is, paradoxically, Boundedness.

Example 2. Let $\mathbb{U}_1$ be the Urysohn sphere, that is, the unique homogeneous separable complete metric space with distances from $[0, 1]$ which is universal for all finite metric spaces with distances from $[0, 1]$. We will denote its metric by d (clearly, one can view $\mathbb{U}_1$ as a relational structure by introducing a binary relation for every distance). Define the relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on finite subsets of $\mathbb{U}_1$ by putting $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if for every $a \in A$ and every $b \in B$ it holds that $d(a, b) = \min(\{1\} \cup \{d(a, c) + d(b, c) : c \in C\})$. One can check that $\mathop{\smile\!\!\!\!\!\!\mid}$ is a 1-supported metric-like SIR, but does not satisfy Boundedness, as for every k one can find a geodesic sequence with k vertices such that the distance of every consecutive pair of them is smaller that $\frac{1}{k-1}$.

Example 3 (k-supported metric-like SIR)**.** Let $k \geq 1$ and $n \geq 3$ be integers. Put $S = \{1, \dots, n\}^k \cup \{0\}^k$, let A be a set and let $d\colon A^2 \to S$ be a function. Let $\preceq$ be the product order on S (i.e. $(a_1, \dots, a_k) \preceq (b_1, \dots, b_k)$ if and only if $a_i \leq b_i$ for every $1 \leq i \leq k$) and let $\oplus$ be the component-wise addition on S capped at n (i.e. $(a_1, \dots, a_k) \oplus (b_1, \dots, b_k) = (c_1, \dots, c_k)$, where $c_i = \min(n, a_i + b_i)$ for every $1 \leq i \leq k$).

We say that (A, d) is an $[n]^k$-metric space if the following holds for every $x, y, z \in A$:

(1) $d(x, y) = d(y, x)$,
(2) $d(x, y) = (0, \dots, 0)$ if and only if $x = y$,
(3) $d(x, z) \preceq d(x, y) \oplus d(y, z)$.

One can verify that the class of all finite $[n]^k$-metric spaces is a Fraïssé class. Consider the structure $\mathbb{M}_k = (M_k, d)$, which is the Fraïssé limit of the class of all $[n]^k$-metric spaces, and define $\mathop{\smile\!\!\!\!\!\!\mid}$ on $\mathbb{M}_k$ by putting $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if for every $a \in A$ and every $b \in B$ it holds that $d(a, b) = \inf_{\preceq}\{d(a, c) \oplus d(c, b) : c \in C\}$. As $\preceq$ has a maximum, the infimum of the empty set is $(n, \dots, n)$.

It is easy to verify that $\mathop{\smile\!\!\!\!\!\!\mid}$ is a bounded metric-like SIR. Moreover, it is k-supported, but not k'-supported for any $k' < k$, which is witnessed by vertices $a, b, c_1, \dots, c_k \in \mathbb{M}_k$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{\{c_1, \dots, c_k\}} b$, $d(a, c_i) = (1, \dots, 1)$ for every i and $d(b, c_i)$ is equal to 1 on the i-th coordinate and equal to 2 everywhere else.

2. Geodesic sequences

In this section we prove some auxiliary results about geodesic sequences which will be used later. Fix a transitive relational structure $\mathbb{F}$ with a metric-like SIR $\mathop{\smile\!\!\!\!\!\!\mid}$.

Lemma 2.1. *Let $a_1, \dots, a_n$ be a geodesic sequence of vertices of $\mathbb{F}$ and let $b \in \mathbb{F} \setminus \{a_n\}$. Then there is $a_{n+1} \models \mathrm{tp}(b/a_n)$ such that $a_1, \dots, a_{n+1}$ is a geodesic sequence.*

Proof. Using Existence, pick $a_{n+1} \models \mathrm{tp}(b/a_n)$ such that $a_1 \cdots a_{n-1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_n} a_{n+1}$. Consider any $1 \leq i < j \leq n-1$. By Monotonicity, $a_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_n} a_{n+1}$ and hence, by Perfect triviality, $a_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_j a_n} a_{n+1}$. Since $a_1, \dots, a_n$ is a geodesic sequence, we know that $a_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_j} a_n$. Transitivity now implies that $a_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_j} a_{n+1}$ and hence $a_1, \dots, a_{n+1}$ is a geodesic sequence.

□

Lemma 2.2. *Let $a, b, c \in \mathbb{F}$ be distinct such that $a \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{\emptyset} b$. There is a geodesic sequence $a_0, a_1, \ldots, a_n \in \mathbb{F}$ satisfying the following:*

(1) $a = a_0$ and $b = a_n$, and
(2) for every $0 \leq i \leq n-1$ it holds that $\mathrm{tp}(a_i a_{i+1}) = \mathrm{tp}(ac)$,
(3) $n = \| \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}} \|$.

Proof. First observe that since all vertices have the same type, for every $v \in \mathbb{F}$ there is $v' \in \mathbb{F}$ such that $\mathrm{tp}(vv') = \mathrm{tp}(ac)$. Put $n = \| \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}} \|$ and use Lemma 2.1 repeatedly to obtain a geodesic sequence $a, x_1, \ldots, x_n$ such that all consecutive pairs of vertices have the type $\mathrm{tp}(ac)$. We know that $a \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{\emptyset} x_n$. By Stationarity, $\mathrm{tp}(x_n/a) = \mathrm{tp}(b/a)$, hence there exists an automorphism f of $\mathbb{F}$ which fixes a and maps x_n to b. By Invariance, $f(a), f(x_1), \ldots, f(x_n)$ has the desired properties. □

Lemma 2.3. *Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ be geodesic sequences of vertices of $\mathbb{F}$ such that for every $1 \leq i < k$ we have $\mathrm{tp}(v_i v_{i+1}) = \mathrm{tp}(w_i w_{i+1})$. Then $\mathrm{tp}(v_1 \cdots v_k) = \mathrm{tp}(w_1 \cdots w_k)$.*

Proof. We shall prove by induction on m that $\mathrm{tp}(v_1 \cdots v_m) = \mathrm{tp}(w_1 \cdots w_m)$. For $m = 2$ this is true by the assumption. Assume now that the statement is true for some m. Using the fact that $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are geodesic sequences and Triviality we get that $v_1 \cdots v_{m-1} \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{v_m} v_{m+1}$ and $w_1 \cdots w_{m-1} \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{w_m} w_{m+1}$. By the assumption we have $\mathrm{tp}(v_m v_{m+1}) = \mathrm{tp}(w_m w_{i+m})$, hence Stationarity together with Invariance give $\mathrm{tp}(v_1 \cdots v_{m+1}) = \mathrm{tp}(w_1 \cdots w_{m+1})$.

□

Proposition 2.4. *Let a, b, c be vertices of $\mathbb{F}$ satisfying the following:*

(1) $a \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{b} c$,
(2) there is a geodesic sequence $a = v_1, \ldots, v_k = b$,
(3) there is a geodesic sequence $b = w_1, \ldots, w_\ell = c$.

Then there is a geodesic sequence $a = x_1, \ldots, x_{k+\ell-1} = c$ such that $\mathrm{tp}(x_1 \cdots x_k) = \mathrm{tp}(v_1 \cdots v_k)$ and $\mathrm{tp}(x_k \cdots x_{k+\ell-1}) = \mathrm{tp}(w_1 \cdots w_\ell)$.

Proof. Use Lemma 2.1 and the fact that all vertices have the same type $\ell - 1$ times to extend $v_1, \ldots, v_k$ by vertices $w'_2, \ldots, w'_\ell$ such that $v_1, \ldots, v_k, w'_2, \ldots, w'_\ell$ is a geodesic sequence and for every $1 \leq i < \ell$ we have $\mathrm{tp}(w'_i w'_{i+1}) = \mathrm{tp}(w_i w_{i+1})$, where we put $w'_1 = v_k$ to simplify the notation.

In particular, $w'_1, \ldots, w'_\ell$ is a geodesic sequence. Using Lemma 2.3 we get that $\mathrm{tp}(w_1 \cdots w_\ell) = \mathrm{tp}(w'_1 \cdots w'_\ell)$, so in particular $\mathrm{tp}(w_1 w_\ell) = \mathrm{tp}(w'_1 w'_\ell)$. Since $w_1 = w'_1 = v_k$, we have that $\mathrm{tp}(w_\ell / v_k) = \mathrm{tp}(w'_\ell / v_k)$. By the hypothesis and the construction, $w_\ell \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{v_k} v_1$ and $w'_\ell \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{v_k} v_1$. Stationarity implies that $w'_\ell \models \mathrm{tp}(w_\ell / v_1 v_k)$, so in particular $w'_\ell \models \mathrm{tp}(w_\ell / v_1)$.

In other words, there is an automorphism g of $\mathbb{F}$ such that $g(v_1) = v_1$ and $g(w'_\ell) = w_\ell$. The image of $v_1, \ldots, v_k, w'_2, \ldots, w'_\ell$ under g then gives the desired geodesic sequence $x_1, \ldots, x_{k+\ell-1}$. □

Let $a, b \in \mathbb{F}$ be distinct. We say that b *is almost free from* a if $a \not\mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{\emptyset} b$ and for every $c \in \mathbb{F}$ different from a, b such that $a \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{b} c$ it holds that $a \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{\emptyset} c$.

Observation 2.5. *Let $a, b \in \mathbb{F}$ be such that b is almost free from a. For every $a', b' \in \mathbb{F}$ such that $\mathrm{tp}(a'b') = \mathrm{tp}(ab)$ it holds that b' is almost free from a'.*

Lemma 2.6. *Suppose that $\mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}$ is bounded. For every $a \in \mathbb{F}$ and every finite $X \subseteq \mathbb{F}$ such that $a \notin X$ there is $b \in \mathbb{F}$ such that a is almost free from b, b is almost free from a, and $b \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{a} X$. In particular, $b \not\mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{\emptyset} a$ and $b \mathop{\smile\hskip -0.9em\lower 0.4ex\hbox{$\mid$}}_{\emptyset} X$.*

Proof. We claim that there exist $a', b' \in \mathbb{F}$ such that b' is almost free from a' and a' is almost free from b'. Suppose that this is true. Since $\mathbb{F}$ is transitive, there is an automorphism f such that $f(a') = a$. Pick $b \models \mathrm{tp}(f(b')/a)$ such that $b \mathop{\smile\!\!\!\!\!\!\;|}_a X$. By Observation 2.5, b is almost free from a and a is almost free from b. The "in particular" part is immediate using Corollary 1.10.

Hence it suffices to prove the claim. Pick $a', b' \in \mathbb{F}$ such that $b' \not\mathop{\smile\!\!\!\!\!\!\;|}_\emptyset a'$ and the length of the longest geodesic sequence starting at a' finishing at b' is as large as possible. (As $\mathop{\smile\!\!\!\!\!\!\;|}$ is bounded, such a', b' exist.) Pick $c \in \mathbb{F}$ such that $a' \mathop{\smile\!\!\!\!\!\!\;|}_{b'} c$. By Proposition 2.4, we can extend the geodesic sequence from a' to b' by some $c' \models \mathrm{tp}(c/b')$. By the properties of a', b' we get that $a' \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset c'$. Invariance and Stationarity then imply that $a' \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset c$ and consequently b' is almost free from a'.

To prove that a' is almost free from b', pick $c \in \mathbb{F}$ such that $b' \mathop{\smile\!\!\!\!\!\!\;|}_{a'} c$. Since the reverse of a geodesic sequence is a geodesic sequence, we extend the geodesic sequence from b' to a' by some $c' \models \mathrm{tp}(c/a')$ as above. Suppose that $b' \not\mathop{\smile\!\!\!\!\!\!\;|}_\emptyset c'$. Since $\mathbb{F}$ is transitive, there is an automorphism f such that $f(b') = a'$. The image of the geodesic sequence from b' to c' is then a geodesic sequence starting at a' which is longer than the geodesic sequence from a' to b' we started with. This is a contradiction, hence $b' \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset c'$. As before, we get that a' is almost free from b' which concludes the proof. □

3. Proof of Theorem 1.1

We will closely follow the proof from the Tent–Ziegler paper on the Urysohn sphere [TZ13a] and use the following result by Tent and Ziegler [TZ13b].

Definition 3.1. Let $\mathbb{F}$ be a countable structure with a stationary independence relation $\mathop{\smile\!\!\!\!\!\!\;|}$, let $g \in \mathrm{Aut}(\mathbb{F})$, let $A \subseteq \mathbb{F}$ be finite and let $p = \mathrm{tp}(a/A)$ be a type. We say that g *moves* p *almost maximally* if there is a realisation $x \models p$ such that

$$x \mathop{\smile\!\!\!\!\!\!\;|}_A g(x).$$

Theorem 3.2 (Corollary 5.4, [TZ13b]). *Let $\mathbb{F}$ be a countable structure with a stationary independence relation and let g be an automorphism of $\mathbb{F}$ which moves every type over every finite set almost maximally. Then every element of* $\mathrm{Aut}(\mathbb{F})$ *is a product of sixteen conjugates of g.*

Throughout the section, we fix $\mathbb{F}$ and $\mathop{\smile\!\!\!\!\!\!\;|}$ as in Theorem 1.1 ($\mathbb{F}$ is a transitive countable relational structure with a bounded 1-supported metric-like stationary independence relation $\mathop{\smile\!\!\!\!\!\!\;|}$) and put $G = \mathrm{Aut}(\mathbb{F})$. As before, we may assume that $\mathbb{F}$ is homogeneous (this will slightly simplify the proof of Lemma 3.6).

Lemma 3.3. *If $g \in G$ is not the identity then there is $a \in \mathbb{F}$ and $h \in G$ which is a product of $\|\mathop{\smile\!\!\!\!\!\!\;|}\|$ conjugates of g such that $a \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset h(a)$.*

Proof. Let $a \in \mathbb{F}$ be such that $a \neq g(a)$ and pick $b \in \mathbb{F}$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset b$ (Existence). Use Lemma 2.2 to obtain a geodesic sequence $a = a_0, \ldots, a_n = b$ such that $n = \|\mathop{\smile\!\!\!\!\!\!\;|}\|$ and for every $0 \leq i \leq n-1$ we have $\mathrm{tp}(a_i a_{i+1}) = \mathrm{tp}(a g(a))$. This means that there are automorphisms $h_0, \ldots, h_{n-1}$ such that $h_i(a) = a_i$ and $h_i(g(a)) = a_{i+1}$. Then $h_i g h_i^{-1}$ moves a_i to a_{i+1} and the statement follows. □

Lemma 3.4. *Let $g \in G$ be such that for some $a \in \mathbb{F}$ we have $a \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset g(a)$. Then for every finite set $A \subset \mathbb{F}$ there is $x \in \mathbb{F}$ with $x \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset A$ and $x \neq g(x)$.*

Proof. We may assume that $a \in A$. Put $Y = A \cup g^{-1}(A)$ and choose $b \in \mathbb{F}$ with $b \neq a$ and $b \not\mathop{\smile\!\!\!\!\!\!\;|}_\emptyset a$ ($\mathop{\smile\!\!\!\!\!\!\;|}$ is metric-like) such that moreover $b \mathop{\smile\!\!\!\!\!\!\;|}_a Y$ (Existence and Invariance). This means that $b \notin g^{-1}(A)$ (if $b \in g^{-1}(A)$, then $b \in Y$, so $b \mathop{\smile\!\!\!\!\!\!\;|}_a b$, which is in contradiction with part (1) of Definition 1.8) and hence $g(b) \notin A$. We know that $a \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g^{-1}(a)$ (by Invariance) and also $b \mathop{\smile\!\!\!\!\!\!\;|}_{a} g^{-1}(a)$, thus $b \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g^{-1}(a)$ (Transitivity) and so $g(b) \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} a$ (Invariance). This means that $b \neq g(b)$ and therefore $g(b) \notin A \cup \{b\}$.

Use Lemma 2.6 to obtain $x \in \mathbb{F}$ such that $x \not\mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(b)$ and $x \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} Ab$. By Monotonicity, $x \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} A$ and $x \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} b$, hence also $g(x) \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(b)$, thus $x \neq g(x)$. □

Let $X \subset \mathbb{F}$ be a finite set and let $a \in \mathbb{F}$ be such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} X$. We call the type $\operatorname{tp}(a/X)$ a *free type*. (It is the unique such type over X.)

Lemma 3.5. *Let $g \in G$ be such that for every free type p there is a realisation $a \models p$ with $g(a) \neq a$. Then for every finite $X \subset \mathbb{F}$ and every type $q = \operatorname{tp}(x/X)$ with $x \notin X$, there is a realisation $c \models q$ such that $g(c) \neq c$.*

Proof. Let a be a vertex such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} X$ and $g(a) \neq a$ (a exists by the assumptions of this lemma) and let $b \models q$ be such that $b \mathop{\smile\!\!\!\!\!\!\;|}_{X} g(a)$.

If $b \not\mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(a)$ then pick $c \models q$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}_{X} ag(a)$. This means that $c \not\mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(a)$ (by Stationarity and Invariance) and $c \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} a$ (by Transitivity), giving us $g(c) \neq c$.

So we have $b \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(a)$. Use Lemma 2.6 to obtain $a' \in \mathbb{F}$ such that $a' \not\mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} b$, $a' \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} X$, and a' is almost free from b. By Stationarity, we have that $a \models \operatorname{tp}(a'/X)$, hence there is $f \in G$ fixing X pointwise such that $f(a') = a$. Put $c' = f(b)$. In particular, $c' \models q$, $a \not\mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} c'$, and a is almost free from c' (Observation 2.5).

Choose $c \models \operatorname{tp}(c'/Xa)$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}_{Xa} g(a)$. In particular, $c \not\mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} a$ (Invariance). By Observation 2.5, a is almost free from c. Using 1-supportedness, $c \mathop{\smile\!\!\!\!\!\!\;|}_{Xa} g(a)$ implies that either $c \mathop{\smile\!\!\!\!\!\!\;|}_{a} g(a)$ (in which case $c \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(a)$ and hence $g(c) \neq c$), or $c \mathop{\smile\!\!\!\!\!\!\;|}_{X} g(a)$. In this case we know that $\operatorname{tp}(c/X) = \operatorname{tp}(b/X)$ and $b \mathop{\smile\!\!\!\!\!\!\;|}_{X} g(a)$ (using Perfect triviality on $b \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(a)$), hence by Stationarity and Invariance, $c \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} g(a)$, thus again $g(c) \neq c$. □

We say that $g \in G$ *moves type p by distance k* if there is $a \models p$ and a geodesic sequence $a = a_0, \ldots, a_k = g(a)$. If $p = \operatorname{tp}(x/X)$ is a type and h is an automorphism or a partial automorphism defined on a finite set such that $X \subseteq \operatorname{Dom}(h)$, we denote $h(p) = \operatorname{tp}(h'(x)/h'(X))$, where h' is some automorphism of $\mathbb{F}$ extending h (remember that we assumed that $\mathbb{F}$ is homogeneous).

Lemma 3.6. *Let $g \in G$ be such that g moves all types almost maximally or by distance n. Then there exists $h \in G$ such that $[g, h] = g^{-1}h^{-1}gh$ moves all types almost maximally or by distance $2n$.*

Proof. As in [TZ13a], we construct h by a "back-and-forth" construction as the union of a chain of finite partial automorphisms. We show the following: Let h' be already defined on a finite set U and let $p = \operatorname{tp}(x/X)$ be a type. Then h' has an extension h such that $[g, h]$ moves p almost maximally or by distance $2n$.

We can assume that $X \cup g^{-1}(X) \subseteq U$. Put $V = h'(U)$. Let a' be a realisation of p such that $a' \mathop{\smile\!\!\!\!\!\!\;|}_{X} Ug^{-1}(U)$ and let b' be a realisation of $h'(\operatorname{tp}(a'/U))$ (which is a type over V). By the hypothesis on g there are realisations $a \models \operatorname{tp}(a'/Ug^{-1}(U))$ and $b \models \operatorname{tp}(b'/V)$ such that either $a \mathop{\smile\!\!\!\!\!\!\;|}_{Ug^{-1}(U)} g(a)$, or there is a geodesic sequence $a = a_0, \ldots, a_n = g(a)$ and similarly for b. We also have

$$a \mathop{\smile\!\!\!\!\!\!\;|}_{X} Ug^{-1}(U) \text{ and } b \mathop{\smile\!\!\!\!\!\!\;|}_{h'(X)} V.$$

Let h_0 be the isomorphism $Ua \simeq Vb$ and let c be a realisation of $h_0^{-1}(\operatorname{tp}(g(b)/Vb))$ (which is a type over Ua) such that $c \mathop{\smile\!\!\!\!\!\!\;|}_{Ua} g(a)$. Put h to be the isomorphism

$Uac \simeq Vbg(b)$. Observe that $[g,h](a) = g^{-1}(c)$. It remains to prove that a witnesses that $[g,h]$ moves p almost maximally or by distance $2n$.

Since $a \mathop{\downarrow}_X g^{-1}(U)$, we know that $g(a) \mathop{\downarrow}_{g(X)} U$. Using Metricity, we get

$$c \mathop{\downarrow}_{g(X)a} g(a),$$

thus from 1-supportedness we know that either $c \mathop{\downarrow}_a g(a)$ or $c \mathop{\downarrow}_{g(X)} g(a)$. In the second case we get $g^{-1}(c) \mathop{\downarrow}_X a$, which implies that $[g,h]$ moves p almost maximally. Hence we can assume that

$$c \mathop{\downarrow}_a g(a).$$

By the choice of a and b we know that one of the following cases occurs:

(1) First suppose that there are geodesic sequences $b = b_0, \ldots, b_n = g(b)$ and $g(a) = a_0, \ldots, a_n = a$ (the reverse of a geodesic sequence is a geodesic sequence by Symmetry). From the construction we know that $\mathrm{tp}(ac) = \mathrm{tp}(bg(b))$. This implies that there is a geodesic sequence $a = c_0, \ldots, c_n = c$. Since $g(a) \mathop{\downarrow}_a c$, Proposition 2.4 gives a geodesic sequence starting at $g(a)$ and finishing at c using $2n+1$ vertices (including c and $g(a)$). Finally, taking the image of this sequence under g^{-1} gives a geodesic sequence starting at a and finishing at $g^{-1}(c) = [g,h](a)$ using $2n+1$ vertices. This means that a witnesses that $[g,h]$ moves p by distance $2n$.

(2) Now assume that $a \mathop{\downarrow}_{Ug^{-1}(U)} g(a)$. Then in fact we have $a \mathop{\downarrow}_X g(a)$, because $a \mathop{\downarrow}_X Ug^{-1}(U)$ (Metricity). As $U \supseteq Xg^{-1}(X)$, $a \mathop{\downarrow}_X U$ also implies $g(a) \mathop{\downarrow}_{g(X)} X$ (by Invariance and Monotonicity), which together with $a \mathop{\downarrow}_X g(a)$ implies $a \mathop{\downarrow}_{g(X)} g(a)$ (Metricity). Thus from $c \mathop{\downarrow}_a g(a)$ we get $c \mathop{\downarrow}_{g(X)} g(a)$ (yet again Metricity) and thus $g^{-1}(c) \mathop{\downarrow}_X a$, i.e. a witnesses that $[g,h]$ moves p almost maximally.

(3) Otherwise we have $b \mathop{\downarrow}_V g(b)$. Using that h is an isomorphism of Uac and $Vbg(b)$ and Invariance we obtain $a \mathop{\downarrow}_U c$. Then we get $a \mathop{\downarrow}_X c$, because $a \mathop{\downarrow}_X U$ (Metricity), and then, combining with $c \mathop{\downarrow}_a g(a)$ using Metricity again, we obtain $c \mathop{\downarrow}_X g(a)$. As in the previous case, $a \mathop{\downarrow}_X U$ implies $g(a) \mathop{\downarrow}_{g(X)} X$ and hence $c \mathop{\downarrow}_{g(X)} g(a)$, or $g^{-1}(c) \mathop{\downarrow}_X a$, i.e. a witnesses that $[g,h]$ moves p almost maximally.

□

Now we prove the following proposition, Theorem 1.1 is then its direct consequence.

Proposition 3.7. *Let $\mathbb{F}$ be a countable relational structure with a bounded 1-supported metric-like stationary independence relation $\mathop{\downarrow}$ and let g be a non-identity automorphism of $\mathbb{F}$. Then there is an automorphism of $\mathbb{F}$ which is a product of at most $2\|\mathop{\downarrow}\|^2$ conjugates of g and g^{-1} and moves every type over every finite set almost maximally.*

Proof. From Lemma 3.3 we get an automorphism g_0 which is a product of at most $\|\mathop{\downarrow}\|$ conjugates of g such that there is $a \in \mathbb{F}$ with $a \mathop{\downarrow}_\emptyset g_0(a)$. Using Lemma 3.4 we get that in fact for every free type there is a realisation which is not fixed by g_0.

Let $p = \mathrm{tp}(x/X)$ be a type. Either $x \in X$ (then $x \mathop{\downarrow}_X g(x)$, hence g_0 moves q almost maximally), or $x \notin X$ and thus by Lemma 3.5 there is a realisation of p which is not fixed by g_0. This means that g_0 moves all types almost maximally or by distance 1.

Put $n = \lceil \log_2(\| \mathop{\smile}\limits^{\mid} \|) \rceil$ and construct a sequence $g_0, g_1, \ldots, g_n$ of automorphisms of $\mathbb{F}$ using Lemma 3.6 such that every g_i moves all types almost maximally or by distance 2^i, and if $i \geq 1$ then g_i is a product of two conjugates of g_{i-1} and g_{i-1}^{-1}. For g_n we get that it moves every type almost maximally or by distance at least $\| \mathop{\smile}\limits^{\mid} \|$. In the latter case, we have for every type p a realisation $a \models p$ and a geodesic sequence $a = a_0, \ldots, a_k = g(a)$, where $k \geq \| \mathop{\smile}\limits^{\mid} \|$. Boundedness (Definition 1.12) implies that $a \mathop{\smile}\limits^{\mid}_{\emptyset} g(a)$, i.e. g_n moves p almost maximally, and hence g_n moves all types almost maximally.

By the construction, g_n is a product of at most $2^{\lceil \log_2(\| \mathop{\smile}\limits^{\mid} \|) \rceil}$ conjugates of g_0 and g_0^{-1}, hence a product of at most $2^{\lceil \log_2(\| \mathop{\smile}\limits^{\mid} \|) \rceil} \| \mathop{\smile}\limits^{\mid} \| \leq 2 \| \mathop{\smile}\limits^{\mid} \|^2$ conjugates of g and g^{-1}. ☐

Proof of Theorem 1.1. Let g be a non-identity automorphism of $\mathbb{F}$. We need to prove that if N is a normal subgroup of G such that $g \in N$, then $N = G$. If $g \in N$, then clearly $g^{-1} \in N$. Let $h \in G$. By Proposition 3.7 and Theorem 3.2, we know that h can be written as a product of conjugates of g and g^{-1}, hence $h \in N$. This is true for every $h \in G$, hence $N = G$ and G is simple. ☐

4. Corollaries

In this section we prove Theorems 1.2 and 1.3.

4.1. **Semigroup-valued metric spaces.** We say that a tuple $\mathfrak{M} = (M, \oplus, \preceq)$ is a *partially ordered commutative semigroup* if the following hold:

1. $(M, \oplus)$ is a commutative semigroup,
2. $(M, \preceq)$ is a partial order which is reflexive ($a \preceq a$ for every $a \in M$),
3. for every $a, b \in M$ it holds that $a \preceq a \oplus b$, and
4. for every $a, b, c \in M$ it holds that if $b \preceq c$ then $a \oplus b \preceq a \oplus c$ ($\oplus$ is monotone with respect to $\preceq$).

$\mathfrak{M}$ is *archimedean* if for every $a, b \in \mathfrak{M}$ there is an integer n such that $n \times a \succeq b$, where by $n \times a$ we mean

$$\underbrace{a \oplus a \oplus \cdots \oplus a}_{n \text{ times}}.$$

Note that if $\mathfrak{M}$ is archimedean and non-trivial, it follows that $\mathfrak{M}$ does not have an identity.

Let L be a set. An *L-edge-labelled graph* is a tuple $\mathbf{A} = (A, E, d)$, where $E \subseteq \binom{A}{2}$ and d is a function $E \to L$. Clearly, the set E can be inferred from the function d and thus we omit it. For simplicity, we write $d(x, y)$ instead of $d(\{x, y\})$ and we put $d(x, x) = 0$, where 0 is a symbol which is not an element of $\mathfrak{M}$. When convenient, we naturally understand 0 as the neutral element with respect to $\oplus$ and as the minimum element of $\preceq$.

We say that $\mathbf{A}$ is *complete* if the graph (A, E) is a complete graph. Note that an L-edge-labelled graph can equivalently be viewed as a relational structure with an irreflexive binary symmetric relation R^m for every $m \in L$ such that every pair of vertices is in at most one relation.

For a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$, a complete $\mathfrak{M}$-edge-labelled graph $\mathbf{A} = (A, d)$ is an *$\mathfrak{M}$-metric space* if for every triple $a, b, c \in A$ of distinct vertices it holds that $d(a, b) \preceq d(a, c) \oplus d(b, c)$ (the *triangle inequality*).

Let $\mathbb{F}$ be an $\mathfrak{M}$-metric space. We say that $\mathbb{F}$ *admits an $\mathfrak{M}$-shortest path independence relation* if for every $a, b \in \mathbb{F}$ and $C \subseteq \mathbb{F}$ finite we have that $\{d(a, c) \oplus d(c, b) : c \in C\}$ has an infimum with respect to $\preceq$ (note that C can be empty which implies that $\mathfrak{M}$ has maximum $\inf_{\preceq}(\emptyset)$). If $\mathbb{F}$ admits an $\mathfrak{M}$-shortest path independence relation, then its *$\mathfrak{M}$-shortest path independence relation* is a ternary relation $\mathop{\smile}\limits^{\mid}$ defined on finite subsets of $\mathbb{F}$ by putting $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ if and only if for every $a \in A$ and every $b \in B$ it holds that $d(a,b) = \inf_{\preceq}\{d(a,c) \oplus d(c,b) : c \in C\}$.

Generalising concepts of Sauer [Sau12], Conant [Con19] (see also [HKN17]) and Braunfeld [Bra17] (see also [KPR18]), Hubička, Konečný and Nešetřil [Kon19, HKN18] introduced the framework of semigroup-valued metric spaces, which served as a motivation for this paper. Given a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ and a "nice" family $\mathcal{F}$ of $\mathfrak{M}$-edge-labelled cycles, the structures of interest are $\mathfrak{M}$-metric spaces which moreover contain no homomorphic images of members of $\mathcal{F}$. We will denote the class of all such finite structures $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$.

The conditions of $\mathcal{F}$ are strong enough that one can then prove that $\mathcal{M}_{\mathfrak{M}}^{\mathcal{F}}$ is a strong amalgamation class, its Fraïssé limit admits an $\mathfrak{M}$-shortest path independence relation which is a SIR (provided that $\mathfrak{M}$ has a maximum, otherwise one can still get a *local* SIR), it has EPPA (see [HKN19, Sin17]) and a precompact Ramsey expansion (see [HN19, NVT15]), but they are general enough that most known binary symmetric homogeneous structures can be viewed as such a semigroup-valued metric space. In fact, it is conjectured that every primitive transitive homogeneous structure in a finite binary symmetric language with trivial algebraic closures admits such an interpretation (Conjecture 1 in [Kon19]).

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We need to prove that $\mathop{\smile\!\!\!\!\!\!\;|}$ is metric-like and bounded. (In fact, we do not need 1-supportedness for this, we only need it later in order to apply Theorem 1.1.)

Since $\mathbb{F}$ is homogeneous, all vertices have the same type. As $d(x,y) = 0$ if and only if $x = y$ and $\oplus$ is monotone with respect to $\preceq$, it follows that if $a \notin A$, then $a \not\mathop{\smile\!\!\!\!\!\!\;|}_A a$. The fact that there are $a \neq b \in \mathbb{F}$ such that $a \not\mathop{\smile\!\!\!\!\!\!\;|}_\emptyset b$ follows from Stationarity, the fact that $\mathfrak{M}$ has at least two elements (remember that $0 \notin \mathfrak{M}$) and the fact that $\mathbb{F}$ realises all distances.

Suppose now that $a \mathop{\smile\!\!\!\!\!\!\;|}_C b$. If there was $c' \in \mathbb{F} \setminus C$ such that $a \not\mathop{\smile\!\!\!\!\!\!\;|}_{Cc'} b$, this would mean that $\inf_{\preceq}\{d(a,c) \oplus d(c,b) : c \in C \cup \{c'\}\} \prec \{d(a,c) \oplus d(c,b) : c \in C\} = d(a,b)$, hence $d(a,c') \oplus d(c',b) \not\succeq d(a,b)$, in other words, abc' violates the triangle inequality which is a contradiction. Consequently, $\mathop{\smile\!\!\!\!\!\!\;|}$ satisfies Perfect triviality and hence $\mathop{\smile\!\!\!\!\!\!\;|}$ is metric-like.

Next we prove that $\mathop{\smile\!\!\!\!\!\!\;|}$ is bounded. Denote by 1 the maximum element of $\mathfrak{M}$ ($\mathfrak{M}$ is finite and hence there is such an element). Assume that there are $a, b \in \mathfrak{M}$ such that $a \oplus b = a$. This means (by associativity) that $a \oplus (n \times b) = a$ for every n. Let $c \in \mathfrak{M}$ be arbitrary. By archimedeanity there is n such that $n \times b \succeq c$. But then $a = a \oplus (n \times b) \succeq c$. Hence $a \succeq c$ for every $c \in \mathfrak{M}$, that is, $a = 1$. In other words, for every $a, b \in \mathfrak{M} \setminus \{1\}$ it holds that $a \oplus b \succ a$, which implies that whenever $a_1, \dots, a_{|\mathfrak{M}|} \in \mathfrak{M}$, then

$$\bigoplus_{i=1}^{|\mathfrak{M}|} a_i = 1.$$

We can use this observation to prove that $\| \mathop{\smile\!\!\!\!\!\!\;|} \| \leq |\mathfrak{M}|$. Indeed, if $a_0, \dots, a_{|\mathfrak{M}|}$ is a geodesic sequence, we know that $d(a_0, a_{i+1}) = d(a_0, a_i) \oplus d(a_i, a_{i+1})$. Using induction we get that

$$d(a_0, a_{|\mathfrak{M}|}) = d(a_1, a_2) \oplus d(a_2, a_3) \oplus \cdots \oplus d(a_{|\mathfrak{M}|-1}, a_{|\mathfrak{M}|}),$$

that is, $d(a_0, a_{|\mathfrak{M}|})$ is a sum of $|\mathfrak{M}|$ elements of $\mathfrak{M}$ and hence $d(a_0, a_{|\mathfrak{M}|}) = 1$, which means that indeed $a_0 \mathop{\smile\!\!\!\!\!\!\;|}_\emptyset a_{|\mathfrak{M}|}$.

We have proved that $\mathop{\smile\!\!\!\!\!\!\;|}$ is bounded and metric-like, hence we can apply Theorem 1.1 to show that $\mathrm{Aut}(\mathbb{F})$ is simple. ∎

Note that whenever $\preceq$ is a linear order, the corresponding $\mathfrak{M}$-shortest path independence relation is necessarily 1-supported. The following theorem is a direct consequence of this fact, Theorem 1.2 and existing results on semigroup-valued metric spaces [Kon19, HKN18].

Let $S \subseteq \mathbb{R}^+$ be a finite subset of positive reals such that the following operation $\oplus_S : S^2 \to S$ is associative:

$$a \oplus_S b = \max\{x \in S : x \leq a + b\}.$$

Delhommé, Laflamme, Pouzet, and Sauer [DLPS07] studied and Sauer later classified [Sau13a, Sau13b] such subsets. Ramsey expansions for all such classes of $(S, \oplus_S, \leq)$-metric spaces were obtained by Hubička and Nešetřil [HN19], and Hubička, Konečný, Nešetřil and Sauer [HKNS20] (Nguyen Van Thé [NVT09] earlier proved some partial results). We contribute to the study of such classes by the following result:

Theorem 4.1. *Let $S \subseteq \mathbb{R}^+$ be a finite subset of positive reals such that $\mathfrak{M}_S = (S, \oplus_S, \leq)$ is an archimedean partially ordered commutative semigroup. Then the automorphism group of the Fraïssé limit of the class of all finite $\mathfrak{M}_S$-metric spaces is simple.*

4.2. **Metrically homogeneous graphs.** A *metrically homogeneous graph* is a graph whose path-metric is a homogeneous metric space. Cherlin [Che11, Che17] gave a list of such graphs by describing the corresponding amalgamation classes of metric spaces. The vast majority of the list is occupied by the 5-parameter classes $\mathcal{A}^{\delta}_{K_1,K_2,C_1,C_2}$, where δ denotes the diameter of such spaces (i.e. they only use distances $\{1, \dots, \delta\}$) and the other four parameters describe four different families of forbidden triangles (for example, all triangles of odd perimeter smaller than $2K_1$ are forbidden).

Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný and Pawliuk [ABWH$^+$17c, ABWH$^+$17a, ABWH$^+$17b] studied EPPA, Ramsey expansions and (local) SIR's for these classes (see also [Kon18, EHKN20, Kon20]). In particular, if $\mathcal{A}^{\delta}_{K_1,K_2,C_1,C_2}$ is primitive (i.e. it is neither antipodal nor bipartite) and δ is finite, it can be shown using another result of Hubička, Kompatscher and Konečný [HKK18] that these (local) stationary independence relations are 1-supported and can be viewed as $\mathfrak{M}$-shortest path independence relations [Kon19] with a finite archimedean $\mathfrak{M}$, which means that Theorem 1.3 is a direct consequence of Theorem 1.2.

5. Conclusion

We conclude with two questions and a conjecture. The first question is a particular instance of the general question whether 1-supportedness is necessary.

Question 5.1. *Consider the structure $\mathbb{M}_k$ from Example 3, that is, the Fraïssé limit of all finite $[n]^k$-metric spaces (which are in fact semigroup-valued metric spaces in the sense of Section 4.1). Is the automorphism group of $\mathbb{M}_k$ simple? (For $k \geq 2$ and n large enough – if, for example, $n = 3$, it is in fact a free amalgamation class, as $(2, \dots, 2)$ is a free relation.)*

The obvious next step is to generalise our results to countable archimedean semigroups which do not have to contain a maximum element, thereby obtaining and analogue of Tent and Ziegler's result on the Urysohn space [TZ13b]. We believe that such a generalisation is quite straightforward. However, there are structures in infinite language which do not even admit a SIR, although they are also very much metric-like. One example is the *sharp Urysohn space*:

Question 5.2. *Let $\mathbb{U}^{\#}$ be the Fraïssé limit of the class of all finite complete $\mathbb{Q}^{+}$-edge-labelled graphs (here $\mathbb{Q}^{+}$ is the set of all positive rational numbers) which contain no triangles a, b, c with $d(a,b) \geq d(a,c) + d(b,c)$ (that is, the triangle inequality is sharp). Is the automorphism group of $\mathbb{U}^{\#}$ simple modulo bounded automorphisms?*

Note that if we consider $\mathbb{N}$ instead of $\mathbb{Q}^{+}$, the resulting structure can be understood as an $\mathfrak{M}$-metric space (putting $a \oplus b = a + b - 1$ and $a \preceq b$ if $a \leq b$).

Remark 5.3. The sharp Urysohn space is a very peculiar structure, because although it does not admit a SIR, it has EPPA, APA and it is Ramsey when equipped with a (free) linear order.

The following conjecture and question are closely related to a conjecture from [Kon19].

Conjecture 5.4. *Every countable homogeneous complete L-edge-labelled graph with $2 \leq |L| < \infty$, primitive automorphism group and trivial algebraic closure admits a metric-like SIR.*

Question 5.5. *Assume that $\mathbb{F}$ is a transitive countable structure with a metric-like SIR $\mathop{\smile\hskip-0.9em\mid\hskip0.5em}$ such that $\mathrm{tp}(ab) = \mathrm{tp}(ba)$ for every $a, b \in \mathbb{F}$. Can one define a partially ordered commutative semigroup $\mathfrak{M}$ on the 2-types of $\mathbb{F}$ such that $\mathop{\smile\hskip-0.9em\mid\hskip0.5em}$ is the $\mathfrak{M}$-shortest path independence relation? If the answer is yes, is it true that for every $a \neq b \neq c \in \mathbb{F}$ it holds that $\mathrm{tp}(ab) \preceq \mathrm{tp}(ac) \oplus \mathrm{tp}(bc)$?*

The obvious special cases of Question 5.5 are for finitely many 2-types, 1-supported $\mathop{\smile\hskip-0.9em\mid\hskip0.5em}$, bounded $\mathop{\smile\hskip-0.9em\mid\hskip0.5em}$, and their combinations. It is not true that the conditions of Question 5.5 imply that the structure at hand is an $\mathfrak{M}$-metric space in the sense of [Kon19, HKN18]. For example, suppose that $\mathbb{F}$ is the Fraïssé limit of the class of all $[n]^1$-metric spaces which also contain a ternary relation R such that if $(a,b,c) \in R$, then $d(a,b) = d(b,c) = d(c,a) = 1$. The standard $([n], +, \leq)$-shortest path independence relation is the desired SIR on $\mathbb{F}$.

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