The Hoffman–Singleton graph and outer automorphisms

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In this note, I show that the Hoffman–Singleton graph can be constructed from a non-trivial outer automorphism of S_6 and vice versa. I have learned from Peter Cameron that this was already known by Higman.

A graph (G, E) is a binary, symmetric, and anti-reflexive relation E on the set G. A *Moore graph* of type (d, D) is a d-regular graph of diameter D with $1 + d \cdot \sum_{i=0}^{D-1} (d-1)^i$ vertices. The Moore graphs are almost completely classified (in [HS], [B], and [D]). The following types exist:

<table>
<tr><td>$(0,0)$:</td><td>one vertex</td><td rowspan="5">} unique up to isomorphism</td></tr>
<tr><td>$(d,1)$ with $d \geqslant 1$:</td><td>the complete graph K_{d+1}</td></tr>
<tr><td>$(2,D)$ with $D > 0$:</td><td>the $(2D+1)$-cycle</td></tr>
<tr><td>$(3,2)$:</td><td>the Petersen graph</td></tr>
<tr><td>$(7,2)$:</td><td>the Hoffman–Singleton graph</td></tr>
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and possibly $(57, 2)$

Fix now $D = 2$ and let $m := d-1$. A $(d, 2)$-Moore graph has $1+d^2$ vertices. An n-*cycle* is a sequence $(b_1, \ldots, b_n)$ of vertices with $(b_i, b_{i+1}) \in E$ and $(b_n, b_1) \in E$. A *triangle* is a 3-cycle and a *quadrangle* a 4-cycle.

If $a_\emptyset$ is some vertex, let $a_0, \ldots, a_m$ be its neighbours and $a_{i1}, \ldots, a_{im}$ the other neighbours of a_i. Moreover, define $A_i := \{a_{i1}, \ldots, a_{im}\}$.

Proposition 1 *A finite d-regular graph G is $(d, 2)$-Moore iff G is triangle- and quadrangle-free and if all vertices have distance at most 2 from some (any) fixed vertex a.*

Proof: Clearly, in a d-regular graph, the $1 + d^2$ elements $a_\emptyset, a_i, a_{ij}$ as above are pairwise distinct, that is the ball $B_2(a_\emptyset)$ of radius 2 around a vertex $a_\emptyset$ has $1 + d^2$ elements, if and only if there are no triangles or quadrangles through $a_\emptyset$.

"$\Rightarrow$" If G is Moore of diameter 2, $|G| = 1 + d^2$ and $B_2(a) = G$ for any vertex a.

"$\Leftarrow$" By assumption and the argument above, $G = B_2(a)$ for some a and $|B_2(a)| = d^2 + 1$ for any a. But this implies $B_2(a) = G$ for any a, hence G has diameter 2. □

Remark: each vertex and each edge lie together on a 5-cycle; G is a union of 5-cycles.

Let G be a $(m+1, 2)$-Moore graph, and fix some vertex $a_\emptyset$. Then

- there are no edges between vertices in A_i (otherwise there would be a triangle);
- each vertex in A_i has to be neighbour to exactly one vertex in A_j for every $j \neq i$:
  Since a_{ik} has distance 2 from a_j, there is some edge (a_{ik}, a_{jl}), and because a_{ik} has valency $m+1$, there can't be a second edge to A_j (alternative argument: a second edge $(a_{ik}, a_{jl'})$ would provide a quadrangle $(a_{ik}, a_{jl}, a_j, a_{jl'})$).
- given a_{ik} and a_{jl} with $i \neq j$, there is some a_{gh} with $(a_{ik}, a_{gh}) \in E$ and $(a_{gh}, a_{jl}) \in E$.

Suppose that the vertices in A_i are numbered in such a way that $(a_{0j}, a_{ij}) \in E$ for all i and j. Then

$$\sigma_{ij} : k \mapsto l \iff (a_{ik}, a_{jl}) \in E \tag{1}$$

defines a permutation $\sigma_{ij} \in S_m$. By definition, $\sigma_{ij} = \sigma_{ji}^{-1}$. Moreover, we let $\sigma_{ii} = \mathrm{id}$ for all i. Composition of permutations will be written from left to right.

Proposition 2 *The existence of a Moore graph of type* $(m+1, 2)$ *is equivalent to the existence of a system of permutations* $\sigma_{ij} \in S_m$ *with*

$$\left\{ \begin{array}{l} \sigma_{ij} = \sigma_{ji}^{-1} \textit{ and } \sigma_{ii} = \mathrm{id}, \\ \textit{if } i \neq k, \textit{ then } \sigma_{ij}\sigma_{jk} \textit{ is fixpoint-free,} \\ \textit{if } i \neq j \neq k \neq i \textit{ and } l \neq j, \textit{ then } \sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li} \textit{ is fixpoint-free.} \end{array} \right\} \tag{2}$$

PROOF: Given the graph, we define the permutations as above, and given the system of permutations, we define a graph via (1). Then this is a $(m+1)$-regular graph and all vertices have distance $\leqslant 2$ from $a_\emptyset$. Using proposition 1, we must show that triangle- and quadrangle-freeness is equivalent to the fixpoint conditions. Triangles and quadrangles through $a_\emptyset$ or some a_i are already excluded by the construction.

A triangle through some vertex a_{0e} is of the form (a_{0e}, a_{ie}, a_{ke}) with $i, k \neq 0$ distinct, and corresponds to the fixpoint e of $\sigma_{ik} = \sigma_{ii}\sigma_{ik}$. The remaining possible triangles are of the form (a_{ie}, a_{jf}, a_{kg}) with $i, j, k \neq 0$ pairwise distinct, and correspond to the fixpoint e of $\sigma_{ij}\sigma_{jk}\sigma_{ki} = \sigma_{ij}\sigma_{jk}\sigma_{ki}\sigma_{ii}$.

Analogously, a quadrangle through some vertex a_{0e} is of the form $(a_{0e}, a_{ie}, a_{jf}, a_{ke})$ with pairwise distinct $i, j, k \neq 0$, and corresponds to the fixpoint e of $\sigma_{ij}\sigma_{jk}$. The remaining possible quadrangles are of the form $(a_{ie}, a_{jgf}, a_{kg}, a_{lh})$ with $i, j, k \neq 0$ pairwise different and $l \neq j$, and correspond to the fixpoint e of $\sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li}$. □

There are three known Moore graphs of diameter 2, namely for $m = 1, 2$ and 6.

- For $m = 1$, the system of permutations is reduced to $\sigma_{11} = \mathrm{id}$.
- For $m = 2$, it consists of $\sigma_{11} = \sigma_{22} = \mathrm{id}$ and $\sigma_{12} = \sigma_{21} = (12)$.
- For $m = 6$, we have the following result:

Proposition 3 *Let $\alpha \in \mathrm{Aut}(S_6) \setminus \mathrm{Inn}(S_6)$. Then $\sigma_{ij} := (ij)^\alpha$ is a system of permutations satisfying (2) and thus defines the Hoffman–Singleton graph. Up to isomorphism over a fixed edge, the construction does not depend on the choice of α.*

Proof: Consider permutations in their cycle decomposition. Let the type of a permutation σ be the multi-set of the cycle lengths $\neq 1$. Then the type determines the conjugation class of σ. A non-trivial outer automorphism α interchanges type $\{2\}$ with type $\{2, 2, 2\}$ and type $\{3\}$ with type $\{3, 3\}$.

$$\text{Let } i \neq k, \text{ then} \qquad (ij)(jk) = \begin{cases} (ikj) & \text{if } i \neq j \neq k \\ (ik) & \text{otherwise} \end{cases}$$

$$\text{Let } |\{i, j, k\}| = 3 \text{ and } l \neq j, \text{ then} \quad (ij)(jk)(kl)(li) = \begin{cases} (jkl) & \text{if } i \neq l \neq k \\ (jk) & \text{otherwise} \end{cases}$$

Hence $\sigma_{ij}\sigma_{jk} = \big((ij)(jk)\big)^\alpha$ and $\sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li} = \big((ij)(jk)(kl)(li)\big)^\alpha$ are without fixpoints.

Finally, composing α with an inner automorphism (on the right or the left side) corresponds to a renumbering of $\{a_1, \ldots, a_6\}$ or the elementsa of A_0. Thus two distinct choices for α yield graphs isomorphic over $\{a_\emptyset, a_0\}$. □

The cases $m = 1$ and $m = 2$ can be considered as coming in the same way from the identity automorphism of S_m.

If we take for granted that the Hoffman–Singleton graph is unique up to isomorphism, for some choice of $a_\emptyset$ and of a_0, the map $(ij) \mapsto \sigma_{ij}$ extends to a non-trivial outer automorphism of S_6. Moreover:

Corollary 1 ([BL]) *The order of the automorphism group of the Hoffman–Singleton graph divides $(1 + d^2) \cdot d! = 50 \cdot 7!$.*

Proof: There are $\frac{1}{2} \cdot 50 \cdot 7$ edges, hence the order of the stabilizer of $(a_\emptyset, a_0)$ divides $50 \cdot 7$. Once $a_\emptyset$ and a_0 fixed, there are $6! = |\mathrm{Inn}(S_6)|$ possibilities for numberings of A_0 and $\{a_1, \ldots, a_6\}$ providing non-identical copies of the Hoffman–Singleton graph. □

Remark 1: In all known cases of Moore graphs of diameter 2, the non-identical permutations σ_{ij} are involutions. Call such a Moore graph *involutional.* A Moore graph is involutional if and only if it is built up from Petersen graphs. An involutional Moore graph of type $(m+1, 2)$ needs $\frac{1}{2}m(m-1)$ different fixpoint-free involutions. On the other hand, S_m contains $(m-1)\cdot(m-3)\cdots$ fixpoint-free involutions. Both numbers are equal exactly for $m = 1, 2, 6$.

Remark 2: There is a presentation of S_m with generators σ_{ij} for $i, j = 1, \ldots, m$, $i \neq j$ and relations

$$\sigma_{ij} = \sigma_{ji} = \sigma_{ij}^{-1} \text{ and } \sigma_{ij}\sigma_{jk} = \sigma_{jk}\sigma_{ik} \text{ for pairwise distinct } i, j, k$$

Hence there is no involutional Moore graph of type $(57, 2)$ such that $\sigma_{ij}\sigma_{jk} = \sigma_{jk}\sigma_{ik}$ for all pairwise distinct i, j, k, since otherwise $\alpha : (ij) \mapsto \sigma_{ij}$ extends to a non-inner automorphism $S_{56} \to S_{56}$.

References

[B] E. Bannai, T. Ito, On finite Moore graphs, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20** (1973), pp. 191–208.

[BL] C. T. Benson, N. E. Losey, On a graph of Hoffman and Singleton, *J. Combinatorial Theory Ser. B* **11** (1971), pp. 67–79.

[D] R. M. Damerell, On Moore graphs. *Proc. Cambridge Philos. Soc.* **74** (1973), pp. 227–236.

[HS] A. J. Hoffman, R. R. Singleton, On Moore Graphs with Diameter 2 and 3, *IBM Journal* **4** (1960), pp. 497–504.