On the $\mathcal{L}^n$-axiomatisability of certain classes of graphs

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Abstract

I study the expressibility of some graph theoretical notions in finite variable logics; in particular whether the class of graphs without K_n as a topological subgraph is axiomatisable in some $\mathcal{L}^k$.

Let $\mathcal{L}$ be the first order language over a binary relation symbol E. A graph is an $\mathcal{L}$-structure $\mathfrak{G}$ where $E^{\mathfrak{G}}$ is symmetric and anti-reflexive. The elements of G are called *vertices*, the elements of $E^{\mathfrak{G}}$ *edges*. For simplicity, I write G instead of $\mathfrak{G}$ and E instead of $E^{\mathfrak{G}}$. Most graph theoretical notations follow [B].

Logical notations Let $\mathcal{L}^n$ be the restriction of $\mathcal{L}$ to n variables, and let $\mathcal{L}^n(z_1, \ldots, z_k)$ be $\mathcal{L}^n$ with k additional constant symbols for variables; I will consider $\mathcal{L}^n(z_1, \ldots, z_k)$ as a sublanguage of $\mathcal{L}^{n+k}$ where the z_i are not allowed to be bounded. I will use x, y, z as variables for variables; in particular they need not to be distinct. If some mathematical property P has been shown to be first order expressible, then I will use $[P]$ in formulas to mean a suitable first order expression for P.

A structure is said to be (finitely) $\mathcal{L}^n$-categorical if it is up to isomorphism the only model of some $\mathcal{L}^n$-theory ($\mathcal{L}^n$-sentence respectively). Let $\equiv^n$ stand for elementary equivalence in $\mathcal{L}^n$.

The main question I am interested in is whether classes of graphs of "low complexity" are axiomatisable in some $\mathcal{L}^n$. Note that the theory of graphs is axiomatised with 2 variables.

Some classes of graphs Let $m \in \omega$ and κ an arbitrary cardinal. Then K_κ stands for the complete graph with κ vertices, K_κ^m for K_κ by m new vertices on each edge. $\mathcal{K}_\kappa^m$ is the class of all subdivisions of K_κ by at most m new vertices on each edge. Note $\mathcal{K}_\kappa^m \subseteq \mathcal{K}_\kappa^{m+1}$. Let $\mathcal{K}_\kappa^\omega := \bigcup_{m \in \omega} \mathcal{K}_\kappa^m$.

The usual graph theoretical notation for $\mathcal{K}_\kappa^\omega$ is TK_κ, and a graph in that class is called a *topological* K_κ, since a graph G has a TK_κ as a subgraph iff there is a continuous map $K_\kappa \to G$.

There are at least three different notions for a graph of being "not complicated" measured by the absence of some complete graphs.

Definition 1 *A graph G is n-*flat *if n is the least number such that G omits $\mathcal{K}_{n+1}^\omega$, i.e. G admits continuous embeddings $K_n \to G$, but not $K_{n+1} \to G$.*

In [HMS], a graph was called *ultraflat* if it is n-flat for some finite n, and those graphs were shown to be superstable. The original notion of flatness comes from [PZ] and equals n-flatness for some $n \leqslant \aleph_0$. If all graphs elementarily equivalent to G are flat, then G is stable.

A graph is a *minor* of G if it can be obtained from G by contracting edges and taking subgraphs.

Definition 2 *A graph G is K_n-*free *if it has no K_n-minor.*

Definition 3 *A graph G has treewidth $\leqslant n$ if there is a family of induced subgraphs $(G_t)_{t\in T}$, each of size $\leqslant n+1$, indexed by a tree T, such that the G_t cover G and such that $\{t \in T \mid g \in G_t\}$ is connected for every $g \in G$.*

A graph of treewidth $\leqslant n$ is K_{n+2}-free, and a K_{n+2}-free graph is $(n+1)$-flat. For $n = 1$ the three notions coincide and define the forests. For $n = 2$, treewidth $\leqslant 2$ and K_4-free coincide.

It is known that treewidth $\leqslant n$ is expressible in $\mathcal{L}^{n+2}$. Supposedly, being K_n-free is also expressible with few variables, but n-flatness is expected not to be $\mathcal{L}^k$-axiomatisable except for small n. I will concentrate on this last question, doing some other work on the way towards it.

Question 1 *Is the class of K_n-free graphs L^k-axiomatisable for some $k = k(n)$?*
Is the class of n-flat graphs L^k-axiomatisable for some $k = k(n)$?

Degrees and complete graphs

The *degree* $d(v)$ of a vertex v is the number of its neighbours. G is *k-regular* if all vertices have degree k. Note that

$$[d(x) \geqslant n] \;:=\; (\exists^{\geqslant n} y)Exy \;=\; \exists y_1 \ldots \exists y_n \big(\bigwedge_{i\neq y} y_i \neq y_j \wedge \bigwedge_{i=1}^{n} Exy_i \big) \qquad \in \mathcal{L}^n(x)$$

$$[d(x) = n] \;:=\; (\exists^{\geqslant n} y)Exy \wedge \neg(\exists^{\geqslant n+1} y)Exy \qquad \in \mathcal{L}^{n+1}(x)$$

Example 1 *K_n is finitely $\mathcal{L}^{n+1}$-categorical, but $K_n \equiv^n K_{n+k}$ for all n and k.*

□ K_n is described by the $\mathcal{L}^{n+1}$-sentence

$$\exists^{=n} x\,(x = x) \wedge \forall x \forall y (x \neq y \rightarrow Exy) \qquad \in \mathcal{L}^{n+1}$$

The elementary equivalence is clear from the pebble game. □

Proposition 1 *The class of graphs with maximal degree k and the class of k-regular graphs are finitely $\mathcal{L}^{k+2}$-axiomatisable, but not $\mathcal{L}^{k+1}$-axiomatisable.*

□ The first by

$$\forall x \exists y_1 \ldots \exists y_k \big(\bigwedge_{i=1}^{k} Exy_i \wedge \forall z (Ezx \rightarrow \bigvee_{i=1}^{k} z = y_i)\big)$$

and the second by

$$\forall x \exists y_1 \ldots \exists y_k \big(\bigwedge_{i\neq j} y_i \neq y_j \wedge \bigwedge_{i=1}^{k} Exy_i \wedge \forall z (Ezx \rightarrow \bigvee_{i=1}^{k} z = y_i)\big)$$

K_{k+1} is k-regular, K_{k+2} is $(k+1)$-regular, and they are $\mathcal{L}^{k+1}$-equivalent, hence $\mathcal{L}^{k+2}$ is best possible. □

Of course, the class of all graphs with finite degree is not even first order axiomatisable.

Paths and connectedness

A *walk* from v to w of length m is a sequence of vertices $(v_0, \ldots, v_m)$ such that $v = v_0$, $w = v_m$ and $Ev_{i-1}v_i$ for all $i = 1, \ldots, m$. A walk is a *path* if the $v_0, \ldots, v_m$ are pairwise distinct. The *distance* $d(v, w)$ between two vertices v and w is the smallest length of a path (walk) between v and w. (So far the terminology follows [B].) A path $(v_0, \ldots, v_m)$ is said to be disjoint from something if the set of inner vertices $\{v_1, \ldots, v_{m-1}\}$ is disjoint from that something.

Lemma 2 *There are formulas $\delta_m(x,y) \in \mathcal{L}^2(x)$ such that $G \vDash \delta_m(a,b)$ iff there is a path of length $\leqslant m$ between a and b.*

$\square$ Note first that there is a path of length $\leqslant m$ between x and y iff there is also a walk of length $\leqslant m$ between x and y (extract a path from the walk). Define per induction

$$\begin{aligned} \delta_0(x,y) &:= x = y \\ \delta_{m+1}(x,y) &:= \exists z_i \big(\delta_m(x,z_i) \wedge E z_i y\big) \quad \text{ where } i \equiv m \pmod 2 \end{aligned}$$

and identify at the end z_0 with y if m is even, and z_1 with y if m is odd. (Write down δ_3 to see that it works!) $\square$

Note that it is possible to take $\delta_0(x,y), \delta_1(x,y) \in \mathcal{L}^0(x,y)$.

Corollary 3 *The distance between x and y is expressible in $\mathcal{L}^3$.*

Now I want to prove a generalisation of lemma 2.

Lemma 4 *Let $\xi_i(y_0,\dots,y_l,\bar{c}) \in \mathcal{L}^k(\bar{c})$ for $i \in \omega$. For $m \geqslant l$ there are formulas*

$$\delta^\xi_m(x,y,\bar{c}) \in \mathcal{L}^{\max\{3,k,l+2\}}(\bar{c})$$

such that $G \vDash \delta^\xi_m(a,b,\bar{c})$ if and only if there is a walk $(v_0,\dots,v_m)$ between a and b such that $G \vDash \xi_i(v_i,\dots,v_{i+l},\bar{c})$ for all $i = 0,\dots,m-l$.

$\square$ Modify the formulas δ_m of lemma 2: We need variables $z_0,\dots,z_l$ to keep track of $l+1$ subsequent vertices in the walk. Let $z_j = z_{j \bmod (l+1)}$ and define

$$\begin{aligned} \delta^\xi_0(x,y,\bar{z},\bar{c}) &:= x = y \wedge \xi_0(z_0,\dots,z_l,\bar{c}) \\ \delta^\xi_{j+1}(x,y,\bar{z},\bar{c}) &:= \exists z_j \big(\delta^\xi_j(x,z_j,\bar{c}) \wedge E z_j y\big) \wedge \xi_{j+1}(z_j,\dots,z_{j+l},\bar{c}) \quad \text{for } j = 0,\dots,m-l-1 \\ \delta^\xi_{j+1}(x,y,\bar{z},\bar{c}) &:= \exists z_j \big(\delta^\xi_j(x,z_j,\bar{c}) \wedge E z_j y\big) \quad \text{for } j = m-l,\dots,m-1 \end{aligned}$$

Then $\delta^\xi_m(x,y,\bar{c})$ has no free variables $\bar{z}$ any more. Moreover, as in the end of the proof of lemma 2, z_i can be identified with y where $m \equiv i \pmod{l+1}$.

The number of variables computes as follows: we need x, the $z_0,\dots,z_l$ (but at least 2 of them if $l = 0$), and the number by which k exceeds this. (Note that x can be re-used in the ξ_i without any harm.) $\square$

Let $d(v,w) \setminus \bar{c}$ be the smallest length of a path between v and w that is disjoint from $\bar{c}$.

Corollary 5 *We can express in $\mathcal{L}^3(\bar{c})$ that there is a path from x to y of length $\leqslant m$ and disjoint from $\bar{c}$, i.e. we can express "$d(x,y) \setminus \bar{c} \leqslant m$".*

$\square$ Applications of lemma 4: Let $l = 0$ and $\xi_i(y,c_1,\dots,c_n) := \bigwedge_{j=1}^n y \neq c_j$. $\square$

A graph G is called *k-connected* if G has at least $k+1$ vertices and if it is connected even after removing $k-1$ of them.

Proposition 6 *The class of graphs all of whose connected components are k-connected is finitely $\mathcal{L}^{k+2}$-axiomatisable.*

(One needs clearly $k+1$ variables. For $k = 2$, the cycle C_4 and the infinite line show that $k+2$ is best possible. What about higher k?)

$\square$ If G has at least $k+1$ vertices and is not k-connected, choose two vertices of minimal distance that get separated after removing $k-1$ vertices; clearly there distance was at most k. Hence G does not satisfy

$$\forall x \forall y \forall z_1 \ldots \forall z_{k-1} \big([d(x,y) \leqslant k] \to [d(x,y) \setminus \bar{z} \leqslant k] \big)$$

but any k-connected graph does. Hence take the negation of this sentence together with

$$\forall x (\exists^{\geqslant k} y)(x \neq y \wedge [d(x,y) \leqslant k])$$

expressing that every connected component has at least $k+1$ vertices. $\square$

Call a graph G *k-separated* if it contains at least $k+2$ vertices and each two of them can be separated by removing k other vertices. (Don't know whether this is standard).

Proposition 7 *The class of k-separated graphs is $\mathcal{L}^{k+3}$-axiomatisable.*

$\square$ Suppose G is not k-separated, witnessed by two vertices v and w. By Menger's theorem, there are at least $k+1$ disjoint paths between v and w. Let m bound the length of $k+1$ of them. Then G does not satisfy the formula

$$\sigma_m^k \;:=\; \forall x \forall y \big(\exists z_1 \ldots \exists z_k [d(x,y) \setminus \bar{z} \leqslant m] \to \exists z_1 \ldots \exists z_k [d(x,y) \setminus \bar{z} > m] \big)$$

Hence $\{\sigma_m^k \mid m \in \omega\}$ axiomatises the class of k-separated graphs. $\square$

Cycles

A walk $v_0, \ldots, v_m$ is a *cycle* if $v_0 = v_m$ and $v_1, \ldots, v_m$ are pairwise distinct.

Lemma 8 *For $m \geqslant 3$, there are formulas $\gamma_m \in \mathcal{L}^4$ describing that there is a cycle of length $\leqslant m$, and formulas $\gamma'_m(x) \in \mathcal{L}^4(x)$ describing that there is a cycle of length $\leqslant m$ through x.*

$\square$ Apply lemma 4 with $l = 2$ and $\xi_i(y_0, y_1, y_2) \;:=\; y_0 \neq y_2$. Then $\delta_m^\xi(x,x)$ describes a walk of length m without *peaks*. Such a walk contains necessarily a cycle; hence $\gamma_m := \exists x\, \delta_m^\xi(x,x)$ works. The only possibility for the walk not to contain a cycle through x is that the first and the last edge together form a peak. With an additional variable one can keep track of that. $\square$

Of course, γ_3 can be taken in $\mathcal{L}^3$ and $\gamma'_3(x)$ in $\mathcal{L}^2(x)$. An immediate consequence of the lemma is:

Proposition 9 *The class of forests is $\mathcal{L}^4$-axiomatisable, but not $\mathcal{L}^3$-axiomatisable.*

$\square$ The axiomatisation is given by $\{\neg\gamma_m \mid m \geqslant 3\}$. On the other hand, C_4, the cycle of length 4, and the infinite line ($\mathbb{Z}$ with the successor relation) are clearly $\mathcal{L}^3$-equivalent. $\square$

Corollary 10 *The class of 2-colourable graphs is $\mathcal{L}^4$-axiomatisable.*

$\square$ A graph G is 2-colourable iff it has no odd cycles. Hence $\{\neg\gamma_m \mid m \text{ odd} \geqslant 3\}$ gives an axiomatisation: if G has an m-cycle, then γ_m holds. Conversely, if γ_m holds for odd m, witnessed by $(v_0, \ldots, v_m)$, then either it is already a cycle, or $v_i = v_j$ for some $i < j < m$. In that case, consider $(v_0, \ldots, v_i, v_{j+1}, \ldots, v_m)$ and $(v_i, \ldots, v_j)$. One is of odd length and we get an odd cycle by induction. $\square$

Menger's theorem allows more or less to express in a sentence that two given vertices lie on a cycle (the "more or less" part means that the length of the cycle has to be bound for one direction). Is it also possible to fix exactly the lengths of the arcs? Is it possible to express that three given vertices lie on a cycle?

Topological minors of complete graphs

Example 2 *The lines K_2^m are finitely $\mathcal{L}^3$-categorical.*

$\square$ Let χ_2^m be the conjunction of the following:

- there exist exactly 2 vertices of degree $\leqslant 1$ and there distance is m:

$$\exists x \exists y \big(x \neq y \wedge [d(x) \leqslant 1] \wedge [d(y) \leqslant 1] \wedge [d(x,y) = m] \wedge \forall z \big([d(z] \leqslant 1 \to (z = x \vee z = y)\big)\big)$$

- the graph is connected: $\exists x \forall y [d(x,y] \leqslant m$
- there are no 3-cycles: $\neg \exists x_1 x_2 x_3 \bigwedge_{i \neq j} E x_i, x_j$
- there are no vertices of degree 3 and higher: $\neg \exists x_1 x_2 x_3 \bigwedge_{i \neq j} [d(x_i, x_j) = 2]$

The last formula is not an exact translation (which would need 4 variables), but it holds in the lines, and implies in the absence of 3-cycles that there are vertices of degree at least 3. $\square$

The example is a special case of the following:

Proposition 11 *There are $\mathcal{L}^{n+1}$-sentences χ_n^m such that $G \vDash \chi_n^m \iff G \in \mathcal{K}_n^m$.*
The class of graphs which are not in K_n^ω is $\mathcal{L}^{n+1}$-axiomatisable.

$\square$ Fix m, n and let χ_n^m be $\exists x_1 \ldots \exists x_n \check{\chi}_n^m(x_1, \ldots, x_k)$ where $\check{\chi}_n^m(x_1, \ldots, x_k)$ is the conjunction over the following formulas:

- the $x_1, \ldots, x_n$ are pairwise distinct: expressible in $\mathcal{L}^0(x_1, \ldots, x_n)$.
- $x_1, \ldots, x_n$ have degree exactly $n-1$: expressible in $\mathcal{L}^n(x_i)$ each.
- all other vertices have degree exactly 2: $\forall y \big(\bigwedge_{i=1}^n y \neq x_i \to [d(y) = 2]\big) \in \mathcal{L}^{\max\{1,4-n\}}(\bar{x})$ since in the part expressing the degree of y, the variables x_i can be re-used, therefore it needs $1 + \max\{0, 3-n\}$ variables in addition to the x_i.
- x_i and x_j are linked by a path of length $\leqslant m$ disjoint to $\{x_1, \ldots, x_n\} \setminus \{x_i, x_j\}$, expressible in $\mathcal{L}^3(x_1 ... \widehat{x}_i .. \widehat{x}_j ... x_n)$ each,

Together we need $n + \max\{1, 4-n\}$ variables, which equals $n+1$ except for $n = 1$ and $n = 2$. But $\mathcal{K}_1^\omega$ contains just the one point graph and is even described by an $\mathcal{L}^2$-sentence. The case $n = 2$ is dealt with in the previous example.

Finally $\{\neg \chi_n^m \mid m \in \omega\}$ axiomatises the class of graphs that are not topological K_n. $\square$

Corollary 12 *Each graph in $\mathcal{K}_n^m$ is finitely $\mathcal{L}^{n+1}$-categorical.*

$\square$ Add to $\check{\chi}_n^m$ in the previous proof the various statements $[d(x_i, x_j) = m_{ij}]$. $\square$

In particular, the cycles C_n are finitely $\mathcal{L}^4$-categorical.

Example 3 $K_n^m \equiv^n K_{n+k}^m$, *but* $K_n^m \not\equiv^{n+1} K_{n+k}^m$ *if* $k \geqslant 1$ *(and one sentence does it uniformly for all m and k).*

$\square$ Note that $K_1^m = K_1$ for all m, therefore they are even $\mathcal{L}^2$-categorical.
The K_2^m are separated from the other K_n^m by the $\mathcal{L}^3$-sentence $\exists x (\exists y\, x \neq y \wedge [d(x) \leqslant 1])$.
If $n \geqslant 3$, then

$$K_{n+k}^m \vDash \exists x_1 \ldots \exists x_{n+1} \big(\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i=1}^{n+1} [d(x_i) \geqslant 3] \big) \quad \in \mathcal{L}^{n+1}$$

and K_n^m does not satisfy this sentence. $\square$

$K_n^m \not\equiv^4 K_n^{m'}$ for $m \neq m'$: if $n \leqslant 3$ they are even $\mathcal{L}^4$-categorical; otherwise K_n^m satisfies for example the sentence saying that the shortest path between two corner vertices all of whose inner points have degree 2 is of length m, i.e.

$$\exists x_1 \exists x_2 \big([d(x_1) > 2] \wedge [d(x_2) > 2] \wedge \delta_m^\xi(x_1, x_2) \wedge \neg\delta_{m-1}^\xi(x_1, x_2)\big)$$

where δ^ξ is as in lemma 4 with $l = 0$ and $\xi_i(y) \; = \; [d(y) = 2]$ for $i = 1, \ldots, m - l - 1$.

n-flat graphs

The 1-flat graphs are exactly the totally disconnected graphs, i.e. the graphs without any edges. They are finitely $\mathcal{L}^2$-axiomatisable by $\neg\exists x \exists y Exy$.

The 2-flat graphs are exactly the forests, hence $\mathcal{L}^4$-axiomatisable.

Let

$$\Xi_m^n \; := \; \exists x_1 \ldots \exists x_n \bigwedge_{i,j=1}^{n} (\forall y_1 \neq x_i, x_j \ldots \forall y_{n-2} \neq x_i, x_j)[d(x_i, x_j) \setminus \{y_1, \ldots, y_{n-2}\} \leqslant m]$$

In words: there are n vertices and no two of them can be separated by $n-2$ other vertices. Note that $\Xi_m^n \in \mathcal{L}^{n+1}$ for all m, and that if a graph G has a TK_n as a subgraph, then $G \vDash \Xi_{2m}^n$.

Proposition 13 *$\{\neg\Xi_m^n \mid m \in \omega\}$ axiomatises $(n-1)$-flatness for $n = 2, 3, 4$, but not for higher n.*

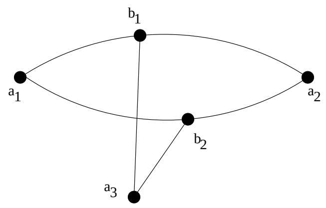
$\square$ This is rather easy to see for $n = 2, 3$. In case $n = 3$ it is not the axiomatisation given above, but it works as well. Not however that if a_1, a_2, a_3 witnessed some Ξ_m^3, they need not to be the corner vertices of the topological K_3.

Now let $n = 4$ and suppose Ξ_m^4 holds for some m. We then want to construct a TK_4 subgraph. Let $a_1, \ldots, a_4$ witness Ξ_m^4.

If a is a vertex outside a cycle C, then the *boundary points* of a on C are all the vertices on C that can be reached from a by paths disjoint from the cycle.

Claim 1: there is a cycle containing a_1 and a_2, a path from a_3 to the circle that does not meet the circle in a_1, a_2, and everything disjoint from a_4:

Consider the paths between a_1, a_2, a_3 disjoint from a_4. By Menger's theorem (applied to $G \setminus \{a_3, a_4\}$), there are two disjoint paths between a_1 and a_2, i.e. a cycle C_1. Again by Menger's theorem (applied to $G / C_1 \setminus \{a_4\}$), there are two disjoint paths from a_3 to the cycle, and it is possible to choose them in such a way that the boundary points are distinct (otherwise a_4 and the one boundary point would separate a_3 from one of a_1, a_2). Now if one of the boundary points is a_1 or a_2, or if the two boundary points are on the same arc of C_1 as divided by a_1, a_2, then one can extract a cycle C_2 containing a_1, a_2, a_3. Otherwise the following situation occurs, which also satisfies the claim:



Let $C := C_2$ if possible, otherwise $C := C_1$, and consider boundary points of a_4 on C.

Claim 2: a_4 has at least three boundary points.

Suppose there are less. In the first case, one of a_1, a_2, a_3 is not a boundary point and would be separated from a_4 by taking the boundary points away: contradiction.

In the second case, the same argument with a_1, a_2 shows that there are at least two boundary points. If there were only two, then they have to be a_1 and a_2: otherwise there is still a path from a_4 to a_1 (or a_2) avoiding the two boundary points. This path meets the circle at last in a_1 (or a_2). But now a_1 and a_2 can't separate a_4 from a_3: hence a path from a_4 through a_3 to one of the b_i provides another boundary point.

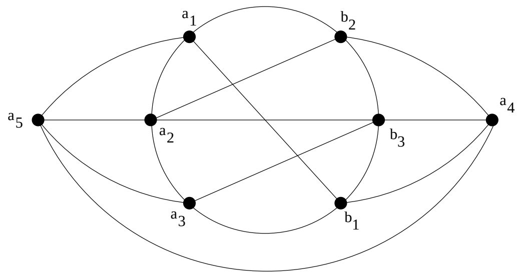
Now choose three boundary points b_1, b_2, b_3 of a_4 and paths P_i from b_i to a_4, disjoint from the cycle.

Claim 3: It is possible to find a vertex b_4 on these paths and to extract pairwise disjoint paths from b_i to b_4 for $i = 1, 2, 3$.

If $b_4 = a_4$ is not possible, assume w.l.o.g. that P_1 is not disjoint from one of the other paths. Go along the path from b_1 to a_4 and let c be the first vertex where P_1 meets another path, P_2 say (w.l.o.g.). Replace then a_4 by c, P_i by $P_i|_{[b_i,c]}$ for $i = 1, 2$ and P_3 by $P_1|_{[c,a_4]}{}^\frown P_3^{-1}$. The total length of the paths reduces and we can apply induction.

Then the b_i are the corner vertices of a TK_4 subgraph.

The graph on the right side is a counter-example for $n = 5$. It satisfies Ξ_3^5 witnessed by $a_1, \ldots, a_5$, but has no topological K_5.



This example can be used to generate counter-examples for higher n. $\square$

In the proof for $n = 4$, a bounded version of Menger's theorem should provide a bound of $3m-2$ on the length of the paths between the b_i. (Better arguments probably provide much better bounds).

Can one arrange to have some of the a_i among the corner vertices?

Here is another axiomatisation excluding this counter-example. It clearly implies the Ξ-axiomatisation and therefore works in the cases $n = 2, 3, 4$. Fix $m \in \omega$ and define recursively $\check{\Theta}_n^m(x_1, \ldots, x_n; \bar{y})$ by

$$\begin{aligned}
\check{\Theta}_0^m(\bar{y}) \;&:=\; \top \\
\check{\Theta}_{n+1}^m(x_1, \ldots, x_{n+1}; \bar{y}) \;&:=\; \bigwedge_{i=1}^{n+1} \Bigg(\forall z_1 \ldots \forall z_{n-1} \Big(\bigvee_{\substack{j=1 \\ j \neq i}}^{n+1} [d(x_i, x_j) \setminus \{\bar{z}, \bar{y}\} \leqslant m] \Big) \;\wedge \\
&\wedge\; \forall z_1 \ldots \forall z_n \Big(\bigwedge_{\substack{j=1 \\ j \neq i}}^{n+1} \big([d(x_i, x_j) \setminus \bar{z}) > m] \wedge \bigwedge_{k=1}^{n} x_j \neq z_k \big) \rightarrow \check{\Theta}_n^m(x_1 ... \widehat{x_i} ... x_{n+1}; \bar{z}) \Big) \Bigg)
\end{aligned}$$

and let $\Theta_n^m \;:=\; \exists x_1 \ldots \exists x_n \check{\Theta}_n^m(x_1, \ldots, x_n)$.

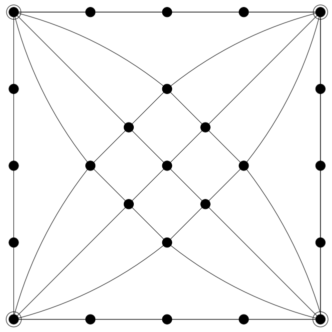
To check whether this works for $n = 5$ or even higher n, one should work by induction. But this seems to be difficult if one cannot ensure that given points are the corner vertices of a topological K_{n-1}, which is another problem.

Topological subgraphs with given corner vertices

Is it possible to express, even for small n, that given vertices $a_1, \ldots, a_n$ are the corner vertices of a topological K_n?

This is of course trivial for $n = 2$. Due to Menger's theorem, one can express that two vertices a_1 and a_2 lie on a common cycle: just say that for some m, for any other vertex b there is a walk of length at most m between a_1 and a_2 avoiding b. This can be done in $\mathcal{L}^3(a_1, a_2)$. (Compare with lemma 8).

What about $n = 3$?



In general, one can always take the negation of all formulas saying that there are n vertices of degree at least $n-1$ and paths of length at most m between them. This axiomatises a large class of n-flat graphs with few variables. But graphs as the one on the right side seem to indicate that it is much more difficult if not impossible to ensure that the paths are disjoint.

One positive result in the other direction, providing a subgraph in $\mathcal{K}^n_m$, is the following observation by Baldwin:

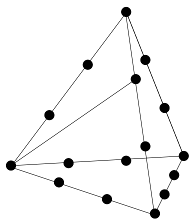
It is expressible in $\mathcal{L}^{n+1}$ that

- there are n vertices $a_1, \ldots, a_n$ of degree at least n;
- for all $i \neq j$ there are walks $(v_0, \ldots, v_m)$ between a_i and a_j such that $d(v_h, a_k) \setminus \{a_i, a_j\} > \frac{m}{2}$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$ and $h = 1, \ldots, m-1$

$$\left(\text{use lemma 4 with } l = 0 \text{ and } \xi_i(y; x_1 ... \widehat{x}_i .. \widehat{x}_j ... x_n) \;=\; \bigwedge_{\substack{k=1 \\ k \neq i,j}}^{n} [d(y, x_k) \setminus \{x_i, x_j\} > \frac{m}{2}] \right)$$

and a graph satisfying this formula contains a $\mathcal{K}^n_m$ as a subgraph, since the condition implies the paths to be disjoint.

The condition of the proposition does not allow "shortcuts" between the paths as in the graph in the picture, which is K^2_4 with an additional edge (and a minimal counter-example). If one adds after the distance condition above "unless $Ev_h a_k$", then a minimal counter-example would be K^4_4 with a shortcut of length 2.



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